Mem. Fac. Lit. & Sci., Shimane Univ.,
Nat. Sci., 10, pp. 1–6 Dec. 20, 1976

Free Products of Orthodox Semigroups*)

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(Received September 6, 1976)

As an analogous construction to inverse semigroups (see [4]), we shall give the free products in the varieties of orthodox semigroups and of bands defined by a given identity. However, if the variety is the set of [left, right] normal bands, we can describe it in a more useful form. We shall discuss it in the later half of this paper. The notations and terminologies are those of [1] and [6], unless otherwise stated.

§1. Orthodox semigroups

At first, we shall study the free product in the class of orthodox semigroups. Let S_i, $i \in I$, be a family of orthodox semigroups, and let S be the free product $\Pi * S_i$ of the S_i in the class of semigroups. For $x = x_{i_1}x_{i_2}\cdots x_{i_k}$, $x_{i_j} \in S_{i_j}$, in a reduced form in S, set

$$W(x) = \{x'_{i_k}\cdots x'_{i_2}x'_{i_1} : x'_{i_j} \in V(x_{i_j})\},$$

where $V(x_i)(=V_{S_i}(x_i))$ denotes the set of inverses of x_i in S_i. If $x \in S_i$, it is obvious $W(x) = V_{S_i}(x)$.

THEOREM 1.1. *Let $\sim$ denote the congruence on S generated by all pairs $(x, xx'x)$ and $(x_1x'_1x_2x'_2\cdots x_rx'_r,\ x_1x'_1x_2x'_2\cdots x_rx'_rx_1x'_1x_2x'_2\cdots x_rx'_r)$ for $x, x_1, x_2, \ldots, x_r$ in S and for $x' \in W(x)$ and $x'_j \in W(x_j)$, $j = 1, 2, \ldots, r$. Then $S/\sim$ is the free product of the S_i in the class of orthodox semigroups.*

PROOF. Let $x = x_{i_1}x_{i_2}\cdots x_{i_k}$ be any element of S. For any element $x' = x'_{i_k}\cdots x'_{i_2}x'_{i_1}$ in $W(x)$, each x_{i_j} is an inverse of x'_{i_j} in S_{i_j}, $j = 1, 2, \ldots, k$. Then $(x', x'xx') \in \sim$, that is, $x'\sim$ is an inverse of $x\sim$ in $S/\sim$. Hence $S/\sim$ is a regular semigroup.

Let e be an element of S and assume that $e\sim$ is an idempotent in $S/\sim$. For any $e' \in W(e)$,

$$\begin{aligned}(e'\sim)^2 &= (e'ee'\sim)(e'ee'\sim)\\ &= (e'\sim)(e\sim)(e'^2\sim)(e\sim)(e'\sim)\\ &= (e'\sim)(e^2\sim)(e'^2\sim)(e^2\sim)(e'\sim)\end{aligned}$$

*) This research was carried out while the author held a Monash Graduate Scholarship.

$$=(e'\sim)(e^2\sim)(e'\sim),$$

since $e'^2\in W(e^2)$,

$$=(e'\sim)(e\sim)(e'\sim)$$
$$=e'\sim.$$

Then $e'\sim$ is an idempotent in $S/\sim$.

We now show that $S/\sim$ is orthodox. For e, f in S, let us assume that $e\sim$ and $f\sim$ are idempotents in $S/\sim$. Then,

$$\begin{aligned} ef\sim &=(e\sim)(f\sim)\\ &=(e\sim)(e'\sim)(e\sim)(f\sim)(f'\sim)(f\sim)\\ &=(e\sim)(e'\sim)^2(e\sim)(f\sim)(f'\sim)^2(f\sim),\end{aligned}$$

since $e'\sim$ and $f'\sim$ are idempotents in $S/\sim$,

$$\begin{aligned} &=((ee')(e'e)(ff')(f'f))\sim\\ &=(ee'e'eff'f'fee'e'eff'f'f)\sim\\ &=(ee'e'eff'f'f\sim)^2\\ &=(ef\sim)^2.\end{aligned}$$

Therefore the set of idempotents in $S/\sim$ forms a band, and hence $S/\sim$ is an orthodox semigroup.

It remains to show that $S/\sim$ is the colimit of the S_i. Let T be any orthodox semigroup and let ψ_i: $S_i\to T$, $i\in I$, be any homomorphisms. Since $S=\Pi * S_i$, there exist monomorphisms ϕ_i: $S_i\to S$, $i\in I$, and a unique homomorphism μ: $S\to T$ such that $\phi_i\mu=\psi_i$, $i\in I$.

