Uniform global asymptotic stability for oscillators with superlinear damping

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Abstract

The present paper deals with the damped superlinear oscillator

$$x'' + h(t)\phi_q(x') + \omega^2 x = 0,$$

where $\omega > 0$ and $\phi_q(z) = |z|^{q-2}z$ with $q \geq 2$. The origin $(x,x') = (0,0)$ is the only equilibrium of this oscillator. We herein establish a sufficient condition for the equilibrium to be uniformly globally asymptotically stable. We conclude that under the assumption that the damping coefficient $h(t)$ is integrally positive, if the integral from σ to $t+\sigma$ of a particular solution of the first-order nonlinear differential equation

$$u' + h(t)\phi_q(u) + 1 = 0$$

diverges to negative infinity uniformly with respect to σ, then the equilibrium is uniformly asymptotically stable. The above-mentioned result is expressed by an implicit condition. We examine when the implicit condition is satisfied and when it is not satisfied. We also give explicit sufficient conditions which assure that the equilibrium is uniformly globally asymptotically stable. Using the obtained result, we present an example of which the equilibrium is uniformly globally asymptotically stable even if $h(t)$ is unbounded.

Key words: Uniform global asymptotic stability; Damped superlinear oscillator; Integrally positive; Uniform divergence condition; Characteristic equation
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1. Introduction

We consider the second-order differential equation

$$x'' + h(t)\phi_q(x') + \omega^2 x = 0, \tag{1.1}$$

where the prime denotes d/dt, the damping coefficient $h(t)$ is continuous and nonnegative for $t \geq 0$, the function $\phi_q(z)$ is defined by

$$\phi_q(z) = |z|^{q-2}z, \qquad z \in \mathbb{R}$$

with $q \geq 2$, and the spring constant ω is positive. It is clear that the only equilibrium of (1.1) is the origin $(x,x') = (0,0)$. The global existence and uniqueness of solutions of (1.1) are guaranteed for the initial value problem. Eq. (1.1) naturally contains the damped linear oscillator

$$x'' + h(t)x' + \omega^2 x = 0 \tag{1.2}$$

as the special case in which $q = 2$. Since $q \geq 2$, we call Eq. (1.1) a *damped superlinear oscillator*. Eq. (1.2) is one of the most famous models which describe a number of physical phenomena.

The purpose of this paper is to present sufficient conditions on the damping coefficient $h(t)$ for the equilibrium of (1.1) to be uniformly globally asymptotically stable (see Section 2 about the exact definition). In time varying differential equations such as Eq. (1.1), it is well-known that the concept of uniform global asymptotic stability greatly differs from the concept of global asymptotic stability; that is, all solutions $x(t)$ satisfy

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0.$$

It is natural that the arrival time from the initial point $(x(t_0), x'(t_0))$ to a neighborhood of the origin $(0,0)$ depends on the initial point, because the longer the distance between the initial point and the origin is, the larger the arrival time will become. In general, it depends on also the initial time t_0. To verify that the equilibrium of (1.1) is uniformly globally asymptotically stable, we have to confirm that each solution of (1.1) approaches near the equilibrium within the same time regardless of the initial time of the solution; namely, the initial time does not affect the asymptotic speed of solutions of (1.1) to the equilibrium. Detailed analysis is required for this verification. However, since we can predict the convergence speed to the equilibrium of solutions, the research on the uniform global asymptotic stability possesses high merit on the application aspects, for example, perturbation problems and control theory.

Very recently, Sugie and Onitsuka [41] have considered Eq. (1.2) and presented some sufficient conditions for the uniform asymptotic stability. To state their result, we need to introduce a family of functions as follows. The damping coefficient $h(t)$ is said to be *integrally positive* if

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n + \lambda < \sigma_n \le \tau_{n+1}$ for some $\lambda > 0$. The integral positivity was introduced by Matrosov [21] (see also [13, 14, 15, 25, 35, 39, 40]). For example, the function $\sin^2 t$ is integrally positive. It is known that $h(t)$ is integrally positive if and only if

$$\liminf_{t\to\infty} \int_t^{t+d} h(s)ds > 0$$

for every $d > 0$. Let $\{I_n\}$ be a sequence of disjoint intervals and suppose the width of I_n is larger than a positive number for all $n \in \mathbb{N}$. As can be seen from the definition above, if $h(t)$ is integrally positive, then the sum from n equals 1 to ∞ of the integral of $h(t)$ on I_n diverges to infinity even if intervals I_n and I_{n+1} gradually part as n increases. Hence, the integral positivity is considerably stronger restriction than

