Mem. Fac. Lit. & Sci., Shimane Univ.,
Nat. Sci., **10**, pp. 7–17 Dec. 20, 1976

Free Products with Amalgamation of Bands*)

Teruo IMAOKA
Department of Mathematics, Shimane University, Matsue, Japan
(Received September 6, 1976)

In this paper, we shall study the question of which classes of bands have the strong or the special amalgamation properties. Let $\mathscr{A}$ be the variety of bands defined by an identity $P=Q$. Then $\mathscr{A}$ has the strong amalgamation property if and only if $P=Q$ is a permutation identity or heterotypical identity. Moreover, we shall show that the varieties of [left, right] regular bands and left [right] quasinormal bands have the special amalgamation property.

§1. Introduction

A class of algebras $\mathscr{A}$ is said to have the *strong amalgamation property* if for any family of algebras $\{A_i: i\in I\}$ from $\mathscr{A}$, each having an algebra $U\in\mathscr{A}$ as a subalgebra, there exist an algebra B in $\mathscr{A}$ and monomorphisms ϕ_i: $A_i\to B$, $i\in I$, such that

(i) $\phi_i|U=\phi_j|U$ for all $i, j\in I$,
(ii) $A_i\phi_i\cap A_j\phi_j=U\phi_i$ for all $i, j\in I$ with $i\neq j$,

where $\phi_i|U$ denotes the restriction of ϕ_i to U. Omitting the condition (ii) gives us the definition of the *weak amalgamation property*. Adding the condition that $A_i=A_j$ for all $i, j\in I$, to the hypothesis of the definition of the strong amalgamation property gives us the definition of the *special amalgamation property*.

Note that in a class of algebras closed under isomorphisms, the weak and the special amalgamation properties together imply the strong amalgamation property. It is well-known (see [4]) that in a class of algebras closed under isomorphisms and the formation of the union of any ascending chain of algebras, each amalgamation property follows from the case in which $|I|=2$. Hence we shall consider in this paper only the case $|I|=2$.

The classes of algebras for which the strong or the weak amalgamation properties is known to hold are "groups, groups with a given operator domain, commutative groups, fields, differential fields of characteristic 0, partially ordered sets, lattices, Boolean algebras, locally finite-dimensional cylindric algebras of a given infinite dimension [7], pseudocomplemented distributive lattices $\mathscr{B}_n$, $n\leqq 2$ or $n=\omega$[3], semilattices, inverse semigroups [4], commutative inverse semigroups [5]". However, it is well-known (see [8], [1, Section 9.4]) that the class of semigroups does not have

*) This research was carried out while the author held a Monash Graduate Scholarship.

even the weak amalgamation property.

In section 2, we shall add the following classes of bands to the list above: "[left, right] normal bands, rectangular bands, left [right] zero semigroups, one element semigroups". Moreover, the variety of bands defined by an identity $P=Q$ has the strong amalgamation property if and only if $P=Q$ is a permutation identity or a hetrotypical identity.

In section 3, we shall consider the congruence extension property of [left, right] normal bands and give the another proof of the class of [left, right] normal bands to have the weak amalgamation property.

In section 4, we shall show that the classes of [left, right] regular bands and left [right] quasinormal bands do not have the weak amalgamation property, due to T. E. Hall, but they have the special amalgamation property.

The notations and terminologies are those of [1] and [12], unless otherwise stated.

§2. Strong amalgamation property

We shall first show that the class of left normal bands has the strong amalgamation property. Let L_1 and L_2 be left normal bands with a common subband U. Let the structure decompositions of L_1, L_2 and U be $L_1 \sim \Sigma\{L_1^\alpha : \alpha \in \Gamma_1\}$, $L_2 \sim \Sigma\{L_2^\alpha : \alpha \in \Gamma_2\}$ and $U \sim \Sigma\{U_\alpha : \alpha \in \Delta\}$, respectively. We can assume without loss of generality that $L_1 \cap L_2 = U$, $\Gamma_1 \cap \Gamma_2 = \Delta$ and $L_1^\alpha \cap L_2^\alpha = U_\alpha$ for all $\alpha \in \Delta$. Let $L = L_1 \cup L_2$ and $\Gamma = (\Gamma_1^{(1)} \times \Gamma_2^{(1)}) \backslash \{(1, 1)\}$. It follows from [6, Corollary 2.2] that

