Mem. Fac. Lit. & Sci., Shimane Univ.,
Nat. Sci., 8, pp. 1—11, March 10, 1975

Note on Regular Extensions of a Band by an Inverse Semigroup

Miyuki YAMADA
Department of Mathematics, Shimane University, Matsue, Japan
(Received November 5, 1974)

This is a supplement to the previous papers [8], [9] and [10]. In [8], [9] and [10], the concept of a complete regular product S of a band B and an inverse semigroup Γ, and the concept of a half-direct product T of a left regular band E, an inverse semigroup Γ' and a right regular band F were introduced. In this paper, we first show that a semigroup M is inversive [quasi-(C)-inversive] if and only if M is isomorphic to a complete regular product of a band and a weakly C-inversive semigroup [a half-direct product of a left regular band, a weakly C-inversive semigroup and a right regular band]. If in particular Γ and Γ' are weakly C-inversive, both the spined product of B, Γ and that of E, Γ', F can be considered. When Γ [Γ'] is weakly C-inversive, we investigate the relationship between the complete regular products of B, Γ and the spined product of B, Γ [the half-direct products of E, Γ', F and the spined product of E, Γ', F].

§ 0. Introduction.

Hereafter, the notation "an inversive semigroup[1)] $G\equiv\sum\{G_\gamma:\gamma\in\Gamma\}$" will mean an inversive semigroup G whose structure semilattice is Γ and whose structure decomposition is $G\sim\sum\{G_\gamma:\gamma\in\Gamma\}$ (see [5]). Since a band is inversive, "a band $G\equiv\sum\{G_\gamma:\gamma\in\Gamma\}$" means a band G whose structure semilattice is Γ and whose structure decomposition is $G\sim\sum\{G_\gamma:\gamma\in\Gamma\}$. If a band T has K as its structure semilattice, T is sometimes denoted by $T(K)$. Similarly, an inverse semigroup M having N as its basic semilattice (see [5]) will be sometimes denoted by $M(N)$. Now, let $\Gamma(\Lambda)\equiv\sum\{\Gamma_\lambda:\lambda\in\Lambda\}$ (where Λ is the basic semilattice (=the structure semilattice) of Γ and each Γ_λ is the greatest subgroup containing λ) be a weakly C-inversive semigroup (that is, an inverse semigroup which is a union of groups), and $B(\Lambda)\equiv\sum\{B_\lambda:\lambda\in\Lambda\}$ a band. Of course, each *λ-kernel* B_λ (see [5]) is a rectangular bubband of B. Let I_λ be a maximal left zero subsemigroup of B_λ, and J_λ a maximal right zero subsemigroup of B_λ. Then, as was shown in [9], $\cup\{I_\lambda:\lambda\in\Lambda\}$ and $\cup\{J_\lambda:\lambda\in\Lambda\}$ are a lower partial chain of the left zero semigroups $\{I_\lambda:\lambda\in\Lambda\}$ and an upper partial chain of the right zero semigroups $\{J_\lambda:\lambda\in\Lambda\}$ respectively with respect to the multiplication in B. We shall denote these lower partial chain $\cup\{I_\lambda:\lambda\in\Lambda\}$, upper partial chain $\cup\{J_\lambda:\lambda\in\Lambda\}$ by $\mathscr{I}=[I_\lambda:\lambda\in\Lambda]$,

1) A regular semigroup G is said to be inversive if the set of idempotents of G is a subsemigroup and if for any element a of G there exists an inverse a^* of a such that $aa^*=a^*a$.

$\mathscr{J}=[J_\lambda: \lambda\in\Lambda]$ respectively. Let u_λ be a representative of B_λ for each $\lambda\in\Lambda$. By [7], each element x of B can be uniquely expressed in the form $x=iu_\lambda j$, $i\in I_\lambda$, $j\in J_\lambda$, $\lambda\in\Lambda$, and B is written in the form $B=\{iu_\lambda j: \lambda\in\Lambda, i\in I_\lambda, j\in J_\lambda\}$. Let $\{u_\lambda: \lambda\in\Lambda\}=U$. Then, the following result follows from Warne [3] (also the author [9]): For each pair (γ, δ) of $\gamma, \delta\in\Gamma$, let $\alpha_{(\gamma,\delta)}, \beta_{(\gamma,\delta)}$ be mappings such that $\alpha_{(\gamma,\delta)}: J_{\gamma^{-1}\gamma}\times I_{\delta\delta^{-1}}\to I_{\gamma\delta(\gamma\delta)^{-1}}$[2] and $\beta_{(\gamma,\delta)}: J_{\gamma^{-1}\gamma}\times I_{\delta\delta^{-1}}\to J_{(\gamma\delta)^{-1}\gamma\delta}$. If the system $\Delta=\{\alpha_{(\gamma,\delta)}: \gamma, \delta\in\Gamma\}\cup\{\beta_{(\gamma,\delta)}: \gamma, \delta\in\Gamma\}$ satisfies the condition

(C1) for $j\in J_{\gamma^{-1}\gamma}$, $p\in I_{\delta\delta^{-1}}$, $q\in J_{\delta^{-1}\delta}$, $m\in I_{\xi\xi^{-1}}$,

$$(j, p)\alpha_{(\gamma,\delta)}((j, p)\beta_{(\gamma,\delta)}q, m)\alpha_{(\gamma\delta,\xi)}=(j, p((q, m)\alpha_{(\delta,\xi)})\alpha_{(\gamma,\delta\xi)}$$

and

$$(j, p((q, m)\alpha_{(\delta,\xi)}))\beta_{(\gamma,\delta\xi)}(q, m)\beta_{(\delta,\xi)}=((j, p)\beta_{(\gamma,\delta)}q, m)\beta_{(\gamma\delta,\xi)},$$

then $S=\{(i, \gamma, j): \gamma\in\Gamma, i\in I_{\gamma\gamma^{-1}}, j\in J_{\gamma^{-1}\gamma}\}$ becomes an orthodox semigroup (see Hall [2]) with respect to the multiplication defined by

$$(i, \gamma, j)(h, \delta, k)=(i((j, h)\alpha_{(\gamma,\delta)}), \gamma\delta, (j, h)\beta_{(\gamma,\delta)}k).$$

Further, it follows from the author [9] that if the subset $\Omega=\{\alpha_{(\xi,\eta)}: \xi, \eta\in\Lambda\}\cup\{\beta_{(\xi,\eta)}: \xi, \eta\in\Lambda\}$ of Δ satisfies the condition

(C2) $u_\lambda jku_\tau=((j, k)\alpha_{(\lambda,\tau)})u_{\lambda\tau}((j, k)\beta_{(\lambda,\tau)})$ for $\lambda, \tau\in\Lambda, j\in J_\lambda, k\in I_\tau$,

then B is embedded as the band of idempotents of S.