$$\lim_{t\to\infty} H(t) = \infty,$$

where

$$H(t) = \int_0^t h(s)ds.$$

Theorem A. *Suppose that $h(t)$ is integrally positive. If*

$$\lim_{t\to\infty} \int_\sigma^{t+\sigma} \frac{\int_\sigma^s e^{H(\tau)}d\tau}{e^{H(s)}}\,ds = \infty \quad \textit{uniformly with respect to } \sigma \ge 0, \tag{1.3}$$

then the equilibrium of (1.2) *is uniformly asymptotically stable.*

Because Eq. (1.2) is linear, the uniform asymptotic stability implies the uniform global asymptotic stability. The double integral (1.3) is the so-called growth condition on $h(t)$. The condition of this type was given for the first time by Smith [34]. He proved that under the assumption that there exists an $\underline{h} > 0$ such that $h(t) \ge \underline{h}$ for $t \ge 0$,

$$\int_0^\infty \frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\,dt = \infty \tag{1.4}$$

is a necessary and sufficient condition for the equilibrium of (1.2) to be (merely) asymptotically stable. Afterwards, Smith's result was improved by many authors by making an effort to remove the lower bound $\underline{h}$ from the assumption of $h(t)$ (for example, see [3, 15, 17, 18, 27, 28, 35, 36, 37, 38, 42]). However, all of them are researches on the asymptotic stability and none of them are researches on the uniform asymptotic stability. Theorem A is a result of developing Smith's result into the uniform asymptotic stability from the asymptotic stability.

Clearly, condition (1.3) is a restriction that is stronger than condition (1.4). It is known that condition (1.4) is satisfied with $h(t) = t$ (refer to [17]). However, the equilibrium of the damped linear oscillator

$$x'' + tx' + x = 0 \tag{1.5}$$

is not uniformly asymptotically stable, because Eq. (1.5) is equivalent to the system

$$x' = y$$
$$y' = -x - ty$$

and a fundamental matrix of the system is given by

$$X(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix},$$

where

$$x_{11}(t) = e^{-t^2/2}, \qquad x_{12}(t) = e^{-t^2/2}\int_0^t e^{s^2/2}ds,$$
$$x_{21}(t) = -t\,e^{-t^2/2}, \qquad x_{22}(t) = 1 - t\,e^{-t^2/2}\int_0^t e^{s^2/2}ds$$

(for detailed calculations, see [41]). This means that condition (1.3) is unchangeable to condition (1.4) in Theorem A.

In Eq. (1.2), the damping force is assumed to be proportional to the velocity of an object. However, this assumption is not necessarily suitable in many phenomena, for instance, a simple pendulum underwater, free rolling motion of a small fishing vessel and damping oscillation by the air resistance. As known well, in those models, the damping force is approximately proportional to the square of the velocity (for example, see [1, 4, 7, 9, 10, 19, 20, 24, 26, 29, 33, 43, 44]). In addition, physical models whose damping force is neither linear nor quadratic have been reported in many papers (for example, see [5, 8, 23, 32]). Thus, it would be reasonable to consider the damped superlinear oscillator (1.1).

Unfortunately, Theorem A cannot be applied to Eq. (1.1) directly when $q > 2$. Then, we will look at condition (1.3) from a different point of view. For this purpose, we consider the scalar linear differential equation

$$u' + h(t)u + 1 = 0. \tag{1.6}$$

By taking into consideration that the solution $u(t;\sigma)$ of (1.6) satisfying the initial condition $u(\sigma;\sigma) = 0$ is given by

$$u(t;\sigma) = -\frac{\int_\sigma^t e^{H(s)}ds}{e^{H(t)}},$$

it turns out that condition (1.3) coincides with

$$\lim_{t\to\infty}\int_\sigma^{t+\sigma} u(s;\sigma)ds = -\infty \quad \text{uniformly with respect to } \sigma \geq 0.$$

Hence, the uniform asymptotic stability for Eq. (1.2) is decided by whether the integral from σ to $t+\sigma$ of $u(t;\sigma)$ diverges to negative infinity as $t \to \infty$ uniformly with respect to σ. Because Eq. (1.6) has a close relation with the damped linear oscillator (1.1) as this fact shows, we will call Eq. (1.6) a *characteristic equation*.