$$E = \{(a, \alpha, \beta) \in L \times \Gamma : a \in L_1^\alpha \cup L_2^\beta,\ \alpha \in \Gamma_1^{(1)},\ \beta \in \Gamma_2^{(1)}\}$$

is the free product of L_1 and L_2 in the variety of left normal bands, if its product is defined by

$$(a, \alpha, \beta)(b, \gamma, \delta) = \begin{cases} (a \cdot e(\alpha\gamma), \alpha\gamma, \beta\delta) & \text{if } a \in L_1^\alpha, \\ (a \cdot e(\beta\delta), \alpha\gamma, \beta\delta) & \text{if } a \in L_2^\beta, \end{cases}$$

where $e(\alpha)$ denotes an element of L_i^α, $i=1, 2$. Hereafter, $e(1)$ means 1.

We define a relation θ on E as follows:

(2.1) For elements (a, α, β), (b, γ, δ) of E, define $(a, \alpha, \beta)\theta_0(b, \gamma, \delta)$ to mean that there exist $\sigma \in \Delta$ and $u \in U_\sigma$ such that

$$(a, \alpha, \beta) = (c_1, \xi_1, \eta_1)(u, \sigma, 1)(c_2, \xi_2, \eta_2),$$

$$(b, \gamma, \delta) = (c_1, \xi_1, \eta_1)(u, 1, \sigma)(c_2, \xi_2, \eta_2),$$

for some (c_1, ξ_1, η_1), $(c_2, \xi_2, \eta_2) \in E^1$. Let $\theta_1 = \theta_0 \cup \theta_0^{-1} \cup \iota$ and let $\theta = \theta_1^t$.

Then of course θ is the congruence on E generated by $\{((u, \sigma, 1), (u, 1, \sigma)): u \in U_\sigma, \sigma \in \Delta\}$. Let $(a, \alpha, \beta)\theta$ denote the θ-class containing (a, α, β). Since any homomorphic image of a left normal band is also a left normal band, E/θ is a left normal band. In order to show that E/θ is the free product of L_1 and L_2 amalgamating U in the variety of left normal bands, we need the following lemma.

LEMMA 2.1. *If* $(a, \alpha, 1)\theta(b, \beta, \gamma)$, *then there exist* $\sigma \in \Delta^1$ *and* $u \in U_\sigma$ *such that*

$$u \cdot e(\gamma) \in U \quad \text{and} \quad a = b(u \cdot e(\gamma)) \quad (\text{in } L_1) \qquad \text{if} \quad b \in L_1^\beta,$$

$$bu \in U \quad \text{and} \quad a = (bu) \cdot e(\beta) \quad (\text{in } L_1) \qquad \text{if} \quad b \in L_2^\gamma,$$

where $U_1 = \{1\}$, $e(1) = 1$.

PROOF. Let $(a, \alpha, 1)\theta(b, \beta, \gamma)$. By the definition (2.1), there exist $(x_1, \delta_1, \varepsilon_1)$, $(x_2, \delta_2, \varepsilon_2), \dots, (x_n, \delta_n, \varepsilon_n)$ in E such that $(a, \alpha, 1) = (x_1, \delta_1, \varepsilon_1)$, $(b, \beta, \gamma) = (x_n, \delta_n, \varepsilon_n)$ and $(x_i, \delta_i, \varepsilon_i)\theta_1(x_{i+1}, \delta_{i+1}, \varepsilon_{i+1})$ for $i = 1, 2, \dots, n-1$.

We use induction on n. For $n = 1$, it is obvious that the statement is true. So we assume the statement is true for $n-1$. First we consider when $(x_{n-1}, \delta_{n-1}, \varepsilon_{n-1})\theta_0(x_n, \delta_n, \varepsilon_n)$; then there exist $\sigma \in \Delta$ and $u \in U_\sigma$ such that

$$(x_{n-1}, \delta_{n-1}, \varepsilon_{n-1}) = (c_1, \xi_1, \eta_1)(u, \sigma, 1)(c_2, \xi_2, \eta_2),$$

$$(x_n, \delta_n, \varepsilon_n) = (c_1, \xi_1, \eta_1)(u, 1, \sigma)(c_2, \xi_2, \eta_2),$$

for some (c_1, ξ_1, η_1), (c_2, ξ_2, η_2) in E^1.