Let ϕ be the natural mapping of S onto $S/\sim$, and let $v_i=\phi_i\phi$, $i\in I$. Let x_i and y_i be elements of S_i such that $x_i\sim y_i$. Then there exist elements $z_1, z_2,\dots, z_n$ in S such that

$$x_i=z_1\longrightarrow z_2\longrightarrow\cdots\longrightarrow z_n=y_i,$$

where $z_j\to z_{j+1}$, $j=1, 2,\dots, n-1$, is an elementary transition determined by $(x, xx'x)$ or $(x_1x'_1x_2x'_2\cdots x_rx'_r,\ x_1x'_1x_2x'_2\cdots x_rx'_r,\ x_1x'_1x_2x'_2\cdots x_rx'_r)$, for some $x, x_1, x_2,\dots, x_r$, $x', x'_1, x'_2,\dots, x'_r$, depending upon i, in S. If $z_j\in S_j$, then any possible transition to z_{j+1} makes that z_{j+1} belong to S_j also. Hence, since $z_1\in S_i$, we must have $z_j\in S_i$ for all j. Within S_i each transition can however replace an element only by an equal element. Then $z_1=z_2=\cdots=z_n$, and hence $x_i=y_i$. Thus we have each $v_i(=\phi_i\phi)$ is a monomorphism, $i\in I$.

Let $x \in S$, and let $x = x_{i_1} x_{i_2} \cdots x_{i_r}$ be in reduced form in S. Then,

$$(xx'x)\mu = (x_{i_1} x_{i_2} \cdots x_{i_r} x'_{i_r} \cdots x'_{i_2} x'_{i_1} x_{i_1} x_{i_2} \cdots x_{i_r})\mu$$

$$= (x_{i_1}\mu)(x_{i_2}\mu) \cdots (x_{i_r}\mu)(x'_{i_r}\mu) \cdots (x'_{i_2}\mu)(x'_{i_1}\mu)(x_{i_1}\mu)(x_{i_2}\mu) \cdots (x_{i_r}\mu)$$

$$= (x_{i_1}\mu)(x_{i_2}\mu) \cdots (x_{i_r}\mu)$$

(since each $x'_{i_j}\mu$ is an inverse of $x_{i_j}\mu$, $j = 1, 2, \ldots, r$, and it follows from [5, Lemma 1.3] that $x({}'_{i_r}\mu) \cdots (x'_{i_2}\mu)(x'_{i_1}\mu)$ is an inverse of $(x_{i_1}\mu)(x_{i_2}\mu) \cdots (x_{i_r}\mu)$ in T)
$= x\mu$.
Similarly $(x_1 x'_1 x_2 x'_2 \cdots x_r x'_r)\mu = (x_1 x'_1 x_2 x'_2 \cdots x_r x'_r x_1 x'_1 x_2 x'_2 \cdots x_r x'_r)\mu$ for any $x_1, x_2, \ldots, x_n$ in S and any $x'_j \in W(x_j)$, $j = 1, 2, \ldots, r$. Then $\sim \subseteq \mu o \mu^{-1}$, and hence we can consider a homomorphism $\theta: S/\sim \to T$ such that $\phi\theta = \mu$. Then,

$$v_i\theta = \phi_i\phi\theta = \phi_i\mu = \psi_i,$$

for all $i \in I$. Thus $S/\sim$ together with the v_i is the free product of the S_i in the class of orthodox semigroups, and hence we have the theorem.

Let $\mathscr{A}$ be the variety of bands defined by an identity $P = Q$. Next, we shall give the free product in $\mathscr{A}$ of a given family of bands $\{S_i: i \in I\}$ from $\mathscr{A}$.

If $P = Q$ is heterotypical, it follows from [3, Theorem 2] that $\mathscr{A}$ is one of the varieties of rectangular bands, left zero semigroups, right zero semigroups and one element semigroups. Let $\mathscr{A}$ be the variety of rectangular bands. By [2, Lemma 2], there exist a left zero semigroup L_i, say, and a right zero semigroup R_i, say, such that $S_i \simeq L_i \times R_i$, for each $i \in I$. We can assume without loss of generality that $L_i \cap L_j = \square$ and $R_i \cap R_j = \square$ if $i \neq j$. Let $L = \cup \{L_i: i \in I\}$, $R = \cup \{R_i: i \in I\}$ and $B = L \times R$. Define a product o on B as follows:

$$(a_1, b_1) o (a_2, b_2) = (a_1, b_2).$$

It is clear that $B(o)$ is the free product of the S_i in $\mathscr{A}$. Similarly, we can easily construct the free products in the other three varieties of bands.