We will extend Theorem A from the viewpoint of characteristic equation. What is the characteristic equation for the damped superlinear oscillator (1.1)? The following result is an answer to the question.

Theorem 1.1. *Suppose that $h(t)$ is integrally positive. Then the equilibrium of* (1.1) *is uniformly globally asymptotically stable provided that*

$$\lim_{t\to\infty}\int_{\sigma}^{t+\sigma} u(s;\sigma)ds = -\infty \quad \text{uniformly with respect to } \sigma \geq 0, \tag{1.7}$$

where $u(t;\sigma)$ is the solution of

$$u' + h(t)\phi_q(u) + 1 = 0 \tag{1.8}$$

satisfying $u(\sigma;\sigma) = 0$.

Let us call (1.7) a *uniform divergence condition*. By the way, if condition (1.3) is satisfied, does the equilibrium of (1.1) become uniformly globally asymptotically stable? We would like to answer about this question in Section 4.

This paper is constituted as follows. In Section 2, we give the proof of the main result, Theorem 1.1. In order to prove uniform global asymptotic stability of the equilibrium, considerably detailed analysis is required. We analyze the asymptotic behavior of solutions of an equivalent nonlinear system to the damped superlinear oscillator (1.1) in detail. The proof of Theorem 1.1 is composed of four parts. The last part is the core of the proof. It is advanced in four steps. The first step is classified into three cases. Before going into the proof, we describe the flow. Since the uniform divergence condition (1.7) is represented implicitly, we cannot judge whether it holds or not from only the damping coefficient $h(t)$. In Section 3, we present an easy sufficient condition which guarantees (1.7). Conversely, we also give necessary conditions for (1.7) to be satisfied, which is easy to check. The characteristic equation (1.8) plays a vital role in Theorem 1.1. In Section 4, we provide some corollaries of not using the characteristic equation (1.8). The first corollary gives an affirmative answer to the question mentioned above; namely, condition (1.3) implies condition (1.7). By virtue of Theorem 1.1 and Proposition 3.1, we see that the equilibrium of (1.1) is uniformly globally asymptotically stable in the case that the damping coefficient $h(t)$ is integrally positive and bounded. In Section 5, by using the second corollary obtained in Section 4, we give an example that the equilibrium of (1.1) is uniformly asymptotically stable even if the damping coefficient $h(t)$ is unbounded. Finally, in order to facilitate an understanding of the example, we attach two graphs concerning $h(t)$ and a phase portrait of solution curves of (1.1).

2. Proof of Theorem 1.1

Let $y = x'/\omega$. Then, the damped superlinear oscillator (1.1) becomes the nonlinear system

$$\begin{aligned} x' &= \omega y \\ y' &= -\omega x - \omega^{q-2} h(t)\phi_q(y). \end{aligned} \tag{2.1}$$

Let $t_0 \geq 0$ and $\mathbf{x}_0 = (x(t_0), y(t_0)) \in \mathbb{R}^2$. We denote the solution of (2.1) passing through a point $\mathbf{x}_0$ at a time t_0 by $\mathbf{x}(t;t_0,\mathbf{x}_0)$. The time t_0 and the point $\mathbf{x}_0$ are the so-called initial time and initial point, respectively. Here, let us give some definitions about the zero solution of (2.1) which is equivalent to the equilibrium of (1.1). The zero solution of (2.1) is said to be *uniformly stable* if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta(\varepsilon)$ imply $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The zero solution is said to be *uniformly globally attractive* if, for any $\rho > 0$ and any $\eta > 0$, there is a $T(\rho,\eta) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \rho$ imply $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T$. The solutions are said to be *uniformly bounded* if, for any $\rho > 0$, there exists a $B(\rho) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \rho$ imply $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < B$ for all $t \geq t_0$. The zero solution is *uniformly globally asymptotically stable* if it is uniformly stable and is uniformly globally attractive, and if the solutions are uniformly bounded. For example, we can refer to the books [2, 6, 11, 12, 22, 30, 31, 45] for those definitions.

In the definition of uniform global asymptotic stability, the numbers $\delta(\varepsilon)$, $T(\rho,\eta)$ and $B(\rho)$ must be independent of t_0. Therefore, for ε, ρ and η given, we have to find positive constants δ, T and B that are independent of t_0 in the proof of Theorem 1.1. This is an important point.

