Acta Math. Hungar., **138 (1-2)** (2013), 156–172.
DOI: 10.1007/s10474-012-0259-7
First published online September 5, 2012

A NECESSARY AND SUFFICIENT CONDITION FOR THE GLOBAL ASYMPTOTIC STABILITY OF DAMPED HALF-LINEAR OSCILLATORS

S. HATA[1] and J. SUGIE[2*]

[1]Department of Mathematics and Computer Science, Shimane University,
Matsue 690-8504, Japan
e-mail: hata@math.shimane-u.ac.jp

[2]Department of Mathematics and Computer Science, Shimane University,
Matsue 690-8504, Japan
e-mail: jsugie@riko.shimane-u.ac.jp

Abstract. This paper is concerned with the global asymptotic stability of the equilibrium of the half-linear differential equation with a damped term,

$$\big(\phi_p(x')\big)' + h(t)\phi_p(x') + (p-1)\phi_p(x) = 0.$$

A necessary and sufficient growth condition on $h(t)$ is obtained by using the generalized Prüfer transformation and the generalized Riccati transformation. Equivalent planar systems to the above-mentioned equation are also considered in the proof of our main result.

1. Introduction

We consider the second-order differential equation

$$(HL) \qquad \big(\phi_p(x')\big)' + h(t)\phi_p(x') + (p-1)\phi_p(x) = 0,$$

where the prime denotes d/dt, the function $\phi_p(z)$ is defined by

$$\phi_p(z) = |z|^{p-2}z, \qquad z \in \mathbb{R}$$

*Corresponding author. Supported in part by Grant-in-Aid for Scientific Research, No. 22540190 from the Japan Society for the Promotion of Science.

Keywords and phrases: Global asymptotic stability, half-linear differential equations, damped oscillator, growth condition, Prüfer transformation, Riccati transformation.

2010 Mathematics Subject Classification: Primary 34D23, 34D45; Secondary 34C15, 37C70.

with $p > 1$, and the damping coefficient $h(t)$ is continuous and nonnegative for $t \geq 0$. The only equilibrium of (HL) is the origin $(x, x') = (0, 0)$.

Equation (HL) includes the damped linear oscillator

$$(L) \qquad x'' + h(t)x' + x = 0$$

as a special case in which $p = 2$. By the demand from a theoretical aspect and a practical aspect, there are a lot of reports concerning equation (L) or its general type. For example, see [1, 3, 6, 11, 12, 15, 16, 18, 20, 22]. As is well known, the solution space of (L) is additive and homogeneous; namely, (i) the sum of two solutions is a solution and (ii) the product of a solution and any constant is also a solution. Since the solution space of (HL) has only the characteristic (ii) mentioned above, equation (HL) is often called *half-linear*. In particular, we will call equation (HL) the *damped half-linear oscillator* because it is a generalization of the damped linear oscillator (L).

Let $\mathbf{x}(t) = (x(t), x'(t))$ and $\mathbf{x}_0 \in \mathbb{R}^2$, and let $\|\cdot\|$ be any suitable norm. We denote the solution of (HL) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. The global existence and uniqueness of solutions of (HL) are guaranteed for the initial value problem. For details, see Došlý [4, p. 170] or Došlý and Řehák [5, pp. 8–10].

The equilibrium of (HL) is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The equilibrium is *uniformly stable* if it is stable and δ can be chosen independent of t_0. The equilibrium is said to be *attractive* if, for any $t_0 \geq 0$, there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \to 0$ as $t \to \infty$. The equilibrium is *uniformly attractive* if δ_0 in the definition of attractivity can be chosen independent of t_0, and for every $\eta > 0$ there is a $T(\eta) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta_0$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for $t \geq t_0 + T(\eta)$. The equilibrium is said to be *globally attractive* if, for any $t_0 \geq 0$, any $\eta > 0$ and any $\mathbf{x}_0 \in \mathbb{R}^2$, there is a $T(t_0, \eta, \mathbf{x}_0) > 0$ such that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T(t_0, \eta, \mathbf{x}_0)$. The equilibrium is said to be *uniformly globally attractive* if, for any $\rho > 0$ and any $\eta > 0$, there is a $T(\rho, \eta) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \rho$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T(\rho, \eta)$. The equilibrium is *globally asymptotically stable* if it is stable and globally attractive. The equilibrium is *uniformly globally asymptotically stable* if it is uniformly stable and is uniformly globally attractive. With respect to the various definitions of stability and boundedness, the reader may refer to the books [2, 8, 9, 17, 21, 27] for example.

The purpose of this paper is to establish a criterion for judging whether the equilibrium of (HL) is globally asymptotically stable or not.

Very recently, Onitsuka and Sugie [19, Theorem 3.1] have considered a system of differential equations of the form

$$\begin{aligned} x' &= -\,a(t)x + b(t)\phi_{p^*}(y), \\ y' &= -\,(p-1)c(t)\phi_p(x) - (p-1)d(t)y, \end{aligned}$$

where p^* is the conjugate number of p; namely,

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

and presented some sufficient conditions for the uniform global asymptotic stability. As a new variable, let y be $\phi_p(x')$. Note that ϕ_{p^*} is the inverse function of ϕ_p. Then, equation (HL) becomes the above system with $a(t) = 0$, $b(t) = c(t) = 1$ and $d(t) = h(t)/(p-1)$. By means of their result, we see that if $h(t)$ is bounded for $t \geq 0$ and

$$\liminf_{t\to\infty} \int_t^{t+\gamma} h(s)ds > 0 \tag{1}$$

holds for every $\gamma > 0$, then the equilibrium of (HL) is uniformly globally asymptotically stable. Condition (1) is a restriction that is considerably stronger than

$$H(t) \stackrel{\text{def}}{=} \int_0^t h(s)ds \to \infty \quad \text{as} \ \ t \to \infty.$$

If $h(t)$ satisfies condition (1), then it is said to be *integrally positive*. For example, the function $\sin^2 t$ is integrally positive. About the integral positivity, see [10, 23–26].

