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On Random Fields on Hyperbolic Spaces

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In this paper, we examine the spectral representation of the covariance function of w-homogeneous random field on hyperbolic spaces and compare it with the case of Euclidean space.

§1. Geometric concepts

1. Let $E_{n,1}$ be a real $(n+1)$-dimensional vector space with a metric ground form

$$[X, X]=X_1^2+\cdots+X_n^2-X_{n+1}^2 \tag{1}$$

A set of $X\in E_{n,1}$ such that $[X, X]<0$ is a cone with vertex at origin. By a **hyperbolic rotation** we shall mean a linear transformation which does not change the distance of points from the origin, preserves the orientation of the space and transforms both halves of the cone into themselves. These transformations form a group and we shall call it **the group of hyperbolic rotations** of $E_{n,1}$, and denote it by $SO(n, 1)$.

2. Let

$$H_{n+1}^{+}=[X\in E_{n,1};\ [X, X]=-1,\ X_{n+1}>0] \tag{2}$$

The group $SO(n, 1)$ acts transitively on the space H_{n+1}^{+} and a subgroup $SO(n)$ leaves the point $\xi=\xi(0,\dots, 0, 1)$ invariant and it is the isotropy subgroup at this point. Thus, the space H_{n+1}^{+} is the coset space $SO(n, 1)/SO(n)$. When $n\neq 3$, we call it **the real hyperbolic space with dimension** $\boldsymbol{n}$.

3. With $X_{n+1}=1$, $[X, X]=0$ is equivalent to $X_1^2+\cdots+X_n^2=1$. The real hyperbolic space can also be interpreted as the interior of the n-dimensional ball of radius 1.

4. When $n=2$, it is also the coset space $SL(2; R)/SO(2)$.

5. The n-dimensional real Euclidean space E_n is the coset space $M(n)/SO(n)$, where $M(n)$ is the group of real matrices of the order $(n+1)$,

$$g=g(a, h)=\begin{pmatrix} h & a \\ 0 & 1 \end{pmatrix},$$

where h is an element of the group $SO(n)$, a is a column vector with n real elements, and 0 is a zero n-row vector.

§2. Irreducible unitary representations and zonal spherical functions

1. We denote by χ the pair $\chi=(l, \varepsilon)$ of complex number l and number ε taking values 0 and 1/2. With each such pair we associate a space $\mathfrak{D}_\chi$ of functions $\phi(z)$ of the complex variable $z=x+iy$, such that:

1) the function $\phi(z)$ is infinitely differentiable with respect to x and y at all points $z=x+iy$, except the point $z=0$;
2) for any positive number a one has the equality

$$\phi(az)=a^{2l}\phi(z)\,;$$

3) the function $\phi(z)$ has given parity:

$$\phi(-z)=(-1)^{2\varepsilon}\phi(z)\,.$$

If Γ is the unit circle, then for $\varepsilon=0$ the space $\mathfrak{D}_\chi$ is realized as the space of infinitely differentiable even functions on Γ, and for $\varepsilon=1/2$, as the space of infinitely differentiable odd functions on Γ. Let us define a function $f(\exp i\theta)$ as the following: for $\phi\in\mathfrak{D}_\chi$,

$$f(\exp i\theta)=\begin{cases}\phi(\exp i\theta/2), & \varepsilon=0\\ \exp(i\theta/2)\phi(\exp i\theta/2), & \varepsilon=1/2.\end{cases}\tag{3}$$

2. Let $C=\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$ and for any $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of $SL(2;R)$, we correspond a matrix $h=C^{-1}gC$, which forms a group $SU(1,1)$, and isomorphic to the group $SL(2;R)$.

3. With each element $g=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$ of $SU(1,1)$ we associate an operator $T_\chi(g)$ in the space $\mathfrak{D}$ of infinitely differentiable functions on the circle, defined by the formula

$$\begin{aligned}&T_\chi(g)f(\exp i\theta)\\ &\quad=(\beta\exp i\theta+\bar{\alpha})^{l+\varepsilon}(\bar{\beta}\exp-i\theta+\alpha)^{l-\varepsilon}f((\alpha\exp i\theta+\bar{\beta})/(\beta\exp i\theta+\bar{\alpha}))\end{aligned}\tag{4}$$

