Mem. Fac. Lit. & Sci., Shimane Univ.,
Nat. Sci., 10, pp. 43–48 Dec. 20, 1976

Laser Oscillation as a Weakly Nonlinear Oscillation System

Toshio TAKEHARA
Department of Physics, Shimane University, Matsue, Japan
(Recieved September 6, 1976)

Laser oscillation is explained by van der Pol equation in nonlinear system. The characteristics of the oscillation and subharmonics in the forced steady-state oscillation are discussed.

§ 1. Introduction

The theory of oscillations of physical systems commonly is based on the assumption that the restoring forces are proportional to the deflections and the damping forces to the velocities. Often the deviation from linearity causes only a small deviation of the linear oscillation without changing the general character of the motion. For example, the simple harmonic oscillation, which we obtain in the linear case, will be accompanied by harmonics of small amplitude if the deviation from linearity is taken into account. However, in some cases the whole character of the oscillatory motion changes. Slight negative damping in the range of small deflections may cause self-excited periodic oscillations, whose period is quite different from the period of the undamped harmonic oscillation. Nonlinearity may change the response of the systems to external periodic forces. Van del Pol[1] investigated the electric oscillations in the triode circuit and found the differential equation of the relaxation oscillation and studied in detail.

Laser is a quantummechanical system. Precisely it must be treated quantum-statisticalmechanically, but fundamental equations of laser in semi-classical theory is reduced to van der Pol equation in fair approximation. Laser oscillation is a self-excited nonlinear oscillation. Subharmonics in the forced steady-state oscillation is also a effect of nonlinearity.

§ 2. Theory of Laser Oscillation by Semi-classical Theory

In semi-classical laser theory, electromagnetic field in a optical resonator is treated classically by Maxwell equations, and the field is expanded by the eigen mode of the resonator, whereas the laser material is described by Schrödinger wave equation or density matrix. The following fundamental equations are obtained for a single mode.[2]

$$\mathrm{d}b/\mathrm{d}t+[\kappa+i(\Omega-\omega)]b=igs \tag{1}$$

$$\mathrm{d}s/\mathrm{d}t+[\gamma_\perp-i(\omega-\omega_0)]s=-igb\sigma \tag{2}$$

$$\mathrm{d}\sigma/\mathrm{d}t-\gamma_\parallel(\sigma_0-\sigma)=2ig(s^*b-sb^*) \tag{3}$$

In these equations, the following notations are used. First, parameters to be given are

- Ω: resonance frequency of resonator
- ω_0: central frequency of transition
- κ: phenomenological attenuation constant of resonator, which is related to the Q value of resonator Q_c with $Q_c=\Omega/2\kappa$
- $\gamma_\parallel$: longitudinal relaxation constant ($=1/T_1$)
- $\gamma_\perp$: transverse relaxation constant ($=1/T_2$)
- g: coupling constant between material and electromagnetic field (characteristic Rabi frequency) and

$$g=(\mu/\hbar)\sqrt{(\hbar\omega_0 A/2\varepsilon_0 V)},\quad V=\int|f|^2dV,\quad A=(1/V)\int|f|^4dV$$

where
- f: normalized mode eigenfunction of resonator
- μ: matrix element of dipole
- ε_0: permittivity of free space
- σ_0: a parameter which represents pumping strengths or corresponds to the number of inverted populations.

The physical meaning of b, s, σ, ω to be solved is as follows.

- b: b and b^* correspond to annihilation and creation operator of photons respectively, and are related with amplitude of electric field operator $\tilde{E}(t)$ by $\tilde{E}(t)=(\hbar\omega_0/2\varepsilon_0 V)^{1/2}b$. Number of photons n for this oscillation mode is $n=b^*b$.
- s: spin flip operator
- σ: effective inverted population for this mode
- ω: oscillation frequency

The fundamental equations (1)~(3) cannot be solved exactly. Fair approximation is needed. The laser is assumed to oscillate in the center of the line for simplicity ($\Omega=\omega_0=\omega$). Then (1)~(3) have the form

$$\mathrm{d}b/\mathrm{d}t+\kappa b=igs \tag{4}$$

$$\mathrm{d}s/\mathrm{d}t+\gamma_\perp s=-igb\sigma \tag{5}$$

$$\mathrm{d}\sigma/\mathrm{d}t+\gamma_\parallel\sigma=\gamma_\parallel\sigma_0+2ig(s^*b-sb^*) \tag{6}$$

Apply $\mathrm{d}/\mathrm{d}t+\gamma_\perp$ to (4) and use (5), and we obtain

$$d^2b/dt^2+(\kappa+\gamma_\perp)db/dt+\kappa\gamma_\perp b=g^2b\sigma. \tag{7}$$

Now, if we eliminate s and s^*, by using (4) and the complex conjugate of it to (6), we have

$$d\sigma/dt+\gamma_\parallel\sigma=\gamma_\parallel\sigma_0-2d(b^*b)/dt-4\kappa b^*b \tag{8}$$