Needless to say, the concept of the global asymptotic stability is stronger than that of the uniform global asymptotic stability. Therefore, to seek a good criterion for judging whether the equilibrium of (HL) is globally asymptotically stable, we have to discuss under the assumption that is weaker than condition (1).

So far as the special case in which $p = 2$ is concerned, Hatvani and Totik [14, Theorem 3.1] have already proved that if there exists a γ_0 with $0 < \gamma_0 < \pi$ such that

$$\liminf_{t\to\infty} \int_t^{t+\gamma_0} h(s)ds > 0, \tag{2}$$

then the equilibrium of (L) is asymptotically stable if and only if

$$\int_0^\infty \frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\, dt = \infty. \tag{3}$$

Besides, they pointed out that $0 < \gamma_0 < \pi$ was the best requirement.

Condition (2) is weaker than condition (1). For example, let

$$h(t) = \max\{0, a + \sin t\}.$$

Then, condition (1) never holds unless $a \geq 1$. On the other hand, condition (2) is satisfied as long as $a > 0$. A strong point of condition (2) is that the set $\{t \geq 0 : h(t) = 0\}$ is permitted to be the union of infinitely many disjoint intervals whose length are less than π. The criterion (3) is the so-called growth condition on $h(t)$. It has first been given by Smith [22, Theorems 1 and 2]. He showed that if there exists an $\underline{h} > 0$ such that $h(t) \geq \underline{h}$ for $t \geq 0$, then condition (3) is necessary and sufficient for the equilibrium of (L) to be asymptotically stable. The above-mentioned result of Hatvani and Totik is its natural generalization. It is known that

(i) if $h(t)$ is bounded for $t \geq 0$ or $h(t) = t$, then condition (3) is satisfied;

(ii) if $h(t) = t^2$, then condition (3) is not satisfied.

(for details, see [12]).

In this paper, we intend to extend their result to be able to apply to the damped half-linear oscillator (HL). To this end, we define ω_p by

$$\omega_p = \begin{cases} \sup\{(\tan_p \theta)^{p-2}(\pi_p - 2\theta) \colon 0 < \theta < \pi_p/2\} & \text{if } p \ge 2, \\ \sup\{(\tan_p(\pi_p/2 - \theta))^{p-2}(\pi_p - 2\theta) \colon 0 < \theta < \pi_p/2\} & \text{if } 1 < p < 2 \end{cases}$$

(the constant π_p and the function $\tan_p \theta$ are explained in Section 2). Then, we can state our main theorem as follows:

Theorem 1. *Suppose that there exists a γ_0 with $0 < \gamma_0 < \omega_p$ such that*

$$\liminf_{t\to\infty} \int_t^{t+\gamma_0} h(s)ds > 0. \tag{4}$$

Then the equilibrium of (HL) is globally asymptotically stable if and only if

$$\int_0^\infty \phi_{p^*}\left(\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)dt = \infty. \tag{5}$$

Remark 1. In the special case in which $p = 2$, it is clear that $\pi_p = \omega_p = \pi$ and $\phi_{p^*}(z) = \phi_2(z) = z$. Hence, assumption (4) becomes assumption (2) and our criterion (5) coincides with the well-known criterion (3). The function $\tan_p \theta$ is continuous and increasing for $\theta \in (-\pi/2, \pi/2)$. Hence, there exists the function $\arctan_p$ that is defined as the inverse function of $\tan_p \theta$. It turns out that ω_p is more than at least $\pi_p - 2\arctan_p 1$ for any $p > 1$.

In the proof of Hatvani and Totik's result, it is important that every solution $x(t)$ of (L) is either oscillatory or monotone for t sufficiently large. This important property is easily led from only the form of (L). Although equation (HL) has the same property as equation (L), we have to use the so-called Riccati technique to confirm this property (see Lemma 2). Riccati's technique is required also in order to show that condition (5) is necessary for the equilibrium of (HL) to be globally asymptotically stable. It is comparatively easy to transform the damped linear oscillator (L) into its polar coordinates system and to examine the asymptotic behavior of solutions of (L). In the damped half-linear oscillator (HL), we adopt the generalized Prüfer transformation mentioned in Section 2. At that time, we need to note whether an angle of each solution of (HL) becomes $n\pi_p$ with $n \in \mathbb{Z}$.

Theorem 1 tells us that Hatvani and Totik's ideas is also available for nonlinear differential equations such as (HL) by combining with other techniques and that the Smith-type growth condition (5) becomes a criterion for the equilibrium of (HL) to be globally asymptotically stable.

From Theorem 1, we can conclude that the growth condition (3) in Smith's result and Hatvani and Totik's result is not caused by the special characteristics of the linear solution space, namely, addition and homogeneity. For this reason, there may be a possibility that a criterion such as the growth condition (5) can be found for more general nonlinear differential equations than (HL).

2. Preliminaries

We introduce the *generalized sine function* $\sin_p \theta$, the *generalized cosine function* $\cos_p \theta$ and the *generalized circular constant* π_p according to the book of Došlý and Řehák [5, pp. 4–5]. Let $S(t)$ be the solution of an elementary half-linear differential equation without the damping term,

$$(\phi_p(x'))' + (p-1)\phi_p(x) = 0$$

satisfying the initial condition $(S(0), S'(0)) = (0, 1)$. Then, it is easy to verify that the generalized Pythagorean identity

$$|S(t)|^p + |S'(t)|^p \equiv 1$$

holds and the function $S(t)$ is positive and increasing on $[0, \pi_p/2]$ with $S(\pi_p/2) = 1$ and $S'(\pi_p/2) = 0$, where

$$\pi_p = \int_0^1 \frac{2}{(1-s^p)^{1/p}}\, ds = \frac{2\pi}{p \sin(\pi/p)}.$$

Combining the arc of $S(t)$ from 0 to $\pi_p/2$ as a basic pattern, we define the generalized sine function $\sin_p \theta$ as follows:

