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On Plane Bundles over Some Elliptic Surfaces

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M. F. Atiyah has given the classification theorem for holomorphic vector bundles over an elliptic curve, (Theorem 7, [2]). In the proof, two lemmas are effective, which are called the uniqueness and existence theorems. These are the motive for this paper. In §1, we prove that, over a product surface of a non singular curve and an elliptic curve, if a line bundle satisfies some condition about a local triviality and the Chern class, then it admits a non trivial extension to a-plane bundle. This fact corresponds to Lemma 16, [2]. In §2, we define a strongly reducible plane bundle and prove that not every plane bundle is strongly reducible over a basic member (8, [4]) on an algebraic curve of genus greater than one. This fact corresponds to Lemma 15, [2].

§ 1. Extensions of line bundles

Let S be the product surface $\Delta \times C$ of a non singular algebraic curve Δ and an elliptic curve C. Let $G \to S$ be a holomorphic line bundle. The surface S admits an open covering $\{U_j \times C_q\}$, where $\{U_j\}$, $\{C_q\}$ are open coverings of Δ, C, respectively. Let $\{h_{j(q)k(r)}(t, z)\}$ be a system of transition functions of G,

$$h_{j(q)k(r)}\colon (U_j \times C_q) \cap (U_k \times C_r) \longrightarrow C^* = C - \{0\},$$

where C^* is the set of complex numbers without the origin O. We call the line bundle G to be *locally Δ-trivial* if and only if

$$\frac{\partial \log h_{j(q)j(r)}(t, z)}{\partial t} = \psi_{j(r)}(t, z) - \psi_{j(q)}(t, z) \quad \text{in} \quad U_j \times C,$$

where $\psi_{j(r)}(t, z)$, $\psi_{j(q)}(t, z)$ are C^∞ in t and holomorphic in z, and the fraction of their exponentials $\exp\int \psi_{j(r)}(t, z)dt / \exp\int \psi_{j(q)}(t, z)dt$ is holomorphic in t. Then we have

$$h_{j(q)j(r)}(t, z) = \frac{\Psi_{j(r)}(t, z)}{\Psi_{j(q)}(t, z)} \bar{h}_{j(q)j(r)}(z), \tag{1}$$

where $\Psi_{j(r)}(t, z) = \exp\int \psi_{j(r)}(t, z)dt$, $\Psi_{j(q)}(t, z) = \exp\int \psi_{j(q)}(t, z)dt$, then $\bar{h}_{j(q)j(r)}(z)$ is holomorphic. It can be seen that

$$\bar{h}_{j(p)j(q)}(z)\bar{h}_{j(q)j(r)}(z) = \bar{h}_{j(p)j(r)}(z),$$

so, for each U_j, $\{h_{j(q)j(r)}(z)\}$ is a system of transition functions of a line bundle over C. We denote this line bundle by G_0. Suppose that first Chern class $C_1(G_0)$ is 1, then by

Lemma 16, [2], there exists an indecomposable plane bundle $E_0 \to C$, unique up to isomorphism, given by an extension

$$0 \longrightarrow I \longrightarrow E_0 \longrightarrow G_0 \longrightarrow 0,$$

where I is the product line bundle and $C_1(E_0)=1$. The system of transition functions of E_0 is given by

$$\begin{pmatrix} 1 & \tilde{h}_{j(q)j(r)}(z) \\ 0 & \bar{h}_{j(q)j(r)}(z) \end{pmatrix}.$$

By the relation

$$\tilde{h}_{j(q)j(r)}(z)+\tilde{h}_{j(p)j(q)}(z)\bar{h}_{j(q)j(r)}(z)=\tilde{h}_{j(p)j(r)}(z),$$

we have

$$\frac{\Psi_{j(r)}(t,z)}{\Psi_{j(p)}(t,z)}\tilde{h}_{j(q)j(r)}(z)+\frac{\Psi_{j(q)}(t,z)}{\Psi_{j(p)}(t,z)}\tilde{h}_{j(p)j(q)}(z)\frac{\Psi_{j(r)}(t,z)}{\Psi_{j(q)}(t,z)}\bar{h}_{j(q)j(r)}$$
$$=\frac{\Psi_{j(r)}(t,z)}{\Psi_{j(p)}(t,z)}\tilde{h}_{j(p)j(r)}(z).$$