4. All representations $T_\chi(g)$, $\chi=(-1/2+i\rho, \varepsilon)$ are unitary and we shall call $T_\chi(g)$, $\chi=(-1/2+i\rho, 0)$ unitary representations of **the first principal series**, and $T_\chi(g)$, $\chi=(-1/2+i\rho, 1/2)$ unitary representations of **the second principal series.**

5. The representations $T_\chi(g)$, $-1<l<0$ and $\varepsilon=0$ are unitary and irreducible and we shall call them unitary representations of **the supplementary series.**

6. Now let $l+\varepsilon$ and $l-\varepsilon$ be both integers. We shall denote by $\mathfrak{D}_l^+$ the subspace in $\mathfrak{D}$ consisting of functions of the form $\exp -i(l-\varepsilon)f(\exp i\theta)$, where $f(z)$ is analytic inside the unit circle and $\mathfrak{D}_l^-$ the subspace in $\mathfrak{D}$ consisting of functions of the form $\exp i(l$

$+\varepsilon)f(\exp i\theta)$, where $f(z)$ is analytic outside the unit circle. For $l-\varepsilon<0$, in $\mathfrak{D}$ there are two nonintersecting invariant subspaces. We shall denote by $T_l^{\pm}(g)$ the representations induced by the representations in subspaces $\mathfrak{D}_l^{\pm}$. These are unitary and irreducible. The representations $T_{-l-1}^{\pm}(g)$, which are equivalent to $T_l^{\pm}(g)$ and act in the factor spaces $\mathfrak{D}_{-l-1}^{\pm}/\mathfrak{D}_{-l-1}^{0}$ respectively, where $\mathfrak{D}_{-l-1}^{0}=\mathfrak{D}_{-l-1}^{+}\cap\mathfrak{D}_{-l-1}^{-}$ are also unitary and irreducible. We shall call them unitary representations of **the discrete series**.

7. In the space $\mathfrak{D}$ of the representation $T_\chi(g)$ we choose a basis, consisting of the functions $[\exp -in\theta]$. We have an expression

$$T_\chi(g)\exp -in\theta=(\beta\exp i\theta+\bar{\alpha})^{l+n+\varepsilon}(\bar{\beta}\exp -i\theta+\alpha)^{l-n-\varepsilon}\exp -in\theta \tag{5}$$

When $\varepsilon=0$, we get a **class 1** representation with respect to the subgroup $SO(2)$ and we get the corresponding zonal spherical function $\mathfrak{P}_l(ch\tau)$, where $g=g(\phi,\tau,\psi)$ in the Euler angles in $SU(1,1)$, the Legendre function with index l;

$$\mathfrak{P}_l(ch\tau)=\frac{1}{2\pi}\int_0^{2\pi}(ch\,\tau+sh\,\tau\cos\theta)^l d\theta \tag{6}$$

In summary, we get class 1 representations for the first principal series, for the supplementary series, and for l integers. With respect to corresponding zonal spherical functions, when l is an integer, the Legendre function $\mathfrak{P}_l(z)$ coincides with the Legendre polynomial $P_l(z)$, and the Legendre functions $\mathfrak{P}_{-1/2+i\rho}(ch\tau)$, are called conical functions.

8. We shall denote by $V^{n+1,s}$ the space of functions $f(X)$ given on the upper half of the cone $[X,X]=0$, $X_{n+1}>0$, and such that

1) the functions $f(X)$ are infinitely differentiable at every point of the upper half of the cone,
2) the function $f(X)$ are homogeneous of degree s:

$$f(aX)=a^s f(X),\qquad a>0.$$

We define an operator $S^{n+1,s}(g)$, $g\in SO(n,1)$, in the space $V^{n+1,s}$ by the formula

$$S^{n+1,s}(g)f(X)=f(g^{-1}X),\qquad g\in SO(n,1), \tag{7}$$

Then $(S^{n+1,s}(g),\ V^{n+1,s})$ is a representation of the group $SO(n,1)$.

With each function f of the space $V^{n+1,s}$ we associate the function F,

$$F(\xi)=Qf(\xi):=f(\xi_1,\xi_2,\ldots,\xi_n,1)$$

on S^{n-1}, $(n-1)$-sphere, and to the operator $S^{n+1,s}(g)$ correspond operator of a representation

$$T^{n+1,s}(g)=QS^{n+1,s}(g)Q^{-1} \tag{8}$$

in the space $\mathfrak{D}$ of infinitely differentiable functions on the sphere S^{n-1}.