In this case, S is called *the complete regular product of* $B(\Lambda)$ *and* $\Gamma(\Lambda)$ *determined by* $\{\mathscr{I}, \mathscr{J}, \{u_\lambda\}, \Delta\}$, and denoted by $C(\Gamma(\Lambda), B(\Lambda); \mathscr{I}, \mathscr{J}, \{u_\lambda\}, \{\alpha_{(\gamma,\delta)}\}, \{\beta_{(\gamma,\delta)}\})$. We shall call Δ (whose subset Ω satisfies (C2)) above *a CR-factor set in* $B=\{iu_\lambda j: \lambda\in\Lambda, i\in I_\lambda, j\in J_\lambda\}$ *belonging to* $\Gamma(\Lambda)$ (see [10]). In [9], it has been shown that every regular extension of $B(\Lambda)$ by $\Gamma(\Lambda)$ can be obtained as a complete regular product of $B(\Lambda)$ and $\Gamma(\Lambda)$ (up to isomorphism). Since $\Gamma(\Lambda)\equiv\sum\{\Gamma_\lambda: \lambda\in\Lambda\}$ is weakly C-inversive (hence each Γ_λ is a group), we can consider the spined product $B\bowtie\Gamma\ (\Lambda)$ (see [5]) of $B(\Lambda)$ and $\Gamma(\Lambda)$ with respect to Λ. In this paper, we shall show a necessary and sufficient condition on $\{\alpha_{(\gamma,\delta)}: \gamma, \delta\in\Gamma\}\cup\{\beta_{(\gamma,\delta)}: \gamma, \delta\in\Gamma\}$ in order that $C(\Gamma(\Lambda), B(\Lambda); \mathscr{I}\ \mathscr{J}\ \{u_\lambda\}, \{\alpha_{(\gamma,\delta)}\}, \{\beta_{(\gamma,\delta)}\})$ be isomorphic to $B\bowtie\Gamma\ (\Lambda)$.

Next, let $E(\Lambda)\equiv\sum\{E_\lambda: \lambda\in\Lambda\}$, $F(\Lambda)\equiv\sum\{F_\lambda: \lambda\in\Lambda\}$ be a left regular band, a right regular band (see [7], [8]) respectively. The concept of a half-direct product (abbrev., an H.D-product) of $E(\Lambda)$, $\Gamma(\Lambda)$ and $F(\Lambda)$ was introduced by the author [8] as follows:

2) If G is an inversive semigroup, then for each element x of G there exists a unique inverse x^* of x such that $xx^*=x^*x$. This x^* is denoted by x^{-1}. If G is in particular a weakly C-inversive semigroup, then G is of course an inverse semigroup and hence x^{-1} is a unique inverse of x for each $x\in G$ (see [1]). The notation "$\varphi: X\to Y$" means "φ is a mapping of X into Y".

Let $\phi: \Gamma \rightarrow \mathrm{End}(E)$ (where $\mathrm{End}(E)$ is the semigroup of all endomorphisms on E), $\psi: \Gamma \rightarrow \mathrm{End}(F)$ be two mappings, and put $\gamma\phi = \rho_\gamma$, $\gamma\psi = \sigma_\gamma$ for all $\gamma \in \Gamma$. If $\{\rho_\gamma: \gamma \in \Gamma\}$, $\{\sigma_\gamma: \gamma \in \Gamma\}$ satisfy

(C 3) each ρ_γ $[\sigma_\gamma]$ maps E_α $[F_\alpha]$ into $E_{(\alpha\gamma)^{-1}\alpha\gamma}[F_{(\alpha\gamma)^{-1}\alpha\gamma}]$ for all $\alpha \in \Lambda$; especially, ρ_γ $[\sigma_\gamma]$ is an inner endomorphism (see [8]) on E $[F]$ for $\gamma \in \Lambda$,

and

(C 4) for any $e \in E_{\beta^{-1}\beta}$ $[F_{\beta^{-1}\beta}]$, $f \in E_{(\alpha\beta)^{-1}\alpha\beta}[F_{(\alpha\beta)^{-1}\alpha\beta}]$,

$$\rho_\alpha \rho_\beta \delta_f \delta_e = \rho_{\alpha\beta} \delta_f \delta_e \ [\sigma_\alpha \sigma_\beta \delta_f \delta_e = \sigma_{\alpha\beta} \delta_f \delta_e]$$

where δ_h denotes the inner endomorphism on E $[F]$ induced by h (see [8]), then $M = \{(e, \gamma, f): \gamma \in \Gamma,\ e \in E_{\gamma\gamma^{-1}},\ f \in F_{\gamma^{-1}\gamma}\ (=F_{\gamma\gamma^{-1}})\}$ becomes a quasi-inverse semigroup with respect to the multiplication defined by

(C 5) $$(e, \gamma, f)(u, \tau, v) = (eu^{\rho_{\gamma^{-1}}}e,\ \gamma\tau,\ vf^{\sigma_\tau}v) = (eu^{\rho_{\gamma^{-1}}},\ \gamma\tau,\ f^{\sigma_\tau}v),$$

where $x^{\rho_{\gamma^{-1}}}[x^{\sigma_\tau}]$ means $x\rho_{\gamma^{-1}}[x\sigma_\tau]$.
(See Theorem 6 of [8]).