Before giving the full proof of Theorem 1.1, it is helpful to mention its broad outline. The proof is divided into three parts. First, we will show that

(a) the zero solution of (2.1) is uniformly stable.

To be precise, we verify that if $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta(\varepsilon) = \varepsilon$, then $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. This part is comparatively easy. We next show that the zero solution of (2.1) is uniformly globally attractive. For this purpose,

(b) we determine $T(\rho,\eta) > 0$ for an arbitrary $\eta > 0$,

and we prove that

(c) $\|\mathbf{x}(t^*;t_0,\mathbf{x}_0)\| < \delta(\eta)$ for some $t^* \in [t_0, t_0 + T]$.

Finally, we show that

(d) the solutions of (2.1) are uniformly bounded.

Let $\mathbf{x}^* = \mathbf{x}(t^*;t_0,\mathbf{x}_0)$. Then, from the conclusion of parts (a) and (c), we have

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| = \|\mathbf{x}(t;t^*,\mathbf{x}^*)\| < \eta \quad \text{for } t \geq t^*.$$

Part (c) is the core of the proof of Theorem 1.1. We prove part (c) by way of contradiction.

Proof of Theorem 1.1. Part (a): For any $\varepsilon > 0$ sufficiently small, we choose

$$\delta(\varepsilon) = \varepsilon.$$

Let $t_0 \geq 0$ and $\mathbf{x}_0 \in \mathbb{R}^2$ be given. We will show that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < \varepsilon$ for $t \geq t_0$. For convenience, we write $(x(t),y(t)) = \mathbf{x}(t;t_0,\mathbf{x}_0)$ and define

$$v(t) = \frac{x^2(t)}{2} + \frac{y^2(t)}{2} = \frac{1}{2}\|\mathbf{x}(t;t_0,\mathbf{x}_0)\|^2.$$

Then, $v'(t) = x(t)x'(t) + y(t)y'(t) = -\omega^{q-2}h(t)|y(t)|^q \leq 0$ for $t \geq t_0$. Since $v(t)$ is decreasing for $t \geq t_0$, we see that

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| = \sqrt{2v(t)} \leq \sqrt{2v(t_0)} = \|\mathbf{x}_0\| < \delta = \varepsilon$$

for $t \geq t_0$; namely, the zero solution of (2.1) is uniformly stable.

Part (b): For every $\rho > 0$ and $\eta > 0$, we decide a number $T(\rho,\eta)$ as follows so that $\|\mathbf{x}_0\| < \rho$ implies $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T$. From condition (1.7) it turns out that there exists a positive number τ_1 depending only on ρ and η such that

$$\int_\sigma^{t+\sigma} u(s;\sigma)ds \leq -\frac{\rho}{\varepsilon_0} \quad \text{for } t \geq \tau_1, \tag{2.2}$$

where

$$\varepsilon_0 = \min\left\{1, \frac{\omega^2\eta}{2}\right\}.$$

As was mentioned in Section 1, since $h(t)$ is integrally positive, the inequality

$$\liminf_{t\to\infty} \int_t^{t+d} h(s)ds > 0$$

holds for every $d > 0$. Hence, we can find an $\ell > 0$ and a $\hat{t} > 0$ such that

$$\int_t^{t+1} h(s)ds \geq \ell \quad \text{for } t \geq \hat{t}.$$

We define

$$\mu = \min\left\{\frac{3\eta^2}{4}, \frac{\varepsilon_0^2}{\omega^2}\right\} \quad \text{and} \quad \tau_2 = \hat{t} + \left[\frac{\rho^2}{2\ell\omega^{q-2}}\left(\frac{2}{\mu}\right)^{q/2}\right] + 1,$$

where $[c]$ means the greatest integer that is less than or equal to the real number c. Since ω, ℓ and $\hat{t}$ are fixed positive constants, the numbers μ and τ_2 depend only on ρ and η. Let

$$\nu = \liminf_{t\to\infty} \frac{\omega^{q-2}}{\rho^2}\left(\frac{\mu}{2}\right)^{q/2} \int_t^{t+\mu/(2\rho^2\omega)} h(s)ds.$$

Note that ν is a positive number and it also depends only on ρ and η. From the definition of ν, we can choose a positive number τ_3 depending only on ρ and η such that

$$\int_t^{t+\mu/(2\rho^2\omega)} h(s)ds \geq \frac{\rho^2\nu}{2\omega^{q-2}}\left(\frac{2}{\mu}\right)^{q/2} \quad \text{for } t \geq \tau_3. \tag{2.3}$$