Case I, $x_{n-1}, x_n \in L_1$. Then $c_1 \in L_1$. By the induction hypothesis, there exist $\tau \in \Delta^1$ and $v \in U_\tau$ such that

$$(2.2) \qquad v \cdot e(\varepsilon_{n-1}) \in U \quad \text{and} \quad a = x_{n-1}(v \cdot e(\varepsilon_{n-1})) \quad (\text{in } L_1).$$

Then we have

$$v \cdot e(\varepsilon_n) = v \cdot e(\eta_1 \sigma \eta_2) = v \cdot e(\eta_1 \eta_2) \cdot u = v \cdot e(\varepsilon_{n-1}) \cdot u \in U,$$

and

$$x_n(v \cdot e(\varepsilon_n)) = x_n(v \cdot e(\varepsilon_{n-1}) \cdot u)$$

$$= c_1 \cdot e(\xi_1 \xi_2) \cdot u \cdot (v \cdot e(\varepsilon_{n-1}))$$

(since u, $v \cdot e(\varepsilon_{n-1}) \in U^1$ and L_1 is left normal)

$$= x_{n-1}(v \cdot e(\varepsilon_{n-1}))$$

$$= a.$$

Case II, $x_{n-1} \in L_1$ and $x_n \in L_2$. Then $(c_1, \xi_1, \eta_1) = 1$, and there exist $\tau \in \Delta^1$

and $v \in U_\tau$ such that the condition (2.2) is satisfied. Then we have

$$x_n v = u \cdot e(\sigma\eta_2) \cdot v = uv \cdot e(\eta_2) = uv \cdot e(\varepsilon_{n-1}) \in U,$$

and

$$\begin{aligned}(x_n v) \cdot e(\delta_n) &= (uv \cdot e(\varepsilon_{n-1})) \cdot e(\xi_2) \\ &= u \cdot e(\xi_2) \cdot (v \cdot e(\varepsilon_{n-1})) \\ &= x_{n-1}(v \cdot e(\varepsilon_{n-1})) \\ &= a.\end{aligned}$$

Case III, $x_{n-1} \in L_2$. Then $c_1 \in L_2$, and hence $x_n \in L_2$. It follows from the induction hypothesis that there exist $\tau \in \Delta^1$ and $v \in U_\tau$ such that

$$x_{n-1} v \in U \quad \text{and} \quad a = (x_{n-1} v) \cdot e(\delta_{n-1}) \quad (\text{in } L_1).$$

Then we have

$$x_n v = c_1 \cdot e(\eta_1 \sigma \eta_2) \cdot v = c_1 \cdot e(\eta_1 \eta_2) \cdot uv = (x_{n-1} v) u \in U,$$

and

$$\begin{aligned}(x_n v) \cdot e(\delta_n) &= (x_{n-1} vu) \cdot e(\xi_1 \xi_2) \\ &= (x_{n-1} v) u \cdot e(\xi_1 \xi_2) \\ &= (x_{n-1} v) \cdot e(\delta_{n-1}) \\ &= a.\end{aligned}$$

Similarly $(x_n, \delta_n, \varepsilon_n)$ satisfies the condition of the lemma when $(x_{n-1}, \delta_{n-1}, \varepsilon_{n-1})\theta_0^{-1} \cup \iota(x_n, \delta_n, \varepsilon_n)$. Hence we have the lemma.

THEOREM 2.2. *We use the notations defined above. Then E/θ is the free product of L_1 and L_2 amalgamating U, in the variety of left normal bands. Moreover, the structure semilattice of E/θ is isomorphic to the free product of Γ_1 and Γ_2 amalgamating Δ, in the variety of semilattices.*