This M is called *the half-direct product* (*the H.D-product*) *of* $E(\Lambda)$, $\Gamma(\Lambda)$ *and* $F(\Lambda)$ *determined by* $\{\phi, \psi\}$, and denoted by $E \underset{\phi}{\times} \Gamma \underset{\psi}{\times} F$. If the band of idempotents of an inversive semigroup H is a regular band (see [8]), then H is said to be *quasi-(C)-inversive*. We shall show in §2 that a semigroup is a quasi-(C)-inversive semigroup if and only if it is isomorphic to an H.D-product of a left regular band, a weakly C-inversive semigroup and a right regular band. On the other hand, we can consider the spined product $E \bowtie \Gamma \bowtie F$ (Λ) of $E(\Lambda)$, $\Gamma(\Lambda)$ and $F(\Lambda)$[3]. In §2, we shall give a necessary and sufficient condition on $\{\phi, \psi\}$ in order that $E \underset{\phi}{\times} \Gamma \underset{\psi}{\times} F$ be isomorphic to $E \bowtie \Gamma \bowtie F$ (Λ).

Throughout this paper, $\Gamma(\Lambda) \equiv \{\Gamma_\lambda: \lambda \in \Lambda\}$, $B(\Lambda) \equiv \sum\{B_\lambda: \lambda \in \Lambda\}$, $E(\Lambda) \equiv \sum\{E_\lambda: \lambda \in \Lambda\}$, $F(\Lambda) \equiv \sum\{F_\lambda: \lambda \in \Lambda\}$ will denote a weakly C-inversive semigroup, a band, a left regular band, a right regular band respectively (their structure decompositions are $\Gamma(\Lambda) \sim \sum\{\Gamma_\lambda: \lambda \in \Lambda\}$ (Λ: the basic semilattice (=the structure semilattice) of Γ; each Γ_λ is the greatest subsemigroup containing λ), $B(\Lambda) \sim \sum\{B_\lambda: \lambda \in \Lambda\}$, $E(\Lambda) \sim \sum\{E_\lambda: \lambda \in \Lambda\}$ and $F(\Lambda) \sim \sum\{F_\lambda: \lambda \in \Lambda\}$). For each $\lambda \in \Lambda$, I_λ, J_λ will denote a maximal left zero subsemigroup of B, a maximal right zero subsemigroup of B_λ respectively. Let $\mathscr{I}$, $\mathscr{J}$ be the lower partial chain of $\{I_\lambda: \lambda \in \Lambda\}$, the upper partial chain of $\{J_\lambda: \lambda \in \Lambda\}$ (with respect to the multiplication in B). Hence, $\mathscr{I} = [I_\lambda: \lambda \in \Lambda]$ and $\mathscr{J} = [J_\lambda: \lambda \in$

3) Let $A_i(\Lambda) \equiv \sum\{A_\lambda^i: \lambda \in \Lambda\}$ $(i = 1, 2, \ldots, n)$ be an inversive semigroup having Λ as its structure semilattice. Then, $A = \{[a_1, a_2, \ldots, a_n]: a_i \in A_\lambda^i\ (i = 1, 2, \ldots, n),\ \lambda \in \Lambda\}$ becomes a semigroup with respect to the multiplication defined by $[a_1, a_2, \ldots, a_n]\ [b_1, b_2, \ldots, b_n] = [a_1 b_1, a_2 b_2, \ldots, a_n b_n]$. This A is called the spined product of $A_1(\Lambda)$, $A_2(\Lambda), \ldots, A_n(\Lambda)$, and denoted by $A_1 \bowtie A_2 \bowtie \cdots \bowtie A_n$ (Λ).

Λ]. Any other notation and terminology should be referred to [5], [8] and [9], unless otherwise stated.

§ 1. Complete regular products.

Let $S=C(\Gamma(\Lambda), B(\Lambda); \mathscr{I}, \mathscr{J}, \{u_\lambda\}, \{\alpha_{(\gamma,\delta)}\}, \{\beta_{(\gamma,\delta)}\})$ be the complete regular product of $B(\Lambda)$ and $\Gamma(\Lambda)$ introduced in § 0. Let B^* be the set of all idempotents of S.

LEMMA 1. *S is an inversive semigroup.*

PROOF. The set B^* of all idempotents of S is $\{(i, \lambda, j): \lambda \in \Lambda, i \in I_\lambda, j \in J_\lambda\}$. It is obvious from Lemma 5 of [9] that B^* is isomorphic to B. Hence, B^* is a band. For any element $(h, \gamma, k) \in S$, the element (h, γ^{-1}, k) is an inverse of (h, γ, k) and satisfies $(h, \gamma, k)(h, \gamma^{-1}, k)=(h, \gamma^{-1}, k)(h, \gamma, k)$ (since for any element γ of Γ the equalities $\gamma\gamma^{-1}=\gamma^{-1}\gamma$ and $(\gamma^{-1})^{-1}=\gamma$ are satisfied in Γ). Therefore, S is an inversive semigroup.

THEOREM 2. *A semigroup is inversive if and only if it is isomorphic to a complete regular product of a band and a weakly C-inversive semigroup.*

PROOF. The "if" part is obvious from Lemma 1. Let T be an inversive semigroup, and η the least inverse semigroup congruence (see [2], [6]) on T. Let A be the band of idempotents of T. Then, it follows from § 6 of [9] that T is isomorphic to a complete regular product of A and T/η (where T/η denotes the factor semigroup of T mod η). Since T is a union of groups and since T/η is a homomorphic image of T, the factor semigroup T/η is also a union of groups. Hence, T/η is a weakly C-inversive semigroup.

According to [4], S is isomorphic to the spined product $B^* \bowtie C\,(\Lambda)$ of the band $B^*(\Lambda)$ and a weakly C-inversive semigroup $C(\Lambda)$ if and only if S is strictly inversive, that is, S satisfies the following condition (1.1).