Using numbers τ_1, τ_2, τ_3 and ν, we define

$$T = T(\rho,\eta) = \tau_3 + \left(\left[\frac{1}{\nu}\right]+1\right)(\tau_1+\tau_2+1).$$

Part (c): Consider a solution $\mathbf{x}(t;t_0,\mathbf{x}_0)$ of (2.1) with $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \rho$. The purpose of part (c) is to prove that there exists a $t^* \in [t_0, t_0+T]$ such that

$$\|\mathbf{x}(t^*;t_0,\mathbf{x}_0)\| < \delta(\eta) = \eta \tag{2.4}$$

for every $\eta > 0$. By way of contradiction, we suppose that $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| \geq \eta$ for $t_0 \leq t \leq t_0+T$. Then, we have

$$\frac{\eta^2}{2} \leq \frac{1}{2}\|\mathbf{x}(t;t_0,\mathbf{x}_0)\|^2 = v(t) \leq v(t_0) = \frac{1}{2}\|\mathbf{x}_0\|^2 < \frac{\rho^2}{2} \tag{2.5}$$

for $t_0 \leq t \leq t_0+T$. Let us pay attention to the behavior of $y^2(t)$, which is the square of the second component of $\mathbf{x}(t;t_0,\mathbf{x}_0)$.

Step 1: For any interval $[\alpha,\beta] \subset [t_0,t_0+T]$, if $y^2(t) \leq \mu$ for $\alpha \leq t \leq \beta$, then the time width $\beta-\alpha$ is less than τ_1+1, where μ and τ_1 are numbers given in part (b). Since $\mu \leq 3\eta^2/4$, by (2.5) we have

$$|x(t)| = \sqrt{2v(t)-y^2(t)} \geq \sqrt{\eta^2-\mu} \geq \frac{\eta}{2}$$

for $\alpha \leq t \leq \beta$. Hence, there are two possibilities: $x(t) \geq \eta/2$ for $\alpha \leq t \leq \beta$ and $x(t) \leq -\eta/2$ for $\alpha \leq t \leq \beta$. We consider only the former, because the latter is carried out in the same way. To show that the beginning sentence of this step is true, we divide our argument into three cases: (i) $0 \leq y(t) \leq \sqrt{\mu}$ for $\alpha \leq t \leq \beta$; (ii) $-\sqrt{\mu} \leq y(t) \leq 0$ for $\alpha \leq t \leq \beta$; (iii) the other case.

Case (i): We have

$$y'(t) = -\omega x(t) - \omega^{q-2}h(t)\phi_q(y(t)) \leq -\frac{\omega\eta}{2}$$

for $\alpha \leq t \leq \beta$. Since $\varepsilon_0 \leq \omega^2\eta/2$ and $\mu \leq \varepsilon_0^2/\omega^2$, we see that

$$-\frac{\omega\eta}{2} \leq -\sqrt{\mu} \leq y(\beta)-y(\alpha) = \int_\alpha^\beta y'(t)dt \leq -\frac{\omega\eta}{2}(\beta-\alpha).$$

Thus, we can conclude that $\beta-\alpha \leq 1$ in this case.

Case (ii): By way of contradiction, we show that $\beta-\alpha < \tau_1$. For this purpose, we suppose that there exists an interval $[\alpha_1,\beta_1] \subset [t_0,t_0+T]$ with $\beta_1-\alpha_1 \geq \tau_1$ such that $x(t) \geq$

$\eta/2$ and $-\sqrt{\mu} \le y(t) \le 0$ for $\alpha_1 \le t \le \beta_1$. Taking into account that $\varepsilon_0 = \min\{1, \omega^2\eta/2\}$ and $\phi_q(\varepsilon_0) \le \varepsilon_0$, we can estimate that

$$\begin{aligned}
\left(\frac{\omega y(t)}{\varepsilon_0}\right)' &= -\frac{\omega^2 x(t)}{\varepsilon_0} - \frac{\omega^{q-1} h(t)\phi_q(y(t))}{\varepsilon_0} \\
&= -\frac{\omega^2 x(t)}{\varepsilon_0} - \frac{h(t)\phi_q(\omega y(t))}{\varepsilon_0} \\
&\le -\frac{\omega^2 \eta}{2\,\varepsilon_0} - \frac{h(t)\,\phi_q(\omega y(t))}{\varepsilon_0} \\
&\le -1 - h(t)\phi_q\left(\frac{\omega y(t)}{\varepsilon_0}\right)
\end{aligned}$$