$$\sin_p \theta = \begin{cases} S(\theta) & \text{if } \ 0 \le \theta \le \pi_p/2 \\ S(\pi_p - \theta) & \text{if } \ \pi_p/2 < \theta \le \pi_p \end{cases}$$

and

$$\sin_p \theta = \begin{cases} -\sin_p(\theta - \pi_p) & \text{if } \ \pi_p \le \theta < 2\pi_p \\ \sin_p(\theta - 2n\pi_p) & \text{if } \ 2n\pi_p \le \theta < 2(n+1)\pi_p \end{cases}$$

for $n \in \mathbb{Z}$. The generalized cosine function $\cos_p \theta$ is defined as

$$\cos_p \theta = \frac{d}{d\theta} \sin_p \theta.$$

Then, from the above Pythagorean identity it follows that

$$|\sin_p \theta|^p + |\cos_p \theta|^p = 1 \quad \text{for } \ \theta \in \mathbb{R}.$$

We define the *generalized tangent function* $\tan_p \theta$ by

$$\tan_p \theta = \frac{\sin_p \theta}{\cos_p \theta}.$$

About generalized trigonometric functions, see also [4, 7, 13].

Before we advance to the proof of Theorem 1, it is very useful to change equation (HL) into planar equivalent systems. Because ϕ_{p^*} is the inverse function of ϕ_p as mentioned in Section 1, by putting $y = \phi_p(x')$ as a new variable, equation (HL) becomes the half-linear system

$$(HS) \qquad \begin{aligned} x' &= \phi_{p^*}(y), \\ y' &= -(p-1)\phi_p(x) - h(t)y. \end{aligned}$$

Let

$$x = r\cos_p\theta \quad \text{and} \quad y = \phi_p(r\sin_p\theta)$$

(this is slightly different from the generalized Prüfer transformation given in [4, pp. 169–170] and [5, p. 7]). Then, by a straightforward calculation, we can transform system (HS) into the system

$$(6) \qquad \begin{aligned} r' &= -\frac{h(t)}{p-1}r|\sin_p\theta|^p, \\ |\sin_p\theta|^{p-2}\theta' &= -|\cos_p\theta|^{p-2} - \frac{h(t)}{p-1}\phi_p(\sin_p\theta)\phi_p(\cos_p\theta). \end{aligned}$$

In particular, when $\theta \neq n\pi_p$ with $n \in \mathbb{Z}$, system (6) becomes the generalized polar coordinates system

$$(HP) \qquad \begin{aligned} r' &= -\frac{h(t)}{p-1}r|\sin_p\theta|^p, \\ \theta' &= -\frac{|\cos_p\theta|^{p-2}}{|\sin_p\theta|^{p-2}} - \frac{h(t)}{p-1}\sin_p\theta\phi_p(\cos_p\theta). \end{aligned}$$

3. Fundamental characteristic of solutions

As is customary, we say that a nontrivial solution $x(t)$ of (HL) is *oscillatory*, if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$. Otherwise, we say that it is *nonoscillatory*. Sturm's separation theorem holds for equation (HL) as well as for equation (L). Equation (HL) can be classified as oscillatory or nonoscillatory according to whether all nontrivial solutions of (HL) have or do not have a sequence of zeros tending to ∞.

LEMMA 2. *Every nonoscillatory solution $x(t)$ of (HL) is monotone for t sufficiently large.*

PROOF. Since $x(t)$ is nonoscillatory, there exists a $T \geq 0$ such that $x(t) \neq 0$ for $t \geq T$. We may assume without loss of generality that $x(t) > 0$ for $t \geq T$. Let $w(t) = \phi_p(x'(t)/x(t))$ for $t \geq T$. Since

$$\frac{d}{dz}\phi_p(z) = (p-1)|z|^{p-2}, \qquad z \in \mathbb{R},$$

we have

$$w'(t) = (p-1)\left|\frac{x'(t)}{x(t)}\right|^{p-2}\frac{x''(t)x(t)-(x'(t))^2}{x^2(t)}$$
$$= (p-1)\left|\frac{x'(t)}{x(t)}\right|^{p-2}\frac{x''(t)}{x(t)} - (p-1)\left|\frac{x'(t)}{x(t)}\right|^{p}.$$

Taking into account that $p = p^*(p-1)$ and

$$(p-1)|x'(t)|^{p-2}x''(t) + h(t)|x'(t)|^{p-2}x'(t) + (p-1)|x(t)|^{p-2}x(t) = 0$$

(this is an alternative expression of (HL)), we obtain the generalized Riccati equation

$$w'(t) + p - 1 + h(t)w(t) + (p-1)|w(t)|^{p^*} = 0 \quad \text{for} \ \ t \geq T. \tag{7}$$

Suppose that there exists a $T_1 \geq T$ such that $w(t) > 0$ for $t \geq T_1$. Then, $w'(t) < -(p-1)$ for $t \geq T_1$. Integrating this inequality from T_1 to t, we obtain

$$w(t) < w(T_1) - (p-1)(t - T_1),$$

which tends to $-\infty$ as $t \to \infty$. This is a contradiction. Thus, we see that $w(t_*) = 0$ for some $t_* \geq T$.

From (7) it turns out that $w'(t_*) = -(p-1) < 0$, and hence, we can choose an $\varepsilon_0 > 0$ so that $w(t) < 0$ for $t_* < t \leq t_* + \varepsilon_0$. Suppose that there exists a $t^* > t_* + \varepsilon_0$ such that $w(t^*) > 0$. Let $\hat{t}$ be the supremum of all $t \in [t_* + \varepsilon_0, t^*]$ for which $w(t) \leq 0$. Then, $w(\hat{t}) = 0$ and

$$w(t) > 0 \quad \text{for} \ \ \hat{t} < t \leq t^*. \tag{8}$$

Using (7) again, we see that $w'(\hat{t}) = -(p-1) < 0$. Hence, $w(t) < 0$ on the right-neighborhood of $\hat{t}$. This contradicts (8). Thus, we get

$$\phi_p\left(\frac{x'(t)}{x(t)}\right) = w(t) \leq 0 \quad \text{for} \ \ t \geq t_*.$$

From the positivity of $x(t)$ it follows that $x'(t) \leq 0$ for $t \geq t_*$; namely, $x(t)$ is monotone for a sufficiently large t. □

As defined in Section 1, the concept of the global asymptotic stability is the union of the concept of the stability and the concept of the global attractivity. We can show the stability of the equilibrium of (HL) only by assuming that $h(t) \geq 0$ for $t \geq 0$.