PROOF. Let ϕ_i: $L_i \rightarrow E/\theta$, $i = 1, 2$, be mappings defined by

$$x\phi_1 = (x, \alpha, 1)\theta \qquad \text{if} \quad x \in L_1^\alpha,$$

$$y\phi_2 = (y, 1, \beta)\theta \qquad \text{if} \quad y \in L_2^\beta.$$

It is clear that ϕ_1 and ϕ_2 are homomorphisms. Let x and y be elements of L_1^α and L_1^β, respectively, such that $x\phi_1 = y\phi_1$. Then $(x, \alpha, 1)\theta(y, \beta, 1)$. By the lemma above, there exist $\sigma, \tau \in \Delta^1$, $u \in U_\sigma$ and $v \in U_\tau$ such that $x = yu$ and $y = xv$. Therefore $\alpha = \beta$ and $x = xy = (yu)y = y$. Hence ϕ_1 is a monomorphism. Similarly ϕ_2 is a monomorphism. By the definition (2.1), it is clear that $\phi_1 | U = \phi_2 | U$.

Next, we shall show that $L_1\phi_1 \cap L_2\phi_2 = U\phi_1$. Let x and y be elements of L_1^α and L_2^β, respectively, such that $x\phi_1 = y\phi_2$. Then $(x, \alpha, 1)\theta(y, 1, \beta)$. By the lemma above and its dual, there exist $\sigma, \tau \in \Delta^1$, $u \in U_\sigma$ and $v \in U_\tau$ such that

$$yu \in U \quad \text{and} \quad (yu)\cdot 1 = x \quad (\text{in } L_1),$$

$$xv \in U \quad \text{and} \quad (xv)\cdot 1 = y \quad (\text{in } L_2).$$

Then $x, y \in U$ and $x = y$. Thus we have $L_1\phi_1 \cap L_2\phi_2 \subseteq U\phi_1$. It is obvious $L_1\phi_1 \cap L_2\phi_2 \supseteq U\phi_1$. Hence $L_1\phi_1 \cap L_2\phi_2 = U\phi_1$.

It is clear that $\langle L_1\phi_1 \cup L_2\phi_2 \rangle = E/\theta$ and that E/θ together with ϕ_1 and ϕ_2 is the colimit of L_1 and L_2 amalgamating U. Hence E/θ is the free product of L_1 and L_2 amalgamating U, in the variety of left normal bands.

The later part of the theorem is clear.

COROLLARY 2.3. *The variety of [left, right] normal bands has the strong amalgamation property.*

PROOF. Let S_1 and S_2 be normal bands with a common subband U. It follows from [14, Theorem 4] that $S_1 = L_1 \bowtie R_1(\Gamma_1)$, $S_2 = L_2 \bowtie R_2(\Gamma_2)$ and $U = U_1 \bowtie U_2(\Delta)$, where L_1, L_2 and U_1 are left normal bands, R_1, R_2 and U_2 are right normal bands and Γ_1, Γ_2 and Δ are semilattices. We can assume without loss of generality that $L_1 \cap L_2 = U_1$, $R_1 \cap R_2 = U_2$ and $\Gamma_1 \cap \Gamma_2 = \Delta$. By Theorem 2.2 and its dual, there exist the free product T_1, say, of L_1 and L_2 amalgamating U_1 in the variety of left normal bands, and the free product T_2, say, of R_1 and R_2 amalgamating U_2 in the variety of right normal bands. Since T_1 and T_2 have the same structure semilattice Ω, say, which is the free product of Γ_1 and Γ_2 amalgamating Δ in the variety of semilattices, let $T_1 \sim \Sigma\{T_1^\alpha : \alpha \in \Omega\}$ and $T_2 \sim \Sigma\{T_2^\alpha : \alpha \in \Omega\}$ be the structure decompositions of T_1 and T_2, respectively. Then it is clear that the spined product $T_1 \bowtie T_2(\Omega)$ is the free product of S_1 and S_2 amalgamating U, in the variety of normal bands. Hence the variety of normal bands has the strong amalgamation property.

COROLLARY 2.4. *The class of $M[L.N, R.N]$-inversive semigroups has the strong amalgamation property.*

PROOF. It follows from [13, Corollary 2 and 3] that an $M[L.N, R.N]$-inversive semigroup is isomorphic to the spined product of a commutative inverse semigroup and a [left, right] normal band. By a similar argument to the proof of Corollary 2.3 we have that the class of $M[L.N, R.N]$-inversive semigroups has the strong amalgamation property.

The following example, due to T. E. Hall, shows that the variety of left [right] regular bands does not have even the weak amalgamation property.