Now, an assumption

$$|db/dt|\ll\gamma_\parallel|b|,\ \kappa|b| \tag{9}$$

is introduced, then d^2b/dt^2 can be omitted against $\kappa\, db/dt$ in (7) and $d(b^*b)/dt$ against $2\kappa\, b^*b$ in (8), (7), (8) become

$$db/dt+[\kappa\gamma_\perp/(\kappa+\gamma_\perp)]b=[g^2/(\kappa+\gamma_\perp)]b\sigma, \tag{10}$$

$$d\sigma/dt+\gamma_\parallel\sigma=\gamma_\parallel\sigma_0-4\kappa\, b^*b. \tag{11}$$

From (10) and (11) and the assumption (9) is used twice, we obtain

$$db/dt-[(g^2\sigma_0-\kappa\gamma_\perp)/(\kappa+\gamma_\perp)]b+[4g^2\kappa/\gamma_\parallel(\kappa+\gamma_\perp)]\,(b^*b)b=0. \tag{12}$$

The coefficients are simplified if we put

$$\alpha=g^2\sigma_0/(\kappa+\gamma_\perp),\quad \gamma=\kappa\gamma_\perp/(\kappa+\gamma_\perp),\quad \beta=[4g^2\kappa/\gamma_\parallel(\kappa+\gamma_\perp)] \tag{13}$$

(12) have the form

$$db/dt+(\gamma-\alpha+\beta b^*b)b=0 \tag{14}$$

where constant γ represents attenuation by the loss of resonator, α the gain by the laser medium, βb^*b saturation of gain. Differential equation of the first order (14) is equivalent in some approximation to the differential equation

$$d^2x/dt^2+2(\gamma-\alpha+\beta x^2)dx/dt+\Omega^2x=0, \tag{15}$$

if we put $x=b\,e^{-i\omega_0 t}+b^*\,e^{i\omega_0 t}$ and apply rotating-wave approximation and omit the term $\exp(\pm 3\omega_0 t)$. (14) and (15) are van der Pol equation for the rotating-wave approximation. Solution of van der Pol equation of formal type is discussed in the next section.

§3. Self-Excited Nonlinear Oscillations

The general form of the equation of free oscillation includes the cases of nonlinear damping and nonlinear restoring force.[3-5] First we consider the linear equation of motion for a single mass m and restoring force $-kx$ with the damping force $-\mu\dot{x}$

$$m\ddot{x}+\mu\dot{x}+kx=0$$

Introducing natural frequency ω_0 of undamped oscillation, we have

$$\ddot{x}+2b\dot{x}+\omega_0^2 x=0 \tag{16}$$

We will generalize (16) in the form

$$\ddot{x}+\omega_0^2 x=f(x, \dot{x}) \tag{17}$$

Let us assume that the function $f(x, \dot{x})$ has the form $f(x, \dot{x})=\varepsilon\phi(x)\dot{x}$, where ε is a positive constant parameter. It is seen that in this case the damping is proportional to the velocity $v=\dot{x}$, but the magnitude of the damping factor is in general a function of deflection. Van der Pol studied an interesting special case assuming $\phi(x)=1-x^2$. Then van der Pol equation is given by

$$\mathrm{d}^2x/\mathrm{d}t^2-\varepsilon(1-x^2)\dot{x}+x=0, \qquad \omega_0^2=1 \tag{18}$$

This equation was first introduced concerning the oscillation in triode circuit. The motion of a mechanical pendulum with nonlinear friction (Froude pendulum), for example, obey the similar differential equation

$$\mathrm{d}^2y/\mathrm{d}t^2-\varepsilon(1-\dot{y}^2)\dot{y}+y=0 \tag{19}$$

which is called Rayleigh equation and is reduced to (18) if we put $x=\sqrt{3}\dot{y}$ and differentiate (19). (18) can be reduced to a differential equation of the first order by considering $\dot{x}=v$ as a function of x. We obtain

$$\mathrm{d}v/\mathrm{d}t=-x/v+\varepsilon(1-x^2) \tag{20}$$

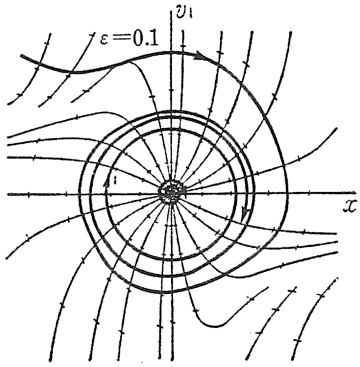
This can be solved by the method of isoclines. Consider first the case $\varepsilon=0$. The integral curve of (20) are concentric circles given by

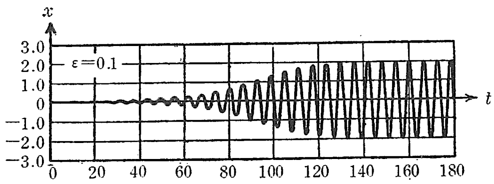
$$v^2+x^2=\text{const.}$$

Since $\mathrm{d}t=\mathrm{d}x/v$, we obtain the x, v values corresponding to the actual motion by proceeding clockwise along the integral curves. The corresponding motion is a harmonic oscillation with constant amplitude.