(1.1) $$e, f \in B^*,\ x \in S,\ xx^{-1}=e,\ f \leqq e \text{ imply } xf=fx.$$

By using this fact, we have

THEOREM 3. *S is isomorphic to the spined product $B \bowtie C\,(\Lambda)$ of the band $B(\Lambda)$ and a weakly C-inversive semigroup $C(\Lambda)$ if and only if the CR-factor set $\Delta=\{\alpha_{(\gamma,\delta)}: \gamma, \delta \in \Gamma\} \cup \{\beta_{(\gamma,\delta)}: \gamma, \delta \in \Gamma\}$ satisfies the following* (1.2):

(1.2) $$\gamma \in \Gamma,\ \lambda \in \Lambda,\ \gamma\gamma^{-1}=\mu \geqq \lambda,\ i \in I_\mu,\ j \in J_\mu \text{ imply}$$
$$\alpha_{(\gamma,\lambda)}\lambda_i=\alpha_{(\mu,\lambda)}\lambda_i \text{ on } J_\mu \times iI_\lambda, \text{ and}$$
$$\beta_{(\lambda,\gamma)}\nu_j=\beta_{(\lambda,\mu)}\nu_j \text{ on } J_\lambda j \times I_\mu$$

where λ_i, ν_j are the left multiplication by i and the right multiplication by j respectively[4]*.*

Further, in this case $\Gamma(\Lambda)$ can be selected as $C(\Lambda)$.

PROOF. Suppose that S is isomorphic to the spined product $B\bowtie C(\Lambda)$ of the band $B(\Lambda)$ and a weakly C-inversive semigroup $C(\Lambda)$. Then, it follows from [4] that S is strictly inversive. Let $\gamma\in\Gamma$, $\lambda\in\Lambda$, $\gamma\gamma^{-1}=\mu\geqq\lambda$, $i\in I_\mu$ and $j\in J_\mu$. For any idempotent $(u,\lambda,v)\in B^*$, $(i,\mu,j)(iu,\lambda,vj)=(i((j,iu)\alpha_{(\mu,\lambda)}),\lambda,((j,iu)\beta_{(\mu,\lambda)})vj)=(i((j,iu)\alpha_{(\mu,\lambda)}),\lambda,vj)$. On the other hand, $i(u_\mu jiuu_\lambda)=i((j,iu)\alpha_{(\mu,\lambda)})u_{\mu\lambda}(j,iu)\beta_{(\mu,\lambda)}$. Hence, $iu=i((j,iu)\alpha_{(\mu,\lambda)})$. Therefore, $(i,\mu,j)(iu,\lambda,vj)=(iu,\lambda,vj)$. Similarly, we have $(iu,\lambda,vj)(i,\mu,j)=(iu,\lambda,vj)$. Since S is strictly inversive and since $(i,\gamma,j)^{-1}=(i,\gamma^{-1},j)$, the equality $(iu,\lambda,vj)(i,\gamma,j)=(i,\gamma,j)(iu,\lambda,vj)$ holds. Hence, $(iu,\lambda\gamma,(vj,i)\beta_{(\lambda,\gamma)}j)=(iu,\lambda,vj)(i,\gamma,j)=(i,\gamma,j)(iu,\lambda,vj)=(i((j,iu)\alpha_{(\gamma,\lambda)}),\gamma\lambda,vj)$, and hence $i((j,iu)\alpha_{(\gamma,\lambda)})=iu$ and $((vj,i)\beta_{(\lambda,\gamma)})j=vj$. That is, the condition (1.2) holds. In this case, if we put $\{(i,\gamma,j)\in S: i\in I_\lambda, j\in J_\lambda, \gamma\in\Gamma \text{ with } \gamma\gamma^{-1}=\lambda\}=S_\lambda$ for each $\lambda\in\Lambda$ then each S_λ is a rectangular group (that is, the direct product of a rectangular band and a group) and S is a semilattice Λ of rectangular groups S_λ. Let B_λ^* be the set of idempotents of S_λ. Then, the structure decomposition of B^* is clearly $B^*\sim\sum\{B_\lambda^*:\lambda\in\Lambda\}$. It follows from the proof of Theorem 4 of [4] that the relation ξ on S defined by

(1.3) $\quad x\xi y$ if and only if $x, y\in S_\tau$ and $x^{-1}y\in B_\tau^*$ for some $\tau\in\Lambda$

is a congruence on S, and S is isomorphic to the spined product of $B^*(\Lambda)$ and $S/\xi(\Lambda)$. Now, for $x=(i,\gamma,j)$, $y=(h,\delta,k)$ it is easily seen that

(1.4) $\quad (i,\gamma,j)\xi(h,\delta,k)$ if and only if $\gamma=\delta$.

Hence the mapping $\varphi: S/\xi\to\Gamma(\Lambda)$ defined by $\overline{(i,\gamma,j)}\varphi=\gamma$ is an isomorphism, where $\overline{(i,\gamma,j)}$ denotes the ξ-class containing (i,γ,j). Since $B\cong B^*$ (where $\cong$ means "isomorphic") and since $S/\xi\cong\Gamma(\Lambda)$, S is isomorphic to the spined product $B\bowtie\Gamma(\Lambda)$.