LEMMA 3. *The equilibrium of* (HL) *is uniformly stable.*

PROOF. Let $x(t)$ be any solution of (HL) with the initial time $t_0 \geq 0$. Define

$$v(t) = |x(t)|^p + |\phi_p(x'(t))|^{p^*}.$$

Recall that ϕ_{p^*} is the inverse function of ϕ_p. Taking into account that

$$\frac{d}{dz}|z|^q = q\phi_q(z), \qquad z \in \mathbb{R}$$

with $q = p$ or p^*, we obtain

$$\begin{aligned} v'(t) &= p\,\phi_p(x(t))x'(t) + p^*\phi_{p^*}\big(\phi_p(x'(t))\big)\big(\phi_p(x'(t))\big)' \\ &= p\,\phi_p(x(t))x'(t) - p^*x'(t)\big(h(t)\phi_p(x'(t)) + (p-1)\phi_p(x(t))\big) \\ &= -\,p^*h(t)\phi_p(x'(t))x'(t) = -\,p^*h(t)|x'(t)|^p \le 0 \end{aligned}$$

for $t \ge t_0$. Hence, it follows that

$$v(t) \le v(t_0) \quad \text{for } \; t \ge t_0. \tag{9}$$

Since $|\phi_p(x'(t))|^{p^*} = |x'(t)|^{p^*(p-1)} = |x'(t)|^p$, we can rewrite $v(t)$ as

$$v(t) = |x(t)|^p + |x'(t)|^p.$$

Let $\|\cdot\|$ be the p-norm $\|\cdot\|_p$. Then, from (9) it is easy to confirm that for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \ge 0$ and $\|\mathbf{x}_0\| < \delta$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \ge t_0$; that is, the equilibrium of (HL) is uniformly stable. □

Remark 2. By virtue of Lemma 3, to prove our main theorem, we have only to show that the growth condition (5) is necessary and sufficient for the equilibrium of (HL) to be globally attractive under the assumption (4).

4. Proof of Theorem 1

We first prove that if the equilibrium of (HL) is globally attractive, then the growth condition (5) holds. We then prove the converse.

Necessity. Suppose that (5) does not hold. Then, we can choose $T \ge 0$ so large that

$$\int_T^\infty \phi_{p^*}\left(\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)dt < \frac{1}{2(p-1)^{p^*-1}}. \tag{10}$$

Consider the solution $x(t)$ of (HL) satisfying the initial condition $(x(T), x'(T)) = (1, 0)$. We will show that $x(t) > 1/2$ for all $t \ge T$.

By way of contradiction, we suppose that there exists a $T_1 > T$ such that $x(T_1) = 1/2$ and $x(t) > 1/2$ for $T \le t < T_1$. Let $w(t) = \phi_p(x'(t)/x(t))$ for $T \le t \le T_1$. Then, $w(T) = 0$. As shown in the proof of Lemma 2, we see that $w(t)$ satisfies the generalized Riccati equation (7). Taking into account that $w(T) = 0$ and following the same process as in the proof of Lemma 2, we conclude that $x(t)$ is nonincreasing, and hence,

$$\frac{1}{2} = x(T_1) \le x(t) \le x(T) = 1 \quad \text{for } \; T \le t \le T_1. \tag{11}$$

From (11) it turns out that

$$\big(\phi_p(x'(t))\big)' + h(t)\phi_p(x'(t)) = -(p-1)\phi_p(x(t)) \geq -(p-1)$$

for $T \leq t \leq T_1$. Hence,

$$\big(\phi_p(x'(t))e^{H(t)}\big)' \geq -(p-1)e^{H(t)} \quad \text{for } \; T \leq t \leq T_1.$$

Integrating both sides of this inequality from T to t, we get

$$\phi_p(x'(t))e^{H(t)} \geq \phi_p(x'(T))e^{H(T)} - (p-1)\int_T^t e^{H(s)}ds = -(p-1)\int_T^t e^{H(s)}ds$$

for $T \leq t \leq T_1$. Since ϕ_{p^*} is the inverse function of ϕ_p, we see that

$$x'(t) \geq -\phi_{p^*}\left(\frac{(p-1)\int_T^t e^{H(s)}ds}{e^{H(t)}}\right) \quad \text{for } \; T \leq t \leq T_1.$$

Integrate both sides of this inequality from T to T_1 to obtain

$$\begin{aligned} x(T_1) &\geq x(T) - \phi_{p^*}(p-1)\int_T^{T_1} \phi_{p^*}\left(\frac{\int_T^t e^{H(s)}ds}{e^{H(t)}}\right)dt \\ &\geq 1 - (p-1)^{p^*-1}\int_T^{\infty} \phi_{p^*}\left(\frac{\int_T^t e^{H(s)}ds}{e^{H(t)}}\right)dt \\ &\geq 1 - (p-1)^{p^*-1}\int_T^{\infty} \phi_{p^*}\left(\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)dt. \end{aligned}$$

Hence, it follows from (10) that $x(T_1) > 1/2$, which contradicts the assumption that $x(T_1) = 1/2$. Thus, the solution $x(t)$ does not tend to zero as $t \to \infty$, and hence, the equilibrium of (HL) is not globally attractive.