For small value of ε, for example, $\varepsilon=0.1$, the closed curve is only slightly different from a circle, but its shape varies quite radically with increasing ε. The corresponding motion for $\varepsilon=0.1$ is shown in figures, that is, deflection is plotted as a function of time. Relaxation oscillation is excited. For $t\to\infty$ the motion in phase space approaches asymptotically to a closed curve (limit cycle). For small ε limit cycle is close to a circle. In such a periodic motion the average of work done by the damping force $\varepsilon(1-x^2)\dot{x}$ over a cycle must be zero. Then we see that

$$\varepsilon\oint(1-x^2)\dot{x}\mathrm{d}x=\varepsilon\oint(1-x^2)\dot{x}^2\mathrm{d}t=0 \tag{21}$$





For small ε, the limit cycle is close to a circle, and $x=a\sin t$. Thus (21) yields

$$0=\int_0^{2\pi}(1-a^2\sin^2 t)a^2\cos^2 t\,dt=\pi a^2(1-a^2/4)$$

from which we have $a=2$. Solution of (18) in the first approximation is obtained as

$$x=\{a_0\exp(\varepsilon t/2)/\sqrt{[1+(a_0^2/4)(\exp\varepsilon t-1)]}\}\sin(t+\varphi) \tag{22}$$

for the initial deflection a_0. The solution tends to the steady motion

$$x=2\sin(t+\varphi) \tag{23}$$

as $t\to\infty$. Small fluctuation grow up to a steady state oscillation.

§4. Eorced Oscillations of Nonlinear Systems

As an example of nonautonomous systems, let us now consider a particular nonlinear oscillator described by Duffing equation

$$\ddot{x}+2b\dot{x}+\omega_0^2x=f\cos\omega t+\varepsilon x^3 \tag{24}$$

Equation of this type occurs frequently when one take into account the nonlinearity of the restoring force. An example is a pendulum for which the restoring force is proportional to $\sin x$. The expansion $\sin x \sim x - x^3/6$ is very accurate for the oscillation.

We shall confine ourselves to the case of weak damping and weak nonlinearity. We assume b, $\varepsilon \ll 1$. It may be expected that the steady state solution of this equation will be related to the solution x_0 of the linear equation obtained as $\varepsilon \to 0$. The steady state solution of the linear equation may be written as

$$x_0 = A(\omega, b)\cos(\omega t + \varphi)$$

$A(\omega, b)$, φ are given by

$$A(\omega, b) = f[(\omega^2 - \omega_0^2)^2 + 4b^2\omega^2]^{-1/2} \tag{25}$$

$$\varphi = \tan^{-1}[2b\omega/(\omega^2 - \omega_0^2)] \tag{26}$$

so that in natural to put

$$x = x_0 + \xi = A\cos(\omega t + \varphi) + \xi \tag{27}$$

Upon substitution into (24), we are led to the following equation for ξ:

$$\ddot{\xi} + 2b\dot{\xi} + \omega_0^2\xi = \varepsilon[(1/4)A^3(\cos 3\omega t + 3\cos\omega t) + (3/2)A^2\xi(1 + \cos 2\omega t) + 3A\xi^2\cos\omega t + \xi^3] \tag{28}$$

For simplicity, the time origin has been shifted by φ/ω so that $\cos(\omega t + \varphi) \to \cos\omega t$, and $\cos^2\omega t$, $\cos^3\omega t$ are expressed in terms of the multiple angle $2\omega t$, $3\omega t$. 3ω is an effect of the nonlinearity. We should like to note that the amplification factor A of the response can produce a response of order one even if the damping b is sufficiently small. We guess that the first term $(1/4)A^3\cos 3\omega t$ would make a contribution of order one when $3\omega \sim \omega_0$. $\omega_0/3$ is a subharmonic oscillation.

References

1) B. VAN DER POL, Phil. Mag. 3 (1927) 65.
2) M. SARGENT III, M. O. SCULLY and W. E. LAMB, Laser Physics (Addison-Wesley Pub. Co.) 1974.
3) K. N. KRYLOFF and N. BOGOLIUBOFF, Introduction to Non-linear Mechanics (Princeton) 1947.
4) J. J. STOKER, Nonlinear Vibrations (Interscience Publishers) 1950.
5) N. W. MCLACHLAN, Ordinary Non-linear Differential Equations in Engineering and Physical Sciences (Oxford) 1956.