Conversely, suppose that $S=C(\Gamma(\Lambda), B(\Lambda); \mathscr{I}, \mathscr{J}, \{u_\lambda\}, \{\alpha_{(\gamma,\delta)}\}, \{\beta_{(\gamma,\delta)}\})$ satisfies the condition (1.2). If S is strictly inversive then S is isomorphic to the spined product of $B^*(\Lambda)$ and a weakly C-inversive semigroup $C(\Lambda)$ (Theorem 4 of [4]). Hence, in this case $S\cong B\bowtie C(\Lambda)$ since $B^*\cong B$. Therefore, we next prove that S is strictly inversive. Let $(i,\gamma,j)\in S$, $(u,\lambda,v)\in B^*$ be two elements such that $(i,\gamma,j)(i,\gamma^{-1},j)=(i,\gamma\gamma^{-1},j)\geqq(u,\lambda,v)$, and put $\gamma\gamma^{-1}=\mu$. Then, $\lambda\leqq\mu$, $i\in I_\mu$ and $j\in J_\mu$. Now, $(i,\gamma,j)(u,\lambda,v)=(i((j,u)\alpha_{(\gamma,\lambda)}),\gamma\lambda,v)$ and $(u,\lambda,v)(i,\gamma,j)=(u,\lambda\gamma,((v,i)\beta_{(\lambda,\gamma)})j)$. On the other hand, $(u,\lambda,v)=(i,\mu,j)(u,\lambda,v)=(i((j,u)\alpha_{(\mu,\lambda)}),\lambda,v)$ and $(u,\lambda,v)=(u,\lambda,v)(i,\mu,j)=(u,\lambda,((v,i)\beta_{(\lambda,\mu)})j)$. Since $i((j,u)\alpha_{(\mu,\lambda)})=u$ and $((v,i)\beta_{(\lambda,\mu)})j=v$, it follows that $iu=u$ and $vj=v$. Hence by (1.2), $i((j,iu)\alpha_{(\gamma,\lambda)})=i((j,iu)\alpha_{(\mu,\lambda)})=iu=u$ and $((vj,i)\beta_{(\lambda,\gamma)})j=((vj,i)\beta_{(\lambda,\mu)})j=vj=v$. Since $\gamma\lambda=\lambda\gamma$, this implies that $(i,\gamma,j)(u,\lambda,v)=(u,\gamma\lambda,v)=$

4) That is, λ_i, ν_j are mappings such that $x\lambda_i=ix$ and $x\nu_j=xj$.

$(u, \lambda, v)(i, \gamma, j)$.

From the theorem above, we obtain the following result.

COROLLARY 4. *If in particular $B(\Lambda)$ is a normal band*[5]*, then a complete regular product of $B(\Lambda)$ and $\Gamma(\Lambda)$ is uniquely determined up to isomorphism and is isomorphic to the spined product of $B(\Lambda)$ and $\Gamma(\Lambda)$.*

PROOF. We need only to show that for any complete regular product $S=C(\Gamma(\Lambda), B(\Lambda); \mathscr{I}\ \mathscr{J}\ \{u_\lambda\}, \{\alpha_{(\gamma,\delta)}\}, \{\beta_{(\gamma,\delta)}\})$ of $B(\Lambda)$ and $\Gamma(\Lambda)$ the system $\Delta=\{\alpha_{(\gamma,\delta)}: \gamma, \delta\in\Gamma\}\cup\{\beta_{(\gamma,\delta)}: \gamma, \delta\in\Gamma\}$ necessarily satisfies the condition (1.2). Let $\gamma\in\Gamma$, $\lambda\in\Lambda$, $\gamma\gamma^{-1}=\mu\geqq\lambda$, $i\in I_\mu$ and $j\in J_\mu$. For $e\in I_\lambda$, we have $iI_\lambda=iI_\lambda e=ieI_\lambda e$ (by the normality of $B(\Lambda)$) $=ie$. Similarly, for $f\in J_\lambda$ we have $J_\lambda j=fj$. Therefore, each of iI_λ and $J_\lambda j$ consists of a single element. Hence, Δ satisfies the condition (1.2).

REMARK. The spined product $B\bowtie\Gamma\ (\Lambda)$ of a band $B(\Lambda)$ and a weakly C-inversive semigroup $\Gamma(\Lambda)$ is always isomorphic to some complete regular product of $B(\Lambda)$ and $\Gamma(\Lambda)$. In fact:

$$(1.5)\qquad \begin{cases} B\bowtie\Gamma\ (\Lambda)=\{[e, \gamma]: e\in B_\lambda, \gamma\in\Gamma_\lambda, \lambda\in\Lambda\}, \text{ and} \\ \text{the multiplication in } B\bowtie\Gamma\ (\Lambda) \text{ is given by} \\ [e, \gamma][f, \delta]=[ef, \gamma\delta] \end{cases}$$

Let u_λ be a representative of B_λ for each λ of Λ. For $e\in B$, e is uniquely expressed in the form $e=e'u_\lambda e''$, $\lambda\in\Lambda$, $e'\in I_\lambda$, $e''\in J_\lambda$. In this case, we shall denote e', e'' by e_l, e_r respectively. Hence, $e=e_lu_\lambda e_r\ (=e_le_r)$. Now, for each pair (λ, τ) of $\lambda, \tau\in\Lambda$, define mappings $\alpha_{(\lambda,\tau)}: J_\lambda\times I_\tau\to I_{\lambda\tau}$ and $\beta_{(\lambda,\tau)}: J_\lambda\times I_\tau\to J_{\lambda\tau}$ by

$$u_\lambda fhu_\tau=((f, h)\alpha_{(\lambda,\tau)})u_{\lambda\tau}((f, h)\beta_{(\lambda,\tau)}) \quad \text{for} \quad f\in J_\lambda, h\in I_\tau.$$

Then, for $e=e_le_r\in B_\lambda$, $f=f_lf_r\in B_\tau$, we have $ef=e_l(e_r, f_l)\alpha_{(\lambda,\tau)}u_{\lambda\tau}(e_r, f_l)\beta_{(\lambda,\tau)}f_r$. Next, for $\gamma, \delta\in\Gamma$, define mappings $\alpha_{(\gamma,\delta)}$, $\beta_{(\gamma,\delta)}$ by $\alpha_{(\gamma,\delta)}=\alpha_{(\gamma\gamma^{-1},\delta\delta^{-1})}$ and $\beta_{(\gamma,\delta)}=\beta_{(\gamma\gamma^{-1},\delta\delta^{-1})}$. Then, $\Delta=\{\alpha_{(\gamma,\delta)}: \gamma, \delta\in\Gamma\}\cup\{\beta_{(\gamma,\delta)}: \gamma, \delta\in\Gamma\}$ becomes a CR-factor set in $B=\{iu_\lambda j: i\in I_\lambda, j\in J_\lambda, \lambda\in\Lambda\}$ belonging to $\Gamma(\Lambda)$. Hence, we can consider the complete regular product $C(\Gamma(\Lambda), B(\Lambda); \mathscr{I}, \mathscr{J}, \{u_\lambda\}, \{\alpha_{(\gamma,\delta)}\}, \{\beta_{(\gamma,\delta)}\})=A$. If we define a mapping $\varphi: A\to B\bowtie\Gamma\ (\Lambda)$ by $(e_l, \gamma, e_r)\varphi=[e_le_r, \gamma]$, then it is easily verified that φ is an isomorphism. Hence, $B\bowtie\Gamma\ (\Lambda)$ is isomorphic to the complete regular product A.