for $\alpha_1 \le t \le \beta_1$. Define $\xi(t) = \omega y(t)/\varepsilon_0$ and let $f(t,u) = -1 - h(t)\phi_q(u)$. Then,

$$\xi'(t) \le f(t, \xi(t))$$

for $\alpha_1 \le t \le \beta_1$. We compare $\xi(t)$ with the solution $u(t;\alpha_1)$ of (1.8) satisfying $u(\alpha_1;\alpha_1) = 0$. Since $\xi(\alpha_1) = \omega y(\alpha_1)/\varepsilon_0 < 0$, by a basic comparison theorem, we see that

$$\frac{\omega y(t)}{\varepsilon_0} = \xi(t) \le u(t;\alpha_1) \le 0$$

for $\alpha_1 \le t \le \beta_1$. Hence, we have

$$x'(t) \le \varepsilon_0 u(t;\alpha_1) \quad \text{for } \alpha_1 \le t \le \beta_1.$$

Integrate both sides of this inequality from α_1 to $\alpha_1 + \tau_1 \le \beta_1$ to obtain

$$x(\alpha_1 + \tau_1) - x(\alpha_1) \le \varepsilon_0 \int_{\alpha_1}^{\alpha_1+\tau_1} u(t;\alpha_1)dt.$$

From (2.2) with $\sigma = \alpha_1$ and $t = \tau_1$, it follows that

$$x(\alpha_1 + \tau_1) - x(\alpha_1) \le \varepsilon_0 \left(-\frac{\rho}{\varepsilon_0}\right) = -\rho.$$

On the other hand, by (2.5) again, we have

$$0 < \frac{\eta}{2} \le x(t) \le \rho \quad \text{for } \alpha_1 \le t \le \beta_1,$$

and therefore,

$$x(\alpha_1 + \tau_1) - x(\alpha_1) > -\rho.$$

This is a contradiction. Thus, we can conclude that $\beta_1 - \alpha_1 < \tau_1$ in this case.

Case (iii): Since $x(t) \ge \eta/2$ for $\alpha \le t \le \beta$, the solution curve of $\mathbf{x} = \mathbf{x}(t;t_0,\mathbf{x}_0)$ stays in the right-hand half-plane $\{(x,y)\colon x > 0 \text{ and } y \in \mathbb{R}\}$. Taking into consideration of the vector field of the orbit on the positive x-axis, we see that the orbit intersects the positive x-axis only once for $\alpha \le t \le \beta$. Hence, there exists a $\gamma \in [\alpha, \beta]$ such that $y(\gamma) = 0$,

$$0 < y(t) \le \sqrt{\mu} \quad \text{for } \alpha \le t < \gamma$$

and

$$-\sqrt{\mu} \le y(t) < 0 \quad \text{for } \gamma < t \le \beta.$$

Repeating the same arguments as cases (i) and (ii), we see that

$$\beta - \alpha = \beta - \gamma + \gamma - \alpha < \tau_1 + 1.$$

In any case, it turns out that the beginning sentence of this step is true.

Step 2: For any interval $[\alpha, \beta] \subset [t_0, t_0 + T]$, if $y^2(t) \ge \mu/2$ for $\alpha \le t \le \beta$, then the time width $\beta - \alpha$ is less than τ_2, where μ and τ_2 are numbers given in part (b). To show this, we suppose that there exists an interval $[\alpha_2, \beta_2] \subset [t_0, t_0 + T]$ with $\beta_2 - \alpha_2 \ge \tau_2$ such that $y^2(t) \ge \mu/2$ for $\alpha_2 \le t \le \beta_2$. Since $v'(t) = -\omega^{q-2} h(t) |y(t)|^q \le 0$ for $t \ge t_0$, by (2.5) we have

$$\begin{aligned} \omega^{q-2} \left(\frac{\mu}{2} \right)^{q/2} \int_{\alpha_2}^{\beta_2} h(t) dt &\le \omega^{q-2} \int_{\alpha_2}^{\beta_2} h(t) |y(t)|^q dt \\ &= -\int_{\alpha_2}^{\beta_2} v'(t) dt = v(\alpha_2) - v(\beta_2) < \frac{\rho^2}{2}; \end{aligned}$$

namely,

$$\int_{\alpha_2}^{\beta_2} h(t) dt < \frac{\rho^2}{2\omega^{q-2}} \left(\frac{2}{\mu} \right)^{q/2}. \tag{2.6}$$

