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On Images of Topological Ordered Spaces under Some Quotient Mappings II

Takuo MIWA
Department of Mathematics, Shimane University, Matsue, Japan
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In section one, we deal with a problem which arises from the theorem in the previous paper [3]. In section two, some examples are established as supplement to [3].

§1. In this paper, we use the terms and notations in our previous paper [3]. In [3, Theorem (3)], we proved that if f is a continuous closed mapping of a normally ordered space $(X, \mathscr{U}, \rho)$ onto a topological ordered space $(Y, \mathscr{V}, \tau)$ where τ is the quotient order of ρ induced by f, then $(Y, \mathscr{V}, \tau)$ is also a normally ordered space. In connection with this theorem, it naturally arises a question whether "normally ordered" can be replaced by "T_4-ordered". First, we show that the answer to this question is negative in the following example.

EXAMPLE 1. Let X be a set $\{(a, x, y): a=0 \text{ or } 1, x, y \in [0, \infty)\}$. The topology $\mathscr{U}$ on X is the usual topology. Next, we define a partial order ρ in X as follows: $(a, x, y)\rho(b, u, v)$ if and only if $a=0$, $b=1$, $x=u\neq 0$, $y=1/x$; or $(a, x, y)=(b, u, v)$. Let A be a set $\{(1, q/p, p): p \text{ and } q \text{ are positive integers}\}$, then A is closed in X. Let Y be the quotient space X/A equipped with the quotient topology $\mathscr{V}$, and f the projection of X onto Y. If we introduce the order τ on Y by the quotient order of ρ induced by f, then f is a closed mapping and $(X, \mathscr{U}, \rho)$ is T_4-ordered. However, $(Y, \mathscr{V}, \tau)$ is not T_1-ordered. Because for $a^*=f(A)\in Y$, $b^*=\{(0, b, 1/b)\}\in Y$ where b is a positive irrational number, $a^* \| b^*$ holds and each increasing neighborhood of b^* necessarily contains a^*.

This example shows that the closed image of T_1-ordered (resp. T_4-ordered) space is not necessarily T_1-ordered (resp. T_4-ordered). So the questions naturally arise: under what conditions hold the generalizations of the well-known facts that the closed image of T_1 (resp. T_4) space is also T_1 (resp. T_4) space? For these questions, we obtain the next theorem.

We call $(X, \mathscr{U}, \rho)$ a *C-space* if, whenever a subset F of X is closed, $i_X(F)$ and $d_X(F)$ are closed (see [4]).

THEOREM. *Suppose* $(X, \mathscr{U}, \rho)$ *is a C-space and* f *a continuous mapping of* $(X, \mathscr{U}, \rho)$ *onto* $(Y, \mathscr{V}, \tau)$ *where* τ *is the quotient order of* ρ *induced by* f.

(1) *If f is a closed mapping and $(X, \mathscr{U}, \rho)$ is a T_1-ordered space, then $(Y, \mathscr{V}, \tau)$ is a T_1-ordered space.*

(2) *If f is a closed mapping and $(X, \mathscr{U}, \rho)$ is a T_4-ordered space, then $(Y, \mathscr{V}, \tau)$ is a T_4-ordered space.*

This theorem is immediately proved by the following two lemmas and [3, Theorem (3)].

LEMMA 1. *Under the assumptions of Theorem, if f is a closed mapping and $(X, \mathscr{U}, \rho)$ is a C-space, then $(Y, \mathscr{V}, \tau)$ is a C-space.*

PROOF. Let F be a closed set of Y. Then $i_Y(F)=f(i_X(f^{-1}(F)))$, $d_Y(F)=f(d_X(f^{-1}(F)))$. Therefore $i_Y(F)$ and $d_Y(F)$ are closed in Y. *Q.E.D.*

LEMMA 2. *If $(X, \mathscr{U})$ is a T_1 space and $(X, \mathscr{U}, \rho)$ is a C-space, then $(X, \mathscr{U}, \rho)$ is T_1-ordered.*

PROOF. Since $\{x\}$ is closed for each $x \in X$, $i_X(x)=[x, \rightarrow]$ and $d_X(x)=[\leftarrow, x]$ are closed in X. Therefore by [2, Theorem 1], $(X, \mathscr{U}, \rho)$ is T_1-ordered. *Q.E.D.*

§2. Suppose f is an open mapping of a T_2-ordered space $(X, \mathscr{U}, \rho)$ onto $(Y, \mathscr{V}, \tau)$ where $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ are T_2 spaces. As was shown in [1, Proposition 5], if f is isotonic and dually isotonic, then $(Y, \mathscr{V}, \tau)$ is a T_2-ordered space. In connection with this proposition, in [3, Example 1] it is shown the hypothesis that f is dually isotonic is essential. The next example shows that the assumption "f is isotonic" is essential.

EXAMPLE 2. Let X be the real numbers, the topology $\mathscr{U}$ on X the usual one and the order ρ in X the natural order. We define a partial order τ in X as follows: $x\tau y$ if and only if $x \leqq y$ for rational numbers x, y; or $x=y$ where $\leqq$ is the natural order of X. If f is the identity mapping of $(X, \mathscr{U}, \rho)$ onto $(X, \mathscr{U}, \tau)$, then all assumptions except that f is isotonic are satisfied. However $(X, \mathscr{U}, \tau)$ is not T_2-ordered.

In Theorem of the previous paper [3], the hypothesis that τ is the quotient order of ρ induced by f is essential. Indeed this hypothesis cannot be replaced with one that either f is dually isotonic or f is isotonic. In case f is dually isotonic, see Example 2. In the other case, see the next example.

EXAMPLE 3. Let X be the real numbers the topology on X the usual one, the order ρ the discrete order on X (i.e. $a\rho b$ if and only if $a=b$) and the order τ in X the same one with Example 2. If f is the identity mapping of $(X, \mathscr{U}, \rho)$ onto $(X, \mathscr{U}, \tau)$, then f is isotonic. However τ is not the quotient order of ρ induced by f. $(X, \mathscr{U}, \rho)$ is T_2-ordered, T_3-ordered and T_4-ordered, but $(X, \mathscr{U}, \tau)$ is neither T_2-ordered nor T_3-ordered nor normally ordered.

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