Sufficiency. Let $x(t)$ be any solution of (HL) with the initial time $t_0 \geq 0$ and let $(r(t), \theta(t))$ be the solution of (6) corresponding to $x(t)$. We prove that if (4) and (5) hold, then $(x(t), x'(t))$ tends to the origin $(0,0)$ as $t \to \infty$. For this purpose, it is enough to show that

$$r(t) = \sqrt[p]{|x(t)|^p + |x'(t)|^p} \to 0 \quad \text{as } \; t \to \infty.$$

Since

$$r'(t) = -\frac{h(t)}{p-1}r(t)|\sin_p\theta(t)|^p \leq 0 \quad \text{for } \; t \geq t_0, \tag{12}$$

$r(t)$ is nonincreasing for $t \geq t_0$ and therefore, it has the limiting value $r_0 \geq 0$. Suppose that r_0 is positive. Then, we get a contradiction as shown below.

If $x(t)$ is nonoscillatory, then by means of Lemma 2 we see that it is monotone for t sufficiently large. Hence, we can find a $T \geq t_0$ so large that

$$x(t) > 0 \quad \text{and} \quad x'(t) \leq 0 \quad \text{for } \ t \geq T,$$

or

$$x(t) < 0 \quad \text{and} \quad x'(t) \geq 0 \quad \text{for } \ t \geq T.$$

Since

$$\tan_p \theta(t) = \frac{\sin_p \theta(t)}{\cos_p \theta(t)} = \frac{x'(t)}{x(t)} \leq 0 \quad \text{for } \ t \geq T,$$

we see that $-\pi_p/2 < \theta(t) \leq 0 \pmod{2\pi_p}$ or $\pi_p/2 < \theta(t) \leq \pi_p \pmod{2\pi_p}$ for $t \geq T$. If $x'(t) = 0$ for $t \geq T$, then $x(t)$ is a nonzero constant. This is impossible, because the only equilibrium of (HL) is the origin $(0, 0)$. Hence, in either case, there exists a $T^* \geq T$ such that $\theta(t) \neq n\pi_p$ for $t \geq T^*$ with $n \in \mathbb{Z}$. Hence, $(r(t), \theta(t))$ is a solution of (HP) for $t \geq T^*$. Since $|x(t)|$ is nonincreasing for $t \geq T^*$, there exists a $c \in \mathbb{R}$ with $0 \leq c \leq r_0$ such that $|x(t)| \searrow c$ as $t \to \infty$. Hence, it follows that

$$|x'(t)|^p \to r_0^p - c^p \quad \text{as } \ t \to \infty.$$

If $c < r_0$, then we can choose a $T_2 \geq T^*$ so that

$$|x'(t)|^p > \frac{r_0^p - c^p}{2} \quad \text{for } \ t \geq T_2.$$

Hence, by (12) we have

$$\begin{aligned}(r^p(t))' = pr^{p-1}(t)r'(t) &= -\frac{p}{p-1}h(t)r^p(t)|\sin_p \theta(t)|^p \\ &= -p^*h(t)|x'(t)|^p < -\frac{p^*(r_0^p - c^p)}{2}h(t)\end{aligned}$$

for $t \geq T_2$. Integrating this inequality from T_2 to t, we obtain

$$r_0^p - r^p(T_2) \leq r^p(t) - r^p(T_2) < -\frac{p^*(r_0^p - c^p)}{2}\int_{T_2}^{t} h(s)ds,$$

which tends to $-\infty$ as $t \to \infty$. This is a contradiction. Thus, we see that $c = r_0$. We therefore conclude that

$$|x(t)| \searrow r_0 \quad \text{and} \quad |x'(t)| \to 0 \quad \text{as } \ t \to \infty.$$

There are two cases that should be considered: (i) $0 < r_0 \leq x(t) \leq x(T)$ and $x'(t) \leq 0$ for $t \geq T$ and (ii) $x(T) \leq x(t) \leq -r_0 < 0$ and $x'(t) \geq 0$ for $t \geq T$. Note that

$$\big(\phi_p(x'(t))e^{H(t)}\big)' = -(p-1)\phi_p(x(t))e^{H(t)} \quad \text{for } \ t \geq 0. \tag{13}$$

In the former case, integrate both sides of (13) from T to t to obtain

$$\phi_p(x'(t))e^{H(t)} \le \phi_p(x'(t))e^{H(t)} - \phi_p(x'(T))e^{H(T)} \le -(p-1)\phi_p(r_0)\int_T^t e^{H(s)}ds$$

for $t \ge T$. Since $\int_0^t e^{H(s)}ds \to \infty$ as $t \to \infty$, there exists a $T_3 > T$ such that

$$\int_T^t e^{H(s)}ds = \int_0^t e^{H(s)}ds - \int_0^T e^{H(s)}ds > \frac{1}{p}\int_0^t e^{H(s)}ds \quad \text{for } t \ge T_3.$$

Hence, we can estimate that

$$\phi_p(x'(t))e^{H(t)} < -\frac{1}{p^*}\phi_p(r_0)\int_0^t e^{H(s)}ds \quad \text{for } t \ge T_3;$$

namely,

$$x'(t) < -\left(\frac{1}{p^*}\right)^{p^*-1} r_0\,\phi_{p^*}\!\left(\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\right) \quad \text{for } t \ge T_3.$$

Integrating this inequality from T_3 to t, we get

$$r_0 - x(T_3) \le x(t) - x(T_3) < -\left(\frac{1}{p^*}\right)^{p^*-1} r_0\int_{T_3}^t \phi_{p^*}\!\left(\frac{\int_0^u e^{H(s)}ds}{e^{H(u)}}\right)du$$

for $t \ge T_3$. This estimation and condition (5) lead to a contradiction. In the latter case, we have

$$-r_0 - x(T_3) \ge x(t) - x(T_3) > \left(\frac{1}{p^*}\right)^{p^*-1} r_0\int_{T_3}^t \phi_{p^*}\!\left(\frac{\int_0^u e^{H(s)}ds}{e^{H(u)}}\right)du,$$

which tends to ∞ as $t \to \infty$. This is a contradiction. Thus, $x(t)$ have to oscillate.