EXAMPLE. Let $S=\{e, f, g, h\}$, $T=\{f, g, h, x, y\}$ and $U=\{e, f, g\}$ be a left regular band, a left normal band and a left zero semigroup, respectively, whose multiplications are defined as follows:

	e	f	g	h
e	e	g	g	h
f	f	f	f	f
g	g	g	g	g
h	h	h	h	h

	f	g	h	x	y
f	f	f	f	x	x
g	g	g	g	y	y
h	h	h	h	x	x
x	x	x	x	x	x
y	y	y	y	y	y

Suppose that there exists a semigroup W such that $S\cup T$ can be embedded in W. Then, since W is associative,

$$ex=e(fx)=(ef)x=gx=y,$$

$$ex=e(hx)=(eh)x=hx=x.$$

Thus the elements x and y must coincide in W, a contradiction.

THEOREM 2.5. *Let $\mathscr{A}$ be the variety of bands defined by an identity $P=Q$. Then $\mathscr{A}$ has the strong amalgamation property if and only if $P=Q$ is a permutation identity or a heterotypical identity.*

PROOF. Suppose that $P=Q$ is neither a permutation identity nor a heterotypical identity. By [2], $\mathscr{A}$ contains the set of left regular bands or the set of right regular bands. Then the example above implies that $\mathscr{A}$ does not have the strong amalgamation property.

If $P=Q$ is a permutation identity, it follows from [14, Theorem 10], [5, Corollary 1] and Corollary 2.3 that $\mathscr{A}$ has the strong amalgamation property. So let $P=Q$ be a heterotypical identity. By [11, Theorem 2], $\mathscr{A}$ is one of the varieties of rectangular bands, left zero semigroups, right zero semigroups and one element semigroups. Then it is clear $\mathscr{A}$ has the strong amalgamation property.

§3. Congruence extension property

THEOREM 3.1. *Let U be any subband of a normal band E and let θ be any con-*

gruence on U. Then θ extends to a congruence on E in the following sense: let Θ be the congruence on E generated by θ, then $\Theta \cap (U \times U) = \theta$.

PROOF. Put

$$\theta_1 = \{(acb, adb) \in E \times E : (c, d) \in \theta,\ a,\ b \in E^1\} \cup \iota_E.$$

Then $\Theta = \theta_1^t = \bigcup_{n=1}^{\infty} \theta_1^n$. We make the convention that $\theta_1^0 = \iota_E$. Then $\Theta = \bigcup_{n=0}^{\infty} \theta_1^n$. Since $\theta \subseteq \Theta \cap (U \times U)$, it remains to show the following: for $n = 0, 1, 2, \ldots$, and for $a, b \in U$, if $(a, b) \in \theta_1^n$ then $(a, b) \in \theta$.

We use induction on n. For $n = 0$, the statement is trivial. So take any integer $n > 0$ and assume the statement for $n-1$ is true. Let $(a, b) \in \theta_1^n \cap (U \times U)$. Then there exist $x, y \in E$ such that

$$a\,\theta_1\,x\,\theta_1^{n-1}\,b, \qquad a\,\theta_1^{n-1}\,y\,\theta_1\,b.$$

Since θ_1 and hence θ_1^{n-1} are compatible, we can obtain

(3.1) $$a\,\theta_1\,(ax)\,\theta_1^{n-1}\,(ab),$$

(3.2) $$b\,\theta_1\,(yb)\,\theta_1^{n-1}\,(ab).$$

Now we prove the following lemma, before continuing the proof of Theorem 3.1.

LEMMA 3.2. *For any $c \in U$ and $d \in E$, if $c\,\theta_1\,(cd)$ then $cd \in U$ and $c\,\theta\,(cd)$.*

PROOF. Since it is obvious for $c = cd$, we assume $c \neq cd$. Then there exist $(x, y) \in \theta$ and $u, v \in E^1$ such that $c = uxv$, $cd = uyv$ and $(u, v) \neq (1, 1)$.