9. The representations $T^{n+1,s}(g)$, $s=-(n-1)/2+i\rho$, are unitary and irreducible and we shall call them the representations of **the principal series** of the group $SO(n, 1)$.

10. The representations $T^{n+1,s}(g)$, s: nonintegers and $-(n-1)<s<0$, are unitary and irreducible and we shall call them the representations of **the supplementary series** of the group $SO(n, 1)$.

11. Let $\mathfrak{D}^{n,k}$ be the space of polynomials of degree k in the variables $\xi_1,\ldots,\xi_n$. When $s=-n-k+1$, where k are nonnegative integers, the representations $(T^{n+1,s}(g), \mathfrak{D}^{n,k})$ are unitary and irreducible, and we shall call them the representations of **the discrete series** of the group $SO(n, 1)$.

12. These representations are representations of class 1 with respect to the subgroup $SO(n)$.

13. In the space $\mathfrak{D}(S^{n-1})$, we introduce the scalar product

$$(F, G)=\int_{S^{n-1}} F(\xi)\overline{G(\xi)}d\xi.$$

We complete it relative to the norm $\|F\|^2=(F, F)$, and we obtain the Hilbert space $L^2(S^{n-1})$. Functions of the form

$$G_K(\xi)=C(K)\prod_{j=0}^{n-3} C_{k_j-k_{j+1}}^{(n-j-2)/2+k_{j+1}}(\cos\phi_{n-j-1})\sin^{k_j+1}\phi_{n-j-1}\cdot \exp \pm ik_{n-2}\phi_1,$$

where $C_m^p(t)$ are Gegenbauer polynomials, $\phi_1,\ldots,\phi_{n-1}$ are geographical coordinates on S^{n-1}, $K=(k_0,\ldots, \pm k_{n-2})$, $k_0\geqslant k_1\geqslant\cdots\geqslant k_{n-2}\geqslant 0$, and $C(K)$ is a normalizing constant, form an orthonormal basis in the space $L^2(S^{n-1})$. The function $G_0(\xi)$, $0=(0,\ldots, 0)$, is equal to 1, and invariant for all h of $SO(n)$. The zonal spherical functions depend only on the Euler angle θ_n^n of $g=g(\theta_j^k;\ 1\leqslant k\leqslant n,\ 1\leqslant j\leqslant k)$ of g in $SO(n, 1)$. Let $g_n(\theta)$ be a hyperbolic rotation by the angle θ in the (X_n, X_{n+1})-plane.

Then, we have an expression for zonal spherical functions

$$Z^{n+1,s}(g_n(\theta)) =\frac{2^{(n-2)/2}\Gamma(n/2)}{sh^{(n-2)/2}\theta}\mathfrak{P}_{s+(n-2)/2}^{-(n-2)/2}(ch\theta), \tag{9}$$

where $\mathfrak{P}_{s+(n-2)/2}^{-(n-2)/2}(z)$ are associated Legendre functions. We have also an integral representation

$$Z^{n+1,s}(g_n(\theta))=\frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n-1)/2)}\int_0^\pi (ch\,\theta-\cos\phi sh\,\theta)^s \sin^{n-2}\phi d\phi \tag{10}$$

In particular, when $n=2$, we have an expression

$$Z^{3,s}(g_2(\theta))=\mathfrak{P}_s^0(ch\theta)=\mathfrak{P}_s(ch\theta).$$

14. Let R be a complex number and with each element $g=g(a, h)$ of the group

$M(n)$ we associate the operator

$$T_R(g)f(\xi)=\exp R(a, \xi)f(h^{-1}\xi),$$

where (a, ξ) is the scalar product, in the space $L^2(S^{n-1})$. When $R\neq 0$, are purely imaginary numbers, the representations are unitary and irreducible, and they are class 1. The zonal spherical functions depend only on the length of a; thus let $g=g(r, e)$, where $r=(0,\dots, r)$, then we get a expression of the zonal spherical functions

$$Z^{iy}(g)=\Gamma(n/2)\frac{J_{(n-2)/2}(yr)}{(yr/2)^{(n-2)/2}} \tag{11}$$

where $J_{(n-2)/2}(yr)$ are Bessel functions.

When $R=0$, we get $T_R(g)f(\xi)=f(h^{-1}\xi)$, $g=g(a, h)$, which is the quasi-regular representation of the subgroup $SO(n)$.