Then the system

$$\begin{pmatrix} \dfrac{\Psi_{j(r)}(t,z)}{\Psi_{j(q)}(t,z)} & \dfrac{\Psi_{j(r)}(t,z)}{\Psi_{j(q)}(t,z)}\tilde{h}_{j(q)j(r)}(z) \\ 0 & \dfrac{\Psi_{j(r)}(t,z)}{\Psi_{j(q)}(t,z)}\bar{h}_{j(q)j(r)}(z) \end{pmatrix}$$

is also a system of transition functions of an indecomposable plane bundle over $U_j\times C$. Thus we obtain an extension of the bundle $G(U_j)$ over $U_j\times C$ which has the system of transition functions in the right hand side of (1),

$$0 \longrightarrow I(U_j) \longrightarrow E(U_j) \longrightarrow G(U_j) \longrightarrow 0,$$

for each j, where $I(U_j)$ is the line bundle with the system of transition functions $\dfrac{\Psi_{j(r)}(t,z)}{\Psi_{j(q)}(t,z)}$. In (1) the equality should be understood as an equivalence.

Now we have holomorphic maps

$$f_{jk}\colon U_j\cap U_k \longrightarrow \text{Isomorphism}\,(G(U_j)|U_j\cap U_k,\ G(U_k)|U_j\cap U_k).$$

For each $t\in U_j\cap U_k$ and C_q, we have the exact sequence of sheaves of germs of holomorphic sections over C_q,

$$0 \longrightarrow I_q \longrightarrow E_q \longrightarrow G_q \longrightarrow 0,$$

which admits a splitting $h_q\colon G_q \to E_q$. Denote by $f_{qr}\colon G_{qr}^{(j)} \to G_{qr}^{(k)}$ the mapping induced from f_{jk}, where $G_{qr}^{(j)}$, $G_{qr}^{(k)}$ are restrictions of $G(U_j)$, $G(U_k)$ on $C_q \cap C_r$ respectively. Then we have the following commutative diagram (1, [1]),

$$\begin{array}{ccccc} I_{qr}^{(j)} \oplus G_{qr}^{(j)} & \xrightarrow{u_q} & E_{qr}^{(j)} & \xleftarrow{u_r} & I_{qr}^{(j)} \oplus G_{qr}^{(j)} \\ \downarrow{\scriptstyle 1\oplus f_{qr}} & & \downarrow{\scriptstyle \hat{f}_{qr}} & & \downarrow{\scriptstyle 1\oplus f_{qr}} \\ I_{qr}^{(k)} \oplus G_{qr}^{(k)} & \xrightarrow{u_q} & E_{qr}^{(k)} & \xleftarrow{u_r} & I_{qr}^{(k)} \oplus G_{qr}^{(k)}, \end{array}$$

where $\hat{f}_{qr}(s' + h_{qr}(s'')) = s' + h_{qr}(f_{qr}(s''))$ for $s' \in I_{qr}^{(j)}$, $s'' \in G_{qr}^{(j)}$. Thus we obtain a holomorphic mapping

$$\hat{f}_{jk}\colon U_j \cap U_k \longrightarrow \text{Isomorphism}\,(E(U_j)|U_j \cap U_k,\ E(U_k)|U_j \cap U_k)$$

such that the next diagram is commutative,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(U_j \cap U_k) & \longrightarrow & E(U_j)\,|\,U_j \cap U_k & \longrightarrow & G(U_j)\,|\,U_j \cap U_k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{jk}|I(U_j\cap U_k)} & & \downarrow{\scriptstyle \hat{f}_{jk}} & & \downarrow{\scriptstyle f_{jk}} & & \\ 0 & \longrightarrow & I(U_j \cap U_k) & \longrightarrow & E(U_k)\,|\,U_j \cap U_k & \longrightarrow & G(U_k)\,|\,U_j \cap U_k & \longrightarrow & 0. \end{array}$$