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5) A band is said to be normal [left normal, right normal] if it satisfies the identity $x_1x_2x_3x_4=x_1x_3x_2x_4$ $[x_1x_2x_3=x_1x_3x_2,\ x_1x_2x_3=x_2x_1x_3]$.

§ 2. H.D-products.

Let $M = E \underset{\phi}{\times} \Gamma \underset{\psi}{\times} F$ be the H.D-product of $E(\Lambda)$, $\Gamma(\Lambda)$ and $F(\Lambda)$ introduced in § 0. Let V be the set of all idempotents of M.

LEMMA 5. *A semigroup is quasi-(C)-inversive if and only if it is isomorphic to an H.D-product of a left regular band, a weakly C-inversive semigroup and a right regular band.*

PROOF. By [8], $M = E \underset{\phi}{\times} \Gamma \underset{\psi}{\times} F$ is a quasi-inverse semigroup. Since $\Gamma(\Lambda)$ is a union of groups, it is easily proved that M is inversive. Hence, M is a quasi-(C)-inversive semigroup. From this result, it follows that if a semigroup A is isomorphic to an H.D-product of a left regular band, a weakly C-inversive semigroup and a right regular band then A is quasi-(C)-inversive. Conversely, assume that a semigroup A is quasi-(C)-inversive. Then, A is of course a quasi-inverse semigroup. Hence, it follows from [8] that if ξ is the least inverse semigroup congruence on A then A is isomorphic to an H.D-product of a left regular band, A/ξ and a right regular band. Since A/ξ is a homomorphic image of A and since A is a union of groups, A/ξ is also a union of groups. Therefore, A/ξ is a weakly C-inversive semigroup.

Next, consider the spined product of $E(\Lambda)$, $\Gamma(\Lambda)$ and $F(\Lambda)$:

$$(2.1)\qquad \begin{cases} E \bowtie \Gamma \bowtie F\ (\Lambda) = \{[e, \gamma, f] : \gamma \in \Gamma_\lambda,\ e \in E_\lambda,\ f \in F_\lambda,\ \lambda \in \Lambda\}, \\ \text{and the multiplication in } E \bowtie \Gamma \bowtie F\ (\Lambda) \text{ is given by} \\ [e, \gamma, f][u, \delta, v] = [eu, \gamma\delta, fv]. \end{cases}$$

For each $\lambda \in \Lambda$, let e_λ, f_λ be representatives of E_λ, F_λ respectively. Define mappings $\varphi_1 : \Gamma \to \mathrm{End}(E)$, $\varphi_2 : \Gamma \to \mathrm{End}(F)$ by $\gamma\varphi_1 = \delta_{e_{\gamma\gamma^{-1}}}$, $\gamma\varphi_2 = \delta_{f_{\gamma\gamma^{-1}}}$ respectively, where $\delta_{e_\lambda}[\delta_{f_\lambda}]$ denotes the inner endomorphism on E [F] induced by e_λ [f_λ]. For each $\gamma \in \Gamma$, put $\gamma\varphi_1 = \rho_\gamma$ and $\gamma\varphi_2 = \sigma_\gamma$. Then, it is easy to see that each of the systems $\{\rho_\gamma : \gamma \in \Gamma\}$ and $\{\sigma_\gamma : \gamma \in \Gamma\}$ satisfies (C 3) and (C 4). Accordingly, we can consider the H.D-product $E \underset{\varphi_1}{\times} \Gamma \underset{\varphi_2}{\times} F$. For any $(e, \gamma, f), (u, \delta, v) \in E \underset{\varphi_1}{\times} \Gamma \underset{\varphi_2}{\times} F$, $(e, \gamma, f)(u, \delta, v) = (eu^{\rho_{\gamma^{-1}}}, \gamma\delta, f^{\sigma_\delta}v) = (ee_{\gamma\gamma^{-1}}ue_{\gamma\gamma^{-1}}, \gamma\delta, f_{\delta\delta^{-1}}ff_{\delta\delta^{-1}}v) = (eu, \gamma\delta, fv)$. Hence, $\Phi : E \bowtie \Gamma \bowtie F\ (\Lambda) \to E \underset{\varphi_1}{\times} \Gamma \underset{\varphi_2}{\times} F$ defined by $[e, \gamma, f]\Phi = (e, \gamma, f)$ is an isomorphism. From this result, we can say that the spined product of $E(\Lambda)$, $\Gamma(\Lambda)$ and $F(\Lambda)$ is isomorphic to an H.D-product of $E(\Lambda)$, $\Gamma(\Lambda)$ and $F(\Lambda)$. Conversely, next we shall investigate about necessary and sufficient conditions on $\{\phi, \psi\}$ in order that $M = E \underset{\phi}{\times} \Gamma \underset{\psi}{\times} F$ be isomorphic to the spined product $E \bowtie \Gamma \bowtie F\ (\Lambda)$.

LEMMA 6. *M is strictly inversive if and only if it satisfies the following* (2.1):

$$(2.1)\qquad \gamma \in \Gamma,\ \lambda \in \Lambda,\ \gamma\gamma^{-1} = \mu \geqq \lambda,\ i \in E_\mu,\ j \in E_\mu \textit{ imply}$$

$\rho_\gamma\lambda_i$ = *the identity mapping on* iE_λ, *and*

$\sigma_\gamma\nu_j$ = *the identity mapping on* $F_\lambda j$.