15. Summary:

(a) Let $(T(g), \mathfrak{H})$ be an irreducible (unitary) representation of a group G and K be a closed subgroup of G. A representation $T(g)$ is called a representation of class 1 relative to K if in its representation space there is a nonzero vector invariant relative to K and the restriction of $T(g)$ to K is unitary. Let f be a normalized invariant vector in the space $\mathfrak{H}$; then a function

$$f(g)=(T(g)f, f)$$

is called a zonal spherical function.

(b) When representation $T(g)$ is unitary, the zonal spherical function is positive definite.

(c) For the case of hyperbolic spaces, the zonal spherical functions are functions of the form

$$\frac{2^{(n-2)/2}\Gamma(n/2)}{sh^{(n-2)/2}\theta}\mathfrak{P}^{-(n-2)/2}_{s+(n-2)/2}(ch\theta),$$

where $s=-(n-1)/2+i\rho$ (principal series),

or s: nonintegers and $-(n-1)<s<0$ (supplementary series),

or $s=-n-k+1$, k nonnegative integers (discrete series).

These are positive definite functions. $\mathfrak{P}^m_l(z)$ are associated Legendre functions.

(d) For the case of Euclidean spaces, the zonal spherical functions are functions of the form

$$\Gamma(n/2)\frac{J_{(n-2)/2}(yr)}{(yr/2)^{(n-2)/2}},$$

where $y\neq 0$ is a real number and $J_{(n-2)/2}(yr)$ is a Bessel function. These functions are positive definite.

§3. *w*-homogeneous random fields on the hyperbolic spaces

1) By a **random field** on the space X, we shall mean a bounded measure on the space $\Omega=R^X$. Taking any $\omega\in R^X$ and $X_1,\dots, X_n$ of X we get a n-tuple $(\omega(X_1),\dots, \omega(X_n))$, and let $E_1,\dots, E_n$ be Borel fields in R, then consider $[\omega;\ \omega(X_1)\in E_1,\dots, \omega(X_n)\in E_n]$. We define a function $F(X_1,\dots, X_n;\ E_1,\dots, E_n)$ by a formula

$$F(X_1,\dots, X_n;\ E_1,\dots, E_n)=\mu[\omega;\ \omega(X_1)\in E_1,\dots, \omega(X_n)\in E_n] \tag{12}$$

where μ is a measure on $(R^X, \mathfrak{B}(R^X))$.

Let X be a homogeneous space G/K. We shall call a random field **homogeneous** when

$$F(gX_1,\dots, gX_n;\ E_1,\dots, E_n)=F(X_1,\dots, X_n;\ E_1,\dots, E_n)\qquad \text{for any}\quad g\in G,$$

and **(K-)isotropic** when

$$F(kX_1,\dots, kX_n;\ E_1,\dots, E_n)=F(X_1,\dots, X_n;\ E_1,\dots, E_n)\qquad \text{for any}\quad k\in K.$$

2) Given random field μ, we define a function

$$f(P, \omega):=\omega(P),\qquad P\in X.$$

We assume that

$$\int_\Omega |f(P, \omega)|^2 d\mu(\omega)=\int_\Omega |\omega(P)|^2 d\mu(\omega)<+\infty \tag{13}$$

We also assume that

$$\lim_{P\to Q}\int_\Omega |f(P, \omega)-f(Q, \omega)|^2 d\mu(\omega)=0,$$

and $\int_\Omega f(P, \omega)d\mu(\omega)=0$.

When the condition (13) holds, we can define a covariance function $C(P, Q)$, $P, Q\in X$, by the formula

$$C(P, Q)=\int_\Omega f(P, \omega)f(Q, \omega)d\mu(\omega). \tag{14}$$

The random field is called ***w*-homogeneous** when $C(gP, gQ)=C(P, Q)$ for any $g\in G$ and ***w*-isotropic** when $C(kP, kQ)=C(P, Q)$ for any $k\in K$.

3) Let $P=gP_0$, where P_0 is an invariant point with respect to the subgroup K

and $f_0(\omega)$ be $f(P_0, \omega)$, then we have a relation $f(P, \omega)=f(gP_0, \omega)$. The random field f spans the Hilbert space $\mathfrak{H}$, with the scalar product (14). The group G acts on $\mathfrak{H}$ by the formula

$$T(g)f(P)=f(g^{-1}P) \tag{15}$$

Then, we have $f(P)=T(g^{-1})f_0$, and for any $k\in K$, $T(k)f_0=f_0$. $(T(g), \mathfrak{H})$ is a representation of the group G.