So we now consider the free product in the variety of bands defined by a homotypical identity.

THEOREM 1.2. *Let $\mathscr{A}$ be the variety of bands defined by a homotypical identity $P(X_1, X_2, \ldots, X_n) = Q(X_1, X_2, \ldots, X_n)$. Let S_i, $i \in I$, be a family of bands belonging to $\mathscr{A}$. Let σ be the congruence on $S = \Pi * S_i$, the free product of the S_i in the class of semigroups, generated by all pairs (x^2, x) and $(P(x_1, x_2, \ldots, x_n), Q(x_1, x_2, \ldots, x_n))$ for $x, x_1, x_2, \ldots, x_n$ in S. Then S/σ is the free product of the S_i in $\mathscr{A}$.*

PROOF. By an argument similar to the proof of Theorem 1.1, we may prove the theorem.

§2. [Left, right] normal bands

We have given the free product in the variety of bands defined by a homotypical identity. However, if $\mathscr{A}$ is the variety of [left, right] normal bands, we can describe it in a more useful form.

Let E_i, $i\in I$, be a family of normal bands. It follows from [7, Theorem 4] that each E_i, $i\in I$, is isomorphic to the spined product $L_i \bowtie R_i(\Gamma_i)$ of a left normal band L_i, say, and a right normal band R_i, say, with respect to a semilattice Γ_i, say. Let $L_i \sim \Sigma\{L_{\alpha_i}: \alpha_i\in\Gamma_i\}$ and $R_i\sim\Sigma\{R_{\alpha_i}: \alpha_i\in\Gamma_i\}$ be the structure decompositions of L_i and R_i, $i\in I$, respectively. Then the structure decomposition of each E_i is $E_i\sim\Sigma\{L_{\alpha_i}\times R_{\alpha_i}: \alpha_i\in\Gamma_i\}$. We identify each E_i with $\Sigma\{L_{\alpha_i}\times R_{\alpha_i}: \alpha_i\in\Gamma_i\}$. To construct the free product of the E_i, we can assume without loss of generality that $E_i\cap E_j=\square$ if $i\neq j$, and we assume $\cup\{L_{\alpha_i}: \alpha_i\in\Gamma_i, i\in I\}=L$, say, and $\cup\{R_{\alpha_i}: \alpha_i\in\Gamma_i, i\in I\}=R$, say, are both disjoint unions. Let $\Gamma=\{(\alpha_i)_{i\in I}: \alpha_i\in\Gamma_i^{(1)}$, only finitely many but at least one α_i are different from 1$\}$. For convenience, we write simply (α_i) instead of $(\alpha_i)_{i\in I}$. Let B be the subset of $L\times R\times\Gamma$ consisting of $(a, b, (\alpha_i))$ such that $a\in L_{\alpha_j}$ and $b\in L_{\alpha_k}$ for some $\alpha_j\neq 1$ and $\alpha_k\neq 1$ of (α_i). Hereafter, we sometimes denote such $(a, b, (\alpha_i))$ by $(a, b, (\alpha_i; j, k))$. By $e(\alpha_i)$ and $f(\beta_j)$, we mean elements e and f of L_{α_i} and R_{β_i} respectively. Define a product o on B as follows:

$$(a, b, (\alpha_i; j, k))o(c, d, (\beta_i; m, n))=(a\cdot e(\alpha_j\beta_j), f(\alpha_n\beta_n)\cdot d, (\alpha_i\beta_i)).$$

At first, we shall show that o is well-defined. Let e' and f' be another elements of $L_{\alpha_j\beta_j}$ and $R_{\alpha_n\beta_n}$, respectively. Since $L_{\alpha_j\beta_j}$ is a left zero semigroup and L_i is left normal, we have

$$ae=aee'=ae'e=ae'.$$

Similarly, we have $fd=f'd$. Hence the product o is well-defined.

