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Forced Linear Harmonic Oscillator in the Interaction Representation

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Quantum mechanical treatment of the forced harmonic oscillator under the influence of an arbitrary time-dependent force is treated. The unitary time development operator, phase factor and excitation probability are calculated. The phase factor is unable to derive in other representations. These quantities depend on the types of forces and are evaluated explicitly for several cases in adiabatic approximation.

The response of a system to an external time-dependent force is treated in the formalism of quantum mechanical perturbation theory.[1] The treatment parallel to the corresponding classical mechanics is the Heisenberg representation. The more general perturbation is the interaction representation,[2] which is also the basic framework in the field of quantum field theory. This article will demonstrate the application to the forced linear harmonic oscillator.

If an external time-dependent force $F_1(t)$ is applied to a generalized harmonic-oscillator, the Hamiltonian for this system is expressed in terms of two canonical obervables p and q. We assume a Hamiltonian of the form

$$H=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2q^2-qF_1(t)\,, \tag{1}$$

where $F_1(t)$ is a real function of t. This is generalized further by introducing a velocity-dependent term:

$$H=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2q^2-qF_1(t)-pF_2(t)\,, \tag{2}$$

where $F_2(t)$ is also a real function of t. The Hermitian operators p and q satisfy the relation

$$qp-pq=i\hbar\,. \tag{3}$$

It is convenient to introduce a operators

$$\begin{aligned} a&=\sqrt{\frac{m\omega}{2\hbar}}\left(q+i\frac{p}{m\omega}\right),\\ a^+&=\sqrt{\frac{m\omega}{2\hbar}}\left(q-i\frac{p}{m\omega}\right). \end{aligned} \tag{4}$$

The commutator of a and a^+ is

$$[a, a^+]=aa^+-a^+a=1\,. \tag{5}$$

By the use of commutation relation, the Hamiltonian (2) may be written in the form

$$H=\hbar\omega\left(a^+a+\frac{1}{2}\right)+f(t)a+f^*(t)a^+\,, \tag{6}$$

provided we define the function $f(t)$ such that

$$f(t)=-\sqrt{\frac{\hbar}{2m\omega}}F_1(t)+i\sqrt{\frac{m\hbar\omega}{2}}F_2(t)\,. \tag{7}$$

The Hamiltonian is split into zero-order part

$$H_0=\hbar\omega\left(a^+a+\frac{1}{2}\right)$$

and the perturbation

$$H'=f(t)a+f^*(t)a^+\,. \tag{8}$$

The equation for the Schödinger representation is

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi=(H_0+H')\psi \tag{9}$$

We are interested in the changes produced by the time-dependent forces in an initially unperturbed linear harmonic oscillator. It is reasonable to assume the disturbance is almost limited in the finite time interval, that is, H' tends to 0 when t goes to $\pm\infty$. H_0 will be assumed time-independent. We define a state function

$$\psi'(t)=\exp\left(\frac{i}{\hbar}H_0t\right)\psi(t)\,. \tag{10}$$

At the time, $t=-\infty$, the system is in the ground state u_0 of the harmonic oscillator and

$$\psi=u_0\,. \tag{11}$$

At $t=+\infty$, the state is in different excited states, $\psi(+\infty)$ will be a linear combination of eigen states

$$u_n=\frac{1}{\sqrt{n!}}(a^+)^n u_0\,. \tag{12}$$

If ψ' isdifferentiated, and the Schrödinger equation is used, we obtain

$$i\hbar\frac{\partial\psi'}{\partial t}=H_I'\psi'\,, \tag{13}$$

where

$$H_I' = \exp\left(\frac{i}{\hbar}H_0 t\right)H' \exp\left(-\frac{i}{\hbar}H_0 t\right) = f(t)ae^{-i\omega t} + f^*(t)a^+e^{i\omega t}\,. \tag{14}$$

Eq. (13) gives

$$\psi'(+\infty) = \exp\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty} H_I'(t')dt'\right)\psi'(-\infty)\,. \tag{15}$$

Eq. (13) can be formally solved by a linear relation

$$\psi'(t) = U(t, -\infty)\psi(-\infty)\,, \tag{16}$$

and the unitary time development operator is

$$U(t, -\infty) = P\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{t} H_I'(t')\,dt'\right) = \exp\left[-i(Ka + K^*a^+)\right], \tag{17}$$

where

$$K = \int_{-\infty}^{+\infty} F(t)e^{-i\omega t}dt \tag{18}$$

and P is the Dyson time ordering operator, which is expressed by the phase factor $\exp[i\delta(t)]$. To prove this, we use the theorem

$$e^{A+B} = e^A e^B e^{C/2} \tag{19}$$

for the rearrangement of the order of pairs of operators A and B, which satisfy the commutation relation $[B, A] = C$(c-number). We find using the group property $U(t_1, t_3)U(t_3, t_2) = U(t_1, t_2)$ for $U(t + \Delta t, -\infty)$ with

$$A = -\frac{i}{\hbar}\Delta t H_I'(t) \tag{20}$$

ane

$$B = -\frac{i}{\hbar}\int_{-\infty}^{t} H_I'(t')dt' \tag{21}$$

that

$$\exp[-i\delta(t)]U(t + \Delta t, -\infty)$$
$$= \exp\left[-\frac{i}{\hbar}\Delta t H_I'(t)\right]\exp[-i\delta(t)]U(t_1 - \infty)$$

$$=\exp\left[-\frac{i}{\hbar}\Delta t H_I'(t)\right]\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{t} H_I'(t)\,dt\right)$$

$$=\exp\left(-\frac{\Delta t}{2\hbar}\int_{-\infty}^{t}[H_I'(t),\,H_I'(t')]\right)$$

$$\times\,\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{t+\Delta t} H_I'(t')\,dt'\right). \tag{22}$$