Case I, $u \neq 1$ and $v \neq 1$:

$$cd = cccd = cuxvuyv = cyuxv = cyc \in U,$$

$$cd = (cyc)\,\theta\,(cxc) = c.$$

Case II, $u \neq 1$ and $v = 1$:

$$cd = ccd = ux\,uy = uxy = cy \in U,$$

$$cd = (cy)\,\theta\,(cx) = c.$$

Case III, $U = 1$ and $v \neq 1$:

$$cd = cccd = cxv\,yv = cyxv = cyc \in U,$$

$$cd = (cyc)\,\theta\,(cxc) = c.$$

This gives us the lemma.

Putting $c=a$, $d=x$ in (3.1), we have

$$a\,\theta(ax)\,\theta_1^{n-1}(ab).$$

Since $ax,\ ab\in U$, it follows from the induction hypothesis that

$$a\,\theta(ax)\,\theta(ab).$$

Similarly it follows from (3.2) and the dual of Lemma 3.2 that

$$b\,\theta(yb)\,\theta(ab).$$

Then $(a, b)\in\theta$, giving the theorem.

COROLLARY 3.3. *Let U be any subband of a left [right] normal band, and let θ be any congruence on U. Then θ extends to a congruence on E in the sense of Theorem* 3.1.

From [3, Theorem 4], [6, Corollary 2.3], Theorem 3.1 and Corollary 3.3, we have the following theorem.

THEOREM 3.4. *The variety of [left, right] normal bands has the weak amalgamation property.*

§4. Special amalgamation property

We have seen that the variety of [left, right] regular bands does not have the weak amalgamation property. In this section, we shall show, however, that it has the special amalgamation property. Let $L\sim\Sigma\{L_\alpha\colon \alpha\in\Gamma\}$ be a left regular band and $U\sim\Sigma\{U_\alpha\colon \alpha\in\Delta\}$ a subband. We can assume without loss of generality that $L\supseteq U$, $\Gamma\supseteq\Delta$ and $L_\alpha\supseteq U_\alpha$ for all $\alpha\in\Delta$. Let L_1 and L_2 be left regular bands which are isomorphic to L such that $L_1\cap L_2=\Box$, and let $\nu_i\colon L\to L_i$, $i=1, 2$, be isomorphisms. Let $U_i=U\nu_i$, $L_i^\alpha=L_\alpha\nu_i$ and $U_i^\beta=U_\beta\nu_i$ for all $\alpha\in\Gamma$, $\beta\in\Delta$ and $i=1, 2$.

Let S be the free product of L_1 and L_2 in the variety of left regular bands. Hereafter, let a_i mean "a_i is an element of L_i", where $i=1$ or 2. Define a relation θ on S as follows:

$a_{i_1}a_{i_2}\cdots a_{i_r}\,\theta_0\,b_{j_1}b_{j_2}\cdots b_{j_s}$ if and only if there exist u in U and $c_{k_1}c_{k_2}\cdots c_{k_p}$, $d_{m_1}d_{m_2}\cdots d_{m_q}$ in S^1 such that

$$a_{i_1}a_{i_2}\cdots a_{i_r}=c_{k_1}c_{k_2}\cdots c_{k_p}(u\nu_1)d_{m_1}d_{m_2}\cdots d_{m_q},$$

$$b_{j_1}b_{j_2}\cdots b_{j_s}=c_{k_1}c_{k_2}\cdots c_{k_p}(u\nu_2)d_{m_1}d_{m_2}\cdots d_{m_q}.$$

Let $\theta_1=\theta_0\cup\theta_0^{-1}\cup\iota$, and let $\theta=\theta_1^t$.

It follows from [1, Section 1.5] that θ is the congruence on S. Then S/θ is a left regular band.

DEFINITION 4.1. Let a be an element of L. A sequence $(x_{i_1}, x_{i_2},\dots, x_{i_n})$ of elements of $L_1 \cup L_2$ is said to *have property* $P_i(a)$, $i=1, 2$, if there exist $u_1, u_2,\dots, u_n, v_1, v_2,\dots, v_n$ in U^1 such that

(i) $u_1=1$,

(ii) $u_j(x_{i_j}v_{\bar{i}_j}^{-1})v_j \in U$ if $i_j \neq i$,

(iii) $a=\prod_{j=1}^{n} u_j(x_{i_j}v_{\bar{i}_j}^{-1})v_j$.