Define a plane bundle E by $E(U_j)/(\hat{f}_{jk})$, then E is a plane bundle over S which is an extension of the line bundle $G \to S$. Hence we have

Proposition 1. *Let Δ be a non singular algebraic curve and C be an elliptic curve, and $G \to \Delta \times C$ be a locally Δ-trivial line bundle. Suppose that the first Chern class $C_1(G_0) = 1$, where $G_0 = G|\{t_o\} \times C$ for a point t_o of Δ, then we have a non trivial extension E of G by a line bundle F,*

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0.$$

Remark. Let G_0 be a line bundle with $C_1(G_0) = 1$ and F_0 be a line bundle over Δ. Denote by π_1, π_2 the projections, $\pi_1\colon \Delta \times C \to \Delta$, $\pi_2\colon \Delta \times C \to C$. Then the line bundle $\pi_1^* F_0 \otimes \pi_2^* G_0$ admits an extension which comes from the extension of G_0.

§2. Irreducibility of plane bundles

Let S be a basic member over a non singular algebraic curve of genus g, (8, [4]). The elliptic surface $\Phi\colon S \to \Delta$ admits a global section $\rho\colon \Delta \to S$. We call a plane bundle E over S to be *strongly reducible* if and only if the bundle E admits a line subbundle F such that $\rho^* F$ is the trivial line bundle over Δ. A plane bundle E is called *strongly irreducible* if E is not strongly reducible. We prove that

Proposition 2. *Let the genus g be greater than* 1. *Then there exists a strongly irreducible plane bundle over S.*

Proof. Let L be a line bundle over Δ with the first Chern class $C_1(L) = 1$. By the Riemann Roch theorem for line bundles, (Theorem 13, [3]),

$$\dim_C H^0(\Delta, O(L^{-1})) - \dim_C H^1(\Delta, O(L^{-1})) - C_1(L^{-1}) = 1 - g,$$

and since $C_1(L^{-1}) = -1$, then $\dim_C H^0(\Delta, O(L^{-1}) = 0$. Thus we have

$$\dim_C H^1(\Delta, O(L^{-1})) = g - 1 - C_1(L^{-1}) = g.$$

On the other hand, by Remark 10.1, [5], $\dim_C S_{2,L} = 3g - 3$, where $S_{2,L}$ is the set of equivalence classes of stable plane bundles over Δ with determinant bundle L. The cohomology group $H^1(\Delta, O(L^{-1}))$ is the set of equivalence classes of extensions of the line bundle L by the trivial line bundle I over Δ,

$$0 \longrightarrow I \longrightarrow E_0 \longrightarrow L \longrightarrow 0.$$

Suppose that every plane bundle over S is strongly reducible, then

$$\dim_C \{\rho^* E;\ E \text{ is a plane bundle and } \det \rho^* E = L\} \leqq \dim_C H^1(\Delta, O(L^{-1})).$$

Since { } of the left hand side in the above inequality includes as a subset $\rho^* \Phi^* S_{2,L} = S_{2,L}$, and $3g - 3 > g$, it is a contradiction.

REMARK 1. M. F. Atiyah has presented an example which is a reducible plane bundle over the product surface $P \times C$ of the projective plane P and an elliptic curve C. His example is

$$0 \longrightarrow [C] \longrightarrow E \longrightarrow [-C] \longrightarrow 0,$$

where $[C]$ is the line bundle given by a divisor $P \times C$ for a point p of P.

REMARK 2. If $g = 1$ and $C_1(L) = 1$, then $\dim_C S_{2,L} = 0$ and $\dim_C H^1(\Delta, O(L^{-1})) = 0$. So we can get no information by this method.

REMARK 3. In 4, [6], it has been proved that not every plane bundle on the ruled surface $P \times P$ is reducible.

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