PROOF. Assume that M is strictly inversive. Let $\gamma\in\Gamma$, $\lambda\in\Lambda$, $\gamma\gamma^{-1}=\mu\geqq\lambda$, $i\in E_\mu$ and $j\in F_\mu$. For an element $(u, \lambda, v)\in M$, $(iu, \lambda, vj)(i, \mu, j)=(iu, \lambda, vj)$. Similarly, $(i, \mu, j)(iu, \lambda, vj)=(iu, \lambda, vj)$. Since M is strictly inversive, we have $(i, \gamma, j)(iu, \lambda, vj)=(iu, \lambda, vj)(i, \gamma, j)$. Therefore, we have $(i(iu)^{\rho_{\gamma^{-1}}}, \gamma\lambda, (j^{\sigma_\lambda})vj)=(iu(i^{\rho_\lambda}), \lambda\gamma, (vj)^{\sigma_\gamma}j)$, whence $i(iu)^{\rho_{\gamma^{-1}}}=iu$ and $(vj)^{\sigma_\gamma}j=vj$. That is, $\rho_{\gamma^{-1}}\lambda_i$ = the identity mapping on iE_λ, while $\sigma_\gamma\nu_j$ = the identity mapping on $F_\lambda j$. Since $\gamma^{-1}\gamma=\gamma\gamma^{-1}$, it follows that $\rho_\gamma\lambda_i$ = the identity mapping on iE_λ and $\sigma_\gamma\nu_j$ = the identity mapping on $F_\lambda j$. Thus, the condition (2.1) holds.

Conversely, assume that M satisfies (2.1). Let (u, λ, v), (i, μ, j) be idempotents of M such that $(u, \lambda, v)(i, \mu, j)=(i, \mu, j)(u, \lambda, v)=(u, \lambda, v)$, and (t, γ, s) an element of M such that $(t, \gamma, s)(t, \gamma, s)^{-1}=(i, \mu, j)$. Since $(ui^{\rho_\lambda}, \lambda\mu, v^{\sigma_\mu}j)=(iu^{\rho_\mu}, \mu\lambda, j^{\sigma_\lambda}v)=(u, \lambda, v)$, we have $\lambda\leqq\mu$, $u=iu^{\rho_\mu}$ and $v^{\sigma_\mu}j=v$. Further, $(t, \gamma, s)(t, \gamma, s)^{-1}=(i, \mu, j)$ implies $\gamma\gamma^{-1}=\mu\geqq\lambda$. Hence by (2.1), $\rho_\gamma\lambda_i$ = the identity mapping on iE_λ and $\sigma_\gamma\nu_j$ = the identity mapping on $F_\lambda j$. Since $iu=u$ and $vj=v$, we have also $iu=i(iu^{\rho_\mu})=i((iu)^{\rho_\mu})$ and $vj=(vj)^{\sigma_\mu}j$. The equality $(t, \gamma, s)(t, \gamma^{-1}, s)=(i, \mu, j)$ implies $(tt^{\rho_{\gamma^{-1}}}, \gamma\gamma^{-1}, s^{\sigma_{\gamma^{-1}}}s)=(i, \mu, j)$, and hence $(t, \mu, s)=(i, \mu, j)$. That is, $(t, \gamma, s)=(i, \gamma, j)$. From this result, it follows that $(i, \gamma, j)(u, \lambda, v)=(iu^{\rho_{\gamma^{-1}}}, \gamma\lambda, j^{\sigma_\lambda}v)=(iu^{\rho_{\gamma^{-1}}}, \gamma\lambda, v)=(i(iu)^{\rho_{\gamma^{-1}}}, \gamma\lambda, v)=(i(iu), \gamma\lambda, v)=(iu, \gamma\lambda, v)=(u, \gamma\lambda, v)$. Similarly, $(u, \lambda, v)(i, \gamma, j)=(u, \lambda\gamma, v)$. Since $\lambda\gamma=\gamma\lambda$ is satisfied, the equality $(i, \gamma, j)(u, \lambda, v)=(u, \lambda, v)(i, \gamma, j)$ holds. Hence, M is strictly inversive.

THEOREM 7. *M is isomorphic to the spined product of a regular band and a weakly C-inversive semigroup if and only if it satisfies* (2.1). *Further, in this case M is isomorphic to* $E\bowtie\Gamma\bowtie F$ (Λ).

PROOF. If M satisfies the condition (2.1), then it follows from Lemma 6 that M is strictly inversive. Hence, in this case M is isomorphic to the spined product of the regular band V (the band of idempotents of M) and a weakly C-inversive semigroup T (Theorem 4 of [4]). It is also easily seen that $V\cong E\bowtie F$ (Λ). For each $\lambda\in\Lambda$, put $M_\lambda=\{(h, \gamma, k): \gamma\gamma^{-1}=\lambda, \gamma\in\Gamma, h\in E_{\gamma\gamma^{-1}}, k\in F_{\gamma\gamma^{-1}}\}$. Then, the structure decomposition of M is $M\sim\sum\{M_\lambda: \lambda\in\Lambda\}$. If we define a relation ξ on M by

(2.2) for any (h, γ, k), $(u, \delta, v)\in M$, $(h, \gamma, k)\ \xi\ (u, \delta, v)$ if and only if

(h, γ, k), $(u, \delta, v)\in M_\lambda$ and $(h, \gamma, k)(u, \delta, v)^{-1}\in E(M_\lambda)$ for some $\lambda\in\Lambda$,

where $E(M_\lambda)$ is the set of idempotents of M_λ,

then $M\cong E\bowtie M/\xi\bowtie F$ (Λ) follows from the proof of Theorem 4 of [4]. On the other hand, it is easily proved that $(h, \gamma, k)\ \xi\ (u, \delta, v)$ if and only if $\gamma=\delta$. Hence, Φ: $M/\xi \to \Gamma$ defined by $\overline{(h, \gamma, k)}\Phi=\gamma$, where $\overline{(h, \gamma, k)}$ is the ξ-class containing (h, γ, k), is an isomorphism. Hence $M/\xi \cong \Gamma$, and accordingly $M \cong E \bowtie \Gamma \bowtie F\ (\Lambda)$.

Conversely, if M is isomorphic to the spined product of a regular band and a weakly C-inversive semigroup then M is clearly strictly inversive (Theorem 4 of [4]). Hence, in this case it follows from Lemma 6 that M satisfies the condition (2.1).