Since $x(t) = r(t)\cos_p\theta(t)$, there exist two sequences $\{\tau_n\}$ and $\{\sigma_n\}$ with $t_0 \le \tau_n < \sigma_n < \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$ such that $\theta(\tau_n) = -\pi_p/2 \pmod{2\pi_p}$ and $\theta(\sigma_n) = \pi_p/2 \pmod{2\pi_p}$. Recall that $y(t) = \phi_p(r(t)\sin_p\theta(t))$ and $(x(t), y(t))$ is the solution of (HS) corresponding to $(r(t), \theta(t))$. From the second equation of (HP) it is clear that

$$\theta' = -\frac{1}{|\tan_p\theta|^{p-2}} - \frac{h(t)}{p-1}\sin_p\theta\,\phi_p(\cos_p\theta) < 0$$

for $0 < \theta < \pi_p/2$ and $-\pi_p < \theta < -\pi_p/2 \pmod{2\pi_p}$; namely, θ' is negative in the first quadrant and the third quadrant. This means that the solution $(x(t), y(t))$ moves clockwise in the first quadrant and the third quadrant. We divide our argument into two cases: (i) $p \ge 2$ and (ii) $1 < p < 2$.

Case (i): $p \ge 2$. From the definition of ω_p and the assumption that $0 < \gamma_0 < \omega_p$, there exists a $\theta^* \in (0, \pi_p/2)$ such that

$$\gamma_0 < (\tan_p\theta^*)^{p-2}(\pi_p - 2\theta^*) < \omega_p. \tag{14}$$

We can choose two divergent sequences $\{t_n\}$ and $\{s_n\}$ with $\tau_n < t_n < \sigma_n < s_n < \tau_{n+1}$ such that $\theta(t_n) = \pi_p - \theta^*$, $\theta(s_n) = \theta^*$ and

$$\theta^* < \theta(t) < \pi_p - \theta^* \quad \text{for } \ t_n < t < s_n.$$

Since θ' is not necessarily negative in the second quadrant and the fourth quadrant, the behavior of $(x(t), y(t))$ may be not so simple. For this reason, the point in the set $\{t \in (\tau_n, \sigma_n) \colon \theta(t) = \pi_p - \theta^*\}$ might not be only one. In such a case, we should select the supremum of all $t \in (\tau_n, \sigma_n)$ for which $\theta(t) \geq \pi_p - \theta^*$ as the point t_n.

Suppose that there exists an $N \in \mathbb{N}$ such that $s_n - t_n \geq \gamma_0$ for $n \geq N$. By (12) we have

$$r'(t) \leq -\frac{r_0(\sin_p \theta^*)^p}{p-1} h(t) \quad \text{for } \ t_n \leq t \leq s_n.$$

Needless to say, $r'(t) \leq 0$ otherwise. Hence, we can estimate that

$$r(s_n) - r(t_n) \leq -\frac{r_0(\sin_p \theta^*)^p}{p-1} \int_{t_n}^{s_n} h(t)dt \leq -\frac{r_0(\sin_p \theta^*)^p}{p-1} \int_{t_n}^{t_n+\gamma_0} h(t)dt$$

for $n \geq N$ and $r(t_{n+1}) - r(s_n) \leq 0$ for $n \in \mathbb{N}$. Adding these two evaluations, we obtain

$$r(t_{n+1}) - r(t_n) \leq -\frac{r_0(\sin_p \theta^*)^p}{p-1} \int_{t_n}^{t_n+\gamma_0} h(t)dt \quad \text{for } \ n \geq N.$$

This inequality yields that

$$r_0 - r(t_N) \leq r(t_{n+1}) - r(t_N) \leq -\frac{r_0(\sin_p \theta^*)^p}{p-1} \sum_{i=N}^{n} \int_{t_i}^{t_i+\gamma_0} h(t)dt.$$

Since t_n tends to ∞ as $n \to \infty$, this is a contradiction. Thus, there exists a sequence $\{n_k\}$ with $n_k \in \mathbb{N}$ and $n_k \to \infty$ as $k \to \infty$ such that

$$s_{n_k} - t_{n_k} < \gamma_0. \tag{15}$$

Since $|\sin_p \theta| \leq 1$ and $|\cos_p \theta| \leq 1$ for $\theta \in \mathbb{R}$, we see that

$$\theta'(t) = -\frac{1}{|\tan_p \theta(t)|^{p-2}} - \frac{h(t)}{p-1} \sin_p \theta(t) \phi_p(\cos_p \theta(t)) \geq -\frac{1}{(\tan_p \theta^*)^{p-2}} - \frac{h(t)}{p-1}$$

for $t_{n_k} \leq t \leq s_{n_k}$. It follows from (15) that

$$\begin{aligned} \theta^* - (\pi_p - \theta^*) &= \theta(s_{n_k}) - \theta(t_{n_k}) \\ &\geq -\frac{s_{n_k} - t_{n_k}}{(\tan_p \theta^*)^{p-2}} - \frac{1}{p-1} \int_{t_{n_k}}^{s_{n_k}} h(t)dt \\ &> -\frac{\gamma_0}{(\tan_p \theta^*)^{p-2}} - \frac{1}{p-1} \int_{t_{n_k}}^{s_{n_k}} h(t)dt \end{aligned}$$

for each $k \in \mathbb{N}$. Hence, from (14) it turns out that

$$\frac{1}{p-1}\int_{t_{n_k}}^{s_{n_k}} h(t)dt > \pi_p - 2\theta^* - \frac{\gamma_0}{(\tan_p \theta^*)^{p-2}} > 0 \quad \text{for } k \in \mathbb{N}.$$

Using (12) and this estimation, we obtain

$$\log \frac{r(s_{n_k})}{r(t_{n_k})} = \int_{t_{n_k}}^{s_{n_k}} \frac{r'(t)}{r(t)} dt = -\frac{1}{p-1}\int_{t_{n_k}}^{s_{n_k}} h(t)|\sin_p \theta(t)|^p dt$$
$$\le -\frac{(\sin_p \theta^*)^p}{p-1}\int_{t_{n_k}}^{s_{n_k}} h(t)dt \le -(\sin_p \theta^*)^p \left(\pi_p - 2\theta^* - \frac{\gamma_0}{(\tan_p \theta^*)^{p-2}}\right);$$

namely,

$$\frac{r(s_{n_k})}{r(t_{n_k})} \le \exp\left\{-(\sin_p \theta^*)^p \left(\pi_p - 2\theta^* - \frac{\gamma_0}{(\tan_p \theta^*)^{p-2}}\right)\right\} < 1$$

for each $k \in \mathbb{N}$. This shows that $r(t)$ tends to zero as $t \to \infty$. Because the limiting value r_0 of $r(t)$ is positive, this is a contradiction.