LEMMA 4.2. (i) *If* $x_{i_1}x_{i_2}\cdots x_{i_r}=y_{j_1}y_{j_2}\cdots y_{j_s}$ *(in S) and* $(x_{i_1}, x_{i_2},\dots, x_{i_r})$ *has property* $P_i(a)$ *for some a in L, then* $(y_{j_1}, y_{j_2},\dots, y_{j_s})$ *also has property* $P_i(a)$.

(ii) *If* $x_{i_1}x_{i_2}\cdots x_{i_r}\,\theta_1\, y_{j_1}y_{j_2}\cdots y_{j_s}$ *(in S) and* $(x_{i_1}, x_{i_2},\dots, x_{i_r})$ *has property* $P_i(a)$, *then* $(y_{j_1}, y_{j_2},\dots, y_{j_s})$ *also has property* $P_i(a)$.

PROOF. In order to show (i), it is sufficient to prove that if $(x_{i_1}, x_{i_2},\dots, x_{i_r}, y_{j_1}, y_{j_2},\dots, y_{j_s}, x_{i_1}, x_{i_2},\dots, x_{i_r})$ has property $P_i(a)$ then $(x_{i_1}, x_{i_2},\dots, x_{i_r}, y_{j_1}, y_{j_2},\dots, y_{j_s})$ also has property $P_i(a)$. Let $(x_{i_1}, x_{i_2},\dots, x_{i_r}, y_{j_1}, y_{j_2},\dots, y_{j_s}, x_{i_1}, x_{i_2},\dots, x_{i_r})$ have property $P_i(a)$. Then there exist $u_1, u_2,\dots, u_{2r+s}, v_1, v_2,\dots, v_{2r+s}$ in U^1 such that $u_1=1$ and

(i) $u_k(x_{i_k}v_{\bar{i}_k}^{-1})v_k,\ u_{r+s+k}(x_{i_k}v_{\bar{i}_k}^{-1})v_{r+s+k} \in U$ if $i_k \neq i$,

(ii) $u_k(y_{j_k}v_{\bar{j}_k}^{-1})v_k \in U$ if $j_k \neq i$,

(iii) $a=\left(\prod_{k=1}^{r} u_k(x_{i_k}v_{\bar{i}_k}^{-1})v_k\right)\left(\prod_{k=r+1}^{r+s} u_k(y_{j_k}v_{\bar{j}_k}^{-1})v_k\right)\left(\prod_{k=r+s+1}^{2r+s} u_k(x_{i_k}v_{\bar{i}_k}^{-1})v_k\right)$.

Let

$$w_k=\begin{cases} u_{r+s+k}v_{r+s+k} & \text{if } i_k=i, \\ u_{r+s+k}(x_{i_k}v_{\bar{i}_k}^{-1})v_{r+s+k} & \text{if } i_k\neq i, \end{cases}$$

and let $v'_{r+s}=v_{r+s}(\prod_{k=1}^{r} w_k)$. Then it is clear that $v'_{r+s} \in U$. Since L is a left regular band, $(x_{i_1}, x_{i_2},\dots, x_{i_r}, y_{j_1}, y_{j_2},\dots, y_{j_s})$ has property $P_i(a)$ with $u_1, u_2,\dots, u_{r+s}, v_1, v_2,\dots, v_{r+s-1}, v'_{r+s}$.

By using (i) and the definition of θ_1, we can easily prove (ii).

The following corollary follows immediately from the lemma above.

COROLLARY 4.3. *Let a be an element of* L_i, $i=1, 2$. *If* $a\,\theta\, x_{j_1}x_{j_2}\cdots x_{j_r}$ *(in S), then* $(x_{j_1}, x_{j_2},\dots, x_{j_r})$ *has property* $P_i(av_{\bar{i}}^{-1})$.

THEOREM 4.4. *We use the notations defined above. Then* S/θ *is the free product of* L_1 *and* L_2 *amalgamating U in the variety of left regular bands. Thus the variety of left regular bands has the special amalgamation property. Moreover, the structure semilattice of* S/θ *is isomorphic to the free product* $\Gamma *_{\Delta} \Gamma$, *say, of* Γ *amalgamating* Δ *in the variety of semilattices.*

PROOF. Let ϕ_i: $L_i \to S/\theta$, $i=1, 2$, be mappings defined by

$$a\phi_i = a\theta \qquad \text{for all} \quad a \in L_i.$$

It is obvious that each ϕ_i, $i=1, 2$, is a homomorphism. Let a and b be any elements of L_i satisfying $a\phi_i = b\phi_i$. Then $a\theta = b\theta$. By the corollary above, there exist u and v in U^1 such that

$$a(uv_i) = b \quad \text{and} \quad b(vv_i) = a.$$

Since L_i is left regular,

$$a = b(vv_i) = bb(vv_i) = ba = bab$$
$$= a(uv_i)ab = a(uv_i)b = b^2 = b,$$

and hence ϕ_i is a monomorphism.