We shall next prove $B(o)$ is a normal band. Let $x_1=(a_1, b_1, (\alpha_i; j, k))$, $x_2=(a_2, b_2, (\beta_i))$ and $x_3=(a_3, b_3, (\gamma_i; m, n))$ be any elements of B. Since L_{α_i} and R_{α_i} are a left zero semigroup and a right zero semigroup, respectively, and Γ_i is a semilattice, $i\in I$, we have

$$\begin{aligned} x_1^2 &= (a_1\cdot e(\alpha_j^2), f(\alpha_k^2)\cdot b_1, (\alpha_i^2)) \\ &= (a_1, b_1, (\alpha_i)) \\ &= x_1, \end{aligned}$$

and
$$\begin{aligned} (x_1ox_2)ox_3 &= (a_1\cdot e(\alpha_j\beta_j\gamma_j), f(\alpha_n\beta_n\gamma_n)b_3, (\alpha_i\beta_i\gamma_i)) \\ &= x_1o(x_2ox_3), \end{aligned}$$

and $((x_1 o x_2) o x_3) o x_4 = (a_1 \cdot e(\alpha_j \beta_j \gamma_j), f(\alpha_k \beta_k \gamma_k) \cdot b_1, (\alpha_i \beta_i \gamma_i))$

$$= ((x_1 o x_3) o x_2) o x_1 .$$

Hence $B(o)$ is a normal band.

THEOREM 2.1. (i) *Let E_i, $i \in I$, be a family of normal bands, and let Γ_i, $i \in I$, be the structure semilattice of E_i. Then $B(o)$ is the free product of the E_i, in the variety of normal bands.*

(ii) *The structure semilattice of $B(o)$ is isomorphic to the free product of the Γ_i, in the variety of semilattices.*

PROOF. We shall use the notations defined above. Let ϕ_i: $E_i \to B$, $i \in I$, be mappings defined by

$$(a, b)\phi_i = (a, b, (\hat{\alpha}_i)) \qquad \text{if} \quad (a, b) \in L_{\alpha_i} \times R_{\alpha_i},$$

where $(\hat{\alpha}_i)$ is the element of Γ with the i-th entry equal to α_i and others equal to 1. It is clear that each ϕ_i is a monomorphism, $i \in I$.

Let T be any normal band, and let ψ_i: $E_i \to T$, $i \in I$, be any homomorphisms. Let us define a mapping μ: $B \to T$ as follows:
For an element $(a, b, (\alpha_i; j, k))$ in B such that only $\alpha_j, \alpha_k, \alpha_{r(1)}, \alpha_{r(2)}, \dots, \alpha_{r(n)}$ are not equal to 1,

$$(a, b, (\alpha_i; j, k))\mu = ((a, c)\psi_j)((g_1, h_1)\psi_{r(1)})((g_2, h_2)\psi_{r(2)}) \times \cdots ((g_n, h_n)\psi_{r(n)})((d, b)\psi_k),$$

where $c \in R_{\alpha_j}$, $d \in L_{\alpha_k}$ and $g_s \in L_{\alpha_{r(s)}}$, $h_s \in R_{\alpha_{r(s)}}$ for $s = 1, 2, \dots, n$.

Since T is normal and each ψ_i, $i \in I$, is a homomorphism, μ is a homomorphism satisfying $\phi_i \mu = \psi_i$, $i \in I$. It is clear that $B = \langle \cup \{E_i \phi_i : i \in I\} \rangle$, and hence $B(o)$ together with the ϕ_i is the free product of the E_i, in the variety of normal bands. Thus we have (*i*).

The later statement of the theorem is obvious.

COROLLARY 2.2. *Let L_i, $i \in I$, be a family of left normal bands whose structure decompositions are $L_i \sim \Sigma\{L_{\alpha_i} : \alpha_i \in \Gamma_i\}$. Let $L = \cup \{L_{\alpha_i} : \alpha_i \in \Gamma_i, i \in I\}$, and let Γ be as defined above. Set*

$$E = \{(a, (\alpha_i; j)) \in L \times \Gamma : a \in L_{\alpha_j}, \alpha_j \neq 1\} .$$

We define a product o on E as follows:

$$(a, (\alpha_i; j)) o (b, (\beta_i; k)) = (a \cdot e(\alpha_j \beta_j), (\alpha_i \beta_i; j))$$

Then $E(o)$ is the free product of the L_i, in the variety of left normal bands. Moreover, the structure semilattice of $E(o)$ is isomorphic to the free product of the Γ_i, in the variety of semilattices.

COROLLARY 2.3. *Let $\mathscr{A}$ be the variety of [left, right] normal bands. Let E_i, $i \in I$, be a family of [left, right] normal bands, and let B together with the monomorphisms ϕ_i be the free product of E_i, in the variety of [left, right] normal bands. If F_i is a subband of E_i, $i \in I$, then*

$$<\{F_i\phi_i : i \in I\}> \simeq \Pi^* F_i.$$

PROOF. It is clear that the corollary follows from the structure of the free product in $\mathscr{A}$.

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