Comparison with Eq. (16) requires that

$$\Delta\delta(t)=\frac{i\Delta t}{2\hbar}\int_{-\infty}^{t}[H_I'(t),\,H_I'(t')]dt'$$

$$=\frac{i\Delta t}{2\hbar}\int_{-\infty}^{t}F_1(t)F_1(t')[q_I(t),\,q_I(t')]$$

$$=\frac{\Delta t}{2m\hbar\omega}\int_{-\infty}^{t}F_1(t)F_1(t')\sin[\omega(t-t')]\,, \tag{23}$$

where we have considered $F_1(t)$ only, and

$$H_I'(t)=-q_I F_1(t) \tag{24}$$

and

$$\begin{aligned} q_I &= \exp(iH_0t/\hbar)q\exp(-iH_0t/\hbar) \\ &= q\cos\omega t+(p/m\omega)\sin\omega t \end{aligned} \tag{25}$$

Integrating with respect to t gives

$$\delta(t)=\frac{1}{2m\hbar\omega}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}F_1(t')F_1(t'')\sin[\omega(t'-t'')]dt''\,. \tag{26}$$

Phase is a common factor for any wave function and it disappears from the transition probability. While our treatment enables to give it.
We can find

$$\delta(t)=\frac{1}{2\hbar m\omega^2}\int_{-\infty}^{t}F_1^2(t')dt' \tag{27}$$

by approximating $F_1(t')\approx F(t'')$ in the adiabatic limit, so that the integration over t'' gives $1/\omega$, and the double integral of Eq. (26) reduces to the single integral.

Next, Eq. (17) is transformed by using the theorem Eq. (19) and the commutation relation into

$$\exp[-i(Ka+K^*a^+)]=\exp(-|K|^2/2)\exp(-iK^*a^+)\exp(-iKa)\,, \tag{28}$$

and thus Eq. (15) is rewritten with $au_0=0$ and with Eq. (12) for u_n

$$\psi(+\infty)=\exp[-i(Ka+K^*a^+)]\psi'(-\infty)=\exp[-i(Ka+K^*a^+)]u_0$$
$$=\exp(-|K|^2/2)\exp(-iK^*a^+)u_0 \tag{29}$$
$$=\exp(-|K|^2/2)\sum_{n=0}^{\infty}\frac{(-iK^*)^n}{\sqrt{n!}}u_0 .$$

The probability amplitude for finding the system in one of the excited states with quantum number n is given by

$$P_n=\psi'(+\infty)\psi'(+\infty)=\exp(-|K|^2)\frac{(|K|^2)^n}{n!} \tag{30}$$

after the interaction ceased. This is a Poisson distribution and satisfy conservation of probability $\sum_{n=0}^{\infty} P_n=1$.

The transition probability depends on constant K, which is the Fourier transform of $f(t)$. We examine the form of K and phase factor $\delta(\infty)$ for several cases. K is assumed to be a real function (velocity-dependent force is neglected)

a) $F(t)=\lambda\exp(-t^2/\tau^2)$

$$K=\int_{-\infty}^{\infty}\lambda e^{-t^2/\tau^2}e^{-i\omega t}dt=\sqrt{\pi}\,\lambda\tau e^{-\omega^2\tau^2/4},$$

$$\delta(\infty)=\frac{\lambda^2}{2m\hbar\omega^2}\int_{-\infty}^{\infty}e^{-2t^2/\tau^2}dt=\frac{\sqrt{\pi}\lambda^2\tau}{2\sqrt{2}\,m\hbar\omega^2} .$$

b) $F(t)=\dfrac{\lambda}{t^2+\tau^2}$

$$K=\int_{-\infty}^{\infty}\frac{\lambda}{t^2+\tau^2}e^{-i\omega t}dt=\pi\lambda\frac{e^{-\omega\tau}}{\tau},$$

$$\delta(\infty)=\frac{\lambda^2}{2m\hbar\omega^2}\int_{-\infty}^{\infty}\frac{dt}{(t^2+\tau^2)^2}=\frac{\pi\lambda^2}{2m\hbar\omega^2\tau^3} .$$

c) $F(t)=\lambda\dfrac{\sin(t/\tau)}{t}$

$$K=\lambda\int_{-\infty}^{\infty}\frac{\sin(t/\tau)}{t}e^{-i\omega t}dt=\pi\lambda,$$

$$\delta(\infty)=\frac{\lambda^2}{2m\hbar\omega^2}\int_{-\infty}^{\infty}\frac{\sin^2(t/\tau)}{t^2}dt=\frac{\pi\lambda^2}{4m\hbar\omega^2} .$$

References

[1] P. A. M. Dirac, Proc. Roy. Soc. **A112**, 661 (1926); **A114**, 243 (1927)
[2] L. I. Schiff, *Quantum Mechanics*, 3rd ed. and other advanced texts.