Case (ii): $1 < p < 2$. From the definition of ω_p and the assumption that $0 < \gamma_0 < \omega_p$, there exists a $\theta_* \in (0, \pi_p/2)$ such that

$$\gamma_0 < \left(\tan_p\left(\frac{\pi_p}{2} - \theta_*\right)\right)^{p-2}(\pi_p - 2\theta_*) < \omega_p.$$

Let $\varepsilon > 0$ be so small that

$$\gamma_0 < \left(\tan_p\left(\frac{\pi_p}{2} - \theta_*\right)\right)^{p-2}(\pi_p - 2\theta_* - 2\varepsilon) < \omega_p. \tag{16}$$

Since the solution $(x(t), y(t))$ of (HS) rotates around the origin $(0,0)$ infinite many times, we can choose four divergent sequences $\{\tilde{t}_n\}$, $\{t_n\}$, $\{s_n\}$ and $\{\tilde{s}_n\}$ with $\tilde{t}_n < t_n < s_n < \tilde{s}_n$ such that $\theta(\tilde{t}_n) = \pi_p - \varepsilon$, $\theta(t_n) = \pi_p/2 + \theta_*$, $\theta(s_n) = \pi_p/2 - \theta_*$, $\theta(\tilde{s}_n) = \varepsilon$,

$$\frac{\pi_p}{2} + \theta_* < \theta(t) < \pi_p - \varepsilon \quad \text{for } \tilde{t}_n < t < t_n,$$

and

$$\varepsilon < \theta(t) < \frac{\pi_p}{2} - \theta_* \quad \text{for } s_n < t < \tilde{s}_n.$$

Here, we select the supremum of all $t \in (\tau_n, \sigma_n)$ for which $\theta(t) \ge \pi_p - \varepsilon$ as the point $\tilde{t}_n$, and the infimum of all $t \in (\tilde{t}_n, \sigma_n)$ for which $\theta(t) \le \pi_p/2 + \theta_*$ as the point t_n. Hence, it naturally follows that

$$\varepsilon < \theta(t) < \pi_p - \varepsilon \quad \text{for } \tilde{t}_n < t < \tilde{s}_n.$$

As in the proof of case (i), we can find a sequence $\{n_k\}$ with $n_k \in \mathbb{N}$ and $n_k \to \infty$ as $k \to \infty$ such that

$$\tilde{s}_{n_k} - \tilde{t}_{n_k} < \gamma_0. \tag{17}$$

Since $|\sin_p \theta| \le 1$ and $|\cos_p \theta| \le 1$ for $\theta \in \mathbb{R}$, we see that

$$\begin{aligned}\theta'(t) &= -\frac{1}{|\tan_p \theta(t)|^{p-2}} - \frac{h(t)}{p-1}\sin_p \theta(t)\phi_p(\cos_p \theta(t))\\ &\ge -\frac{1}{(\tan_p(\pi_p/2-\theta_*))^{p-2}} - \frac{h(t)}{p-1}\end{aligned}$$

for $\tilde{t}_{n_k} \le t \le t_{n_k}$ and $s_{n_k} \le t \le \tilde{s}_{n_k}$. Integrating this inequality, we obtain

$$\begin{aligned}\frac{\pi_p}{2} + \theta_* - (\pi_p - \varepsilon) &= \theta(t_{n_k}) - \theta(\tilde{t}_{n_k})\\ &\ge -\frac{t_{n_k} - \tilde{t}_{n_k}}{(\tan_p(\pi_p/2-\theta_*))^{p-2}} - \frac{1}{p-1}\int_{\tilde{t}_{n_k}}^{t_{n_k}} h(t)dt\end{aligned}$$

and

$$\begin{aligned}\varepsilon - (\pi_p/2 - \theta_*) &= \theta(\tilde{s}_{n_k}) - \theta(s_{n_k})\\ &\ge -\frac{\tilde{s}_{n_k} - s_{n_k}}{(\tan_p(\pi_p/2-\theta_*))^{p-2}} - \frac{1}{p-1}\int_{s_{n_k}}^{\tilde{s}_{n_k}} h(t)dt\end{aligned}$$

for each $k \in \mathbb{N}$. Hence, it follows from (16) and (17) that

$$\frac{1}{p-1}\int_{\tilde{t}_{n_k}}^{\tilde{s}_{n_k}} h(t)dt > \pi_p - 2\theta_* - 2\varepsilon - \frac{\gamma_0}{(\tan_p(\pi_p/2-\theta_*))^{p-2}} > 0 \quad \text{for } k \in \mathbb{N}.$$