4) (a) A necessary and sufficient condition for the random field f to be w-homogeneous is that the representation $(T(g), \mathfrak{H})$ is unitary and which has an invariant vector f_0 with respect to the subgroup K.

(b) A necessary and sufficient condition for the random field f to be w-isotropic is that the representation $(T(k), \mathfrak{H})$ restricted to the subgroup K is unitary and which has a invariant vector f_0.

5) **In the following we assume that the random field f is w-homogeneous.** Then the covariance function $C(P, Q)$ is of the form $C(P, Q)=C(g_2^{-1}g_1P_0, P_0)$, where $P=g_1P_0$, $Q=g_2P_0$. We define a function $B(g)$ on G by the formula

$$B(g)=C(gP_0, P_0)=(T(g^{-1})f_0, f_0)\,.$$

The function $B(g)$ is a positive definite and K-biinvariant functions.

6) **Now let X be a hyperbolic space.** Then, we get the spectral representation

$$B(g)=\frac{2^{(n-2)/2}\Gamma(n/2)}{sh^{(n-2)/2}\theta}\left[\int_{-\infty}^{\infty}\mathfrak{P}^{-(n-2)/2}_{-1/2+i\rho}(ch\theta)dF_1(\rho)\right.$$
$$\left.+\int_{-n/2}^{(n-2)/2}\mathfrak{P}_l^{-(n-2)/2}(ch\theta)dF_2(l)+\sum_{k=0}^{\infty}c_k\mathfrak{P}^{-(n-2)/2}_{-n/2-k}(ch\theta)\right] \tag{16}$$

, where c_k are nonnegative and $g=g_n(\theta)$. Since $\mathfrak{P}^{-(n-2)}_{-1/2-i\rho}(ch\theta)=\mathfrak{P}^{-(n-2)/2}_{-1/2+i\rho}(ch\theta)$, we get a formula

$$B(g)=\frac{2^{(n-2)/2}\Gamma(n/2)}{sh^{(n-2)/2}\theta}\left[\int_0^{\infty}\mathfrak{P}^{-(n-2)/2}_{-1/2+i\rho}(ch\theta)dG(\rho)+\int_{-n/2}^{(n-2)/2}\right.$$
$$\left.\mathfrak{P}_l^{-(n-2)/2}(ch\theta)dF_2(l)+\sum_{k=0}^{\infty}c_k\mathfrak{P}^{-(n-2)/2}_{n/2+k-1}(ch\theta)\right] \tag{16'}$$

7) **In the cace of X being a Euclidean space**, we get the spectral representation

$$B(g)=\Gamma(n/2)\int_0^{\infty}\frac{J_{(n-2)/2}(yr)}{(yr/2)^{(n-2)/2}}dF(y). \tag{17}$$

Thus, with $P=g_1P_0$, $Q=g_2P_0$ and $g_i=g(a_i, h_i)$ $(i=1, 2)$, we have an expression

$$C(P, Q)=B(g_2^{-1}g_1)=\Gamma(n/2)\int_0^{\infty}\frac{J_{(n-2)/2}(y\|a_1-a_2\|)}{(y\|a_1-a_2\|/2)^{(n-2)/2}}dF(y). \tag{18}$$

8) **Let X be the real hyperbolic plane.** In this case we have a formula

$$B(g)=\int_0^\infty \mathfrak{P}_{-1/2+i\rho}(ch\theta)dF(\rho)+\int_{-1}^0 \mathfrak{P}_l(ch\theta)dG(l)+\sum_{k=0}^\infty c_k P_k(ch\theta), \tag{19}$$

for $g=g(\phi, \theta, \psi)$. We have also when $B(g)\in L^2(G)$, $G=SU(1, 1)$,

$$B(g)=\frac{1}{4\pi^2}\int_0^\infty a(y)\mathfrak{P}_{-1/2+iy}(g)yth(\pi y)dy, \tag{20}$$

$$\text{where } a(y)=\int B(g)\overline{\mathfrak{P}_{-1/2+iy}(g)}dg,$$

$$dg=sh\theta d\theta d\phi d\psi.$$

When $P=g_1P_0$, and $Q=g_2P_0$, to get the function $C(P, Q)$ we substitute in the formula (19), $ch\theta=ch\theta_1ch\theta_2-sh\theta_1sh\theta_2\cos(\phi_1-\phi_2)$ with $g_i=g(\phi_i, \theta_i, \psi_i)$, $i=1, 2$.

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