COROLLARY 8. *If in particular $E(\Lambda)$, $F(\Lambda)$ are a left normal band, a right normal band respectively, then an H.D-product of $E(\Lambda)$, $\Gamma(\Lambda)$ and $F(\Lambda)$ is uniquely determined up to isomorphism and is isomorphic to the spined product of $E(\Lambda)$, $\Gamma(\Lambda)$ and $F(\Lambda)$.*

PROOF. We need only to show that for any H.D-product $E \underset{\phi}{\times} \Gamma \underset{\psi}{\times} F$, $\{\phi, \psi\}$ (hence, $\{\rho_\gamma: \gamma \in \Gamma\} \cup \{\sigma_\gamma: \gamma \in \Gamma\}$, where $\rho_\gamma=\gamma\phi$ and $\sigma_\gamma=\gamma\psi$) satisfies the condition (2.1). For μ, λ, e, f, i such that λ, $\mu \in \Lambda$, $\mu \geqq \lambda$, $i \in E_\mu$ and $e, f \in E_\lambda$, we have $ie=ief=ife=if$. Hence, iE_λ consists of a single element. Therefore, $\rho_\gamma\lambda_i=$the identity mapping on iE_λ if $\gamma\gamma^{-1}=\mu$. Similarly, $\sigma_\gamma\nu_j=$the identity mapping on $F_\lambda j$ if $j \in F_\mu$, $\mu \geqq \lambda$, $\gamma \in \Gamma$, $\gamma\gamma^{-1}=\mu$.

§ 3. Special cases.

In [8], the concept of *an L.H.D-product* of a left regular band and an inverse semigroup [*an R.H.D-product* of an inverse semigroup and a right regular band] has been introduced. For L.H.D-products and R.H.D-products, we can obtain the following results in the same manner as we established Theorem 7 and Corollary 8.

THEOREM 9. *An L.H.D-product $E \underset{\phi}{\times} \Gamma$ [an R.H.D-product $\Gamma \underset{\psi}{\times} F$] is isomorphic to the spined product of a left regular band and a weakly C-inversive semigroup [a weakly C-inversive semigroup and a right regular band] if and only if it satisfies the following* (3.1):

(3.1) $\gamma \in \Gamma$, $\lambda \in \Lambda$, $\gamma\gamma^{-1}=\mu \geqq \lambda$, $i \in E_\mu$ *imply*

$\rho_\gamma\lambda_i=$*the identity mapping on* iE_λ, *where* $\rho_\gamma=\gamma\phi$.

[$\gamma \in \Gamma$, $\lambda \in \Lambda$, $\gamma\gamma^{-1}=\mu \geqq \lambda$, $j \in F_\mu$ *imply*

$\sigma_\gamma\nu_j=$*the identity mapping on* $F_\lambda j$, *where* $\sigma_\gamma=\gamma\psi$].

Further, in this case $E \underset{\phi}{\times} \Gamma \cong E \bowtie \Gamma\ (\Lambda)$ [$\Gamma \underset{\psi}{\times} F \cong \Gamma \bowtie F\ (\Lambda)$].

COROLLARY 10. *If $E(\Lambda)$ [$F(\Lambda)$] is a left normal band [a right normal band], then an L.H.D-product of $E(\Lambda)$ and $\Gamma(\Lambda)$ [an R.H.D-product of $\Gamma(\Lambda)$ and $F(\Lambda)$] is* *uniquely determined up to isomorphism and is isomorphic to the spined product of* $E(\Lambda)$ *and* $\Gamma(\Lambda)$ [$\Gamma(\Lambda)$ *and* $F(\Lambda)$].

EXAMPLE. Let Ω and K be the weakly C-inversive semigroup and the band given by the multiplication tables (D 1) and (D 2) respectively. The basic semilattice Π of Ω consists of two elements 0, 1, and the structure decomposition of Ω is $\Omega \sim \sum\{\Omega_\lambda: \lambda \in \Pi\}$, where $\Omega_0=\{0\}$ and $\Omega_1=\{1, \gamma\}$. On the other hand, the structure decomposition of K is $K \sim \sum\{K_\lambda: \lambda \in \Pi\}$, where $K_0=\{e, f\}$, $K_1=\{1\}$. K is clearly a right regular band. Now, $\Omega \bowtie K\,(\Pi)=\{[0, e], [0, f], [1, 1], [\gamma, 1]\}$. Let σ_0, σ_1 be the inner

Ω	0	1	γ
0	0	0	0
1	0	1	γ
γ	0	γ	1

(D 1)

K	e	f	1
e	e	f	e
f	e	f	f
1	e	f	1

(D 2)

endomorphisms on $K(\Pi)$ induced by 0, 1, and σ_γ an endomorphism on $K(\Pi)$ such that σ_γ maps 1, e, f to 1, f, e respectively. Define $\varphi: \Omega \to \mathrm{End}(K)$ by $\tau\varphi=\rho_\tau$ ($\tau=0, 1, \gamma$). Since $\{\sigma_0, \sigma_1, \sigma_\gamma\}$ satisfies (C 3) and (C 4), we can consider the R.H.D-product $\Omega \underset{\varphi}{\times} K$. Of course, $\Omega \underset{\varphi}{\times} K=\{(0, e), (0, f), (1, 1), (\gamma, 1)\}$. However, $\Omega \underset{\varphi}{\times} K$ is not strictly inversive. In fact: $(1, 1)>(0, e)$ and $(\gamma, 1)(\gamma, 1)^{-1}=(1, 1)$, but $(0, e)(\gamma, 1)=(\gamma, e^{\sigma_\gamma})=(0, f)\neq(0, e)=(0, 1^{\sigma_0}e)=(\gamma, 1)(0, e)$. Hence, $\Omega \underset{\varphi}{\times} K \not\cong \Omega \bowtie K\,(\Pi)$. Thus, we can say that an R.H.D-product of a weakly C-inversive semigroup C and a right regular band R is not necessarily isomorphic to the spined product of C and R.

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