Using (12) and this estimation, we obtain

$$\begin{aligned}\log\frac{r(\tilde{s}_{n_k})}{r(\tilde{t}_{n_k})} &= \int_{\tilde{t}_{n_k}}^{\tilde{s}_{n_k}} \frac{r'(t)}{r(t)}dt\\ &= -\frac{1}{p-1}\int_{\tilde{t}_{n_k}}^{\tilde{s}_{n_k}} h(t)|\sin_p \theta(t)|^p dt \le -\frac{(\sin_p \varepsilon)^p}{p-1}\int_{\tilde{t}_{n_k}}^{\tilde{s}_{n_k}} h(t)dt\\ &\le -(\sin_p \varepsilon)^p\left(\pi_p - 2\theta_* - 2\varepsilon - \frac{\gamma_0}{(\tan_p(\pi_p/2-\theta_*))^{p-2}}\right);\end{aligned}$$

namely,

$$\frac{r(\tilde{s}_{n_k})}{r(\tilde{t}_{n_k})} \le \exp\left\{-(\sin_p \varepsilon)^p\left(\pi_p - 2\theta^* - 2\varepsilon - \frac{\gamma_0}{(\tan_p(\pi_p/2-\theta_*))^{p-2}}\right)\right\} < 1$$

for each $k \in \mathbb{N}$. This shows that $r(t)$ tends to zero as $t \to \infty$. Because the limiting value r_0 of $r(t)$ is positive, this is a contradiction.

The proof of Theorem 1 is thus complete. □

References

[1] Z. Artstein and E. F. Infante, On the asymptotic stability of oscillators with unbounded damping, *Quart. Appl. Math.*, **34** (1976/77), 195–199.

[2] A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory*, 2nd ed. Springer-Verlag (Berlin, Heidelberg and New York, 2005).

[3] R. J. Ballieu and K. Peiffer, Attractivity of the origin for the equation $\ddot{x}+f(t,x,\dot{x})|\dot{x}|^{\alpha}\dot{x}$ $+g(x)=0$, *J. Math. Anal. Appl.*, **65** (1978), 321–332.

[4] O. Došlý, Half-linear differential equations, in *Handbook of Differential Equations: Ordinary Differential Equations I* (eds. A. Cañada, P. Drábek and A. Fonda), Elsevier (Amsterdam, 2004), pp. 161–357.

[5] O. Došlý and P. Řehák, *Half-linear Differential Equations*, North-Holland Mathematics Studies **202**, Elsevier (Amsterdam, 2005).

[6] L. H. Duc, A. Ilchmann, S. Siegmund and P. Taraba, On stability of linear time-varying second-order differential equations, *Quart. Appl. Math.*, **64** (2006), 137–151.

[7] Á. Elbert, A half-linear second order differential equation, in *Qualitative Theory of Differential Equations I* (ed. M. Farkas), *Colloq. Math. Soc.* János Bolyai, **30**, North-Holland (Amsterdam and New York, 1981), pp. 153–180.

[8] J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience (New York, London and Sydney, 1969); (revised) Krieger (Malabar, 1980).

[9] A. Halanay, *Differential Equations*: *Stability*, *Oscillations*, *Time Lags*, Academic Press (New York and London, 1966).

[10] L. Hatvani, On the asymptotic stability for a two-dimensional linear nonautonomous differential system, *Nonlinear Anal.*, **25** (1995), 991–1002.

[11] L. Hatvani, Integral conditions on the asymptotic stability for the damped linear oscillator with small damping, *Proc. Amer. Math. Soc.*, **124** (1996), 415–422.

[12] L. Hatvani, T. Krisztin and V. Totik, A necessary and sufficient condition for the asymptotic stability of the damped oscillator, *J. Differential Equations*, **119** (1995), 209–223.

[13] L. Hatvani and L. Székely, Asymptotic stability of two dimensional systems of linear difference equations and of second order half-linear differential equations with step function coefficients, *E. J. Qualitative Theory of Diff. Equ.*, **38** (2011), 1–17.

[14] L. Hatvani and V. Totik, Asymptotic stability of the equilibrium of the damped oscillator, *Diff. Integral Eqns.*, **6** (1993), 835–848.

[15] A. O. Ignatyev, Stability of a linear oscillator with variable parameters, *Electron. J. Differential Equations*, **1997** (1997), No. 17, pp. 1–6.

[16] J. J. Levin and J. A. Nohel, Global asymptotic stability for nonlinear systems of differential equations and applications to reactor dynamics, *Arch. Rational Mech. Anal.*, **5** (1960), 194–211.

[17] A. N. Michel, L. Hou and D. Liu, *Stability Dynamical Systems*: *Continuous*, *Discontinuous*, *and Discrete Systems*, Birkhäuser (Boston, Basel and Berlin, 2008).

[18] M. Onitsuka, Non-uniform asymptotic stability for the damped linear oscillator, *Nonlinear Anal.*, **72** (2010), 1266–1274.

[19] M. Onitsuka and J. Sugie, Uniform global asymptotic stability for half-linear differential systems with time-varying coefficients, to appear in *Proc. Roy. Soc. Edinburgh Sect. A* **141** (2011), 1083–1101.

[20] P. Pucci and J. Serrin, Precise damping conditions for global asymptotic stability for nonlinear second order systems, *Acta Math.*, **170** (1993), 275–307.

[21] N. Rouche, P. Habets and M. Laloy, *Stability Theory by Liapunov's Direct Method*, Applied Math. Sciences **22**, Springer-Verlag (New York, Heidelberg and Berlin, 1977).

[22] R. A. Smith, Asymptotic stability of $x'' + a(t)x' + x = 0$, *Quart. J. Math. Oxford* (2), **12** (1961), 123–126.

[23] J. Sugie, Convergence of solutions of time-varying linear systems with integrable forcing term, *Bull Austral. Math. Soc.*, **78** (2008), 445–462.

[24] J. Sugie, Global asymptotic stability for damped half-linear oscillators, *Nonlinear Anal.*, **74** (2011), 7151–7167.

[25] J. Sugie and S. Hata, Global asymptotic stability for half-linear differential systems with generalized almost periodic coefficients, to appear in *Monatsh. Math.*

[26] J. Sugie and M. Onitsuka, Integral conditions on the uniform asymptotic stability for two-dimensional linear systems with time-varying coefficients, *Proc. Amer. Math. Soc.*, **138** (2010), 2493–2503.

[27] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, Math. Soc. Japan (Tokyo, 1966).