By the definition of θ, it is obvious that $v_1\phi_1|U = v_2\phi_2|U$. Let a and b be elements of L_1 and L_2, respectively, such that $a\phi_1 = b\phi_2$. Then we have $a\theta = b\theta$. By the corollary above, there exist u and v in U^1 such that $(av_1^{-1})u \in U$, $(bv_2^{-1})v \in U$, $bv_2^{-1} = (av_1^{-1})u$ and $av_1^{-1} = (bv)v_2^{-1}$. Then we have $av_1^{-1} = bv_2^{-1} \in U$, and hence $L_1\phi_1 \cap L_2\phi_2 \subseteq Uv_1\phi_1$. It is obvious $L_1\phi_1 \cap L_2\phi_2 \supseteq Uv_1\phi_1$. Therefore $L_1\phi_1 \cap L_2\phi_2 = Uv_1\phi_1$.

By the definition of θ, it is clear that S/θ is the free product of L_1 and L_2 amalgamating U in the variety of left regular bands, and that the structure semilattice of S/θ is isomorphic to $\Gamma *_\Delta \Gamma$.

It follows from [10, Theorem 1] and [9, Theorem 2] that a regular band is the spined product of a left regular band and a right regular band, and that a left [right] quasinormal band is the spined product of a left [right] regular band and a right [left] normal band. By a similar argument to the proof of Corollary 2.3, we have the following corollary.

COROLLARY 4.5. *The varieties of [right] regular bands and left [right] quasinormal bands have the special amalgamation property.*

Acknowledgements

I wish to express my thanks to Dr. T. E. Hall, my supervisor, for suggesting many improvements.

References

[1] A. H. CLIFFORD and G. B. PRESTON, The algebraic theory of semigroups, Math. Surveys, No. 7, Amer. Math. Soc., Providence, R. I., Vol. I (1961), Vol. II (1967).

[2] C. F. FENNEMORE, All varieties of bands, Doctoral dissertation, Pennsylvania State University, 1969.

[3] G. GRATZER and H. LAKSER, The structure of pseudocomplemented distributive lattices II: congruence extension and amalgamation, Trans. Amer. Math. Soc. **156** (1971), 343–357.

[4] T. E. HALL, Free products with amalgamation of inverse semigroups, J. Algebra **34** (1975), 375–385.

[5] T. IMAOKA, Free products with amalgamation of commutative inverse semigroups, J. Austral. Math. Soc. (to appear).

[6] T. IMAOKA, Free products of orthodox semigroups, Mem. Fac. Lit. & Sci., Shimane Univ., Nat. Sci. (to appear).

[7] Bjarni JONSSON, Extensions of relational structures, Theory of models, Proceedings of 1963 international symposium at Berkely, edited by J. W. Addision, L. Henkin and A. Tarski, North Holland, Amsterdam, 1965, 146–157.

[8] N. KIMURA, On semigroups, Doctoral dissertation, The Tulane University of Louisiana, 1957.

[9] N. KIMURA, Note on idempotent semigroups III, Proc. Japan Acad. **34** (1958), 113–114.

[10] N. KIMURA, The structure of idempotent semigroups (I), Pacific J. Math. **8** (1958), 257–275.

[11] M. PETRICH, A construction and classification of bands, Math. Nach. 4 (1971), 263–274.

[12] M. YAMADA, The structure of separatice bands, Doctoral dissertation, University of Utah, 1962.

[13] M. YAMADA, Regular semigroups whose idempotents satisfy permutation identities, Pacific J. Math. **21** (1967), 371–392.

[14] M. YAMADA and N. KIMURA, Note on idempotent semigroups II, Proc. Japan Acad. **34** (1958), 110–112.