On the other hand, since $\tau_2 = \hat{t} + [\rho^2 (2/\mu)^{q/2} / (2\ell\omega^{q-2})] + 1$, we see that

$$\begin{aligned} \int_{\alpha_2}^{\beta_2} h(t) dt \ge \int_{\alpha_2}^{\alpha_2 + \tau_2} h(t) dt &= \int_{\alpha_2}^{\alpha_2 + \hat{t}} h(t) dt + \int_{\alpha_2 + \hat{t}}^{\alpha_2 + \tau_2} h(t) dt \\ &\ge \int_{\alpha_2 + \hat{t}}^{\alpha_2 + \hat{t} + [\rho^2 (2/\mu)^{q/2} / (2\ell\omega^{q-2})] + 1} h(t) dt \\ &= \sum_{i=0}^{[\rho^2 (2/\mu)^{q/2} / (2\ell\omega^{q-2})]} \int_{\alpha_2 + \hat{t} + i}^{\alpha_2 + \hat{t} + i + 1} h(t) dt \\ &\ge \left(\left[\frac{\rho^2}{2\ell\omega^{q-2}} \left(\frac{2}{\mu} \right)^{q/2} \right] + 1 \right) \ell > \frac{\rho^2}{2\omega^{q-2}} \left(\frac{2}{\mu} \right)^{q/2}. \end{aligned}$$

This contradicts (2.6). Thus, it turns out that the beginning sentence of this step is true.

From Steps 1 and 2, we conclude that $y^2(t)$ cannot remain in the range from $\mu/2$ to μ for a long time and passes through this range many times. Then, how much is time for $y^2(t)$ to stay in this range? To answer this question, we divide the interval $[t_0 + \tau_3, t_0 + T]$ into some small intervals J_i whose width is $\tau_1 + \tau_2 + 1$, where

$$J_i = \big[t_0 + \tau_3 + (i-1)(\tau_1 + \tau_2 + 1), t_0 + \tau_3 + i(\tau_1 + \tau_2 + 1) \big]$$

for any $i \in \mathbb{N}$. Then, we can describe

$$[t_0 + \tau_3, t_0 + T] = J_1 \cup J_2 \cup \cdots \cup J_{[1/\nu]+1}.$$

Step 3: Let us examine the behavior of $y^2(t)$ in the interval J_1 in detail. For this purpose, we subdivide J_1 into the intervals $[t_0 + \tau_3, t_0 + \tau_2 + \tau_3]$ and $[t_0 + \tau_2 + \tau_3, t_0 + \tau_1 + \tau_2 + \tau_3 + 1]$. Since the width of $[t_0 + \tau_3, t_0 + \tau_2 + \tau_3]$ is τ_2, it turns out from the the conclusion of Step 2 that there exists a $\underline{t} \in [t_0 + \tau_3, t_0 + \tau_2 + \tau_3]$ such that $y^2(\underline{t}) < \mu/2$. Since the width of $[t_0 + \tau_2 + \tau_3, t_0 + \tau_1 + \tau_2 + \tau_3 + 1]$ is $\tau_1 + 1$, it also turns out from the the conclusion of Step 1 that there exists a $\bar{t} \in [t_0 + \tau_2 + \tau_3, t_0 + \tau_1 + \tau_2 + \tau_3 + 1]$ such that $y^2(\bar{t}) > \mu$. From the continuity of $y^2(t)$, we can find numbers t_1 and t_2 with $\underline{t} \le t_1 < t_2 \le \bar{t}$ such that $y^2(t_1) = \mu/2$, $y^2(t_2) = \mu$ and

$$\frac{\mu}{2} \le y^2(t) \le \mu \quad \text{for } t_1 \le t \le t_2. \tag{2.7}$$

In fact, we have only to define t_2 and t_1 as $\inf\{t \in [\underline{t}, \bar{t}] : y^2(t) > \mu\}$ and $\sup\{t \in [\underline{t}, t_2] : y^2(t) < \mu/2\}$, respectively. Hence, we have

$$\begin{aligned} \frac{\mu}{2} &= y^2(t_2) - y^2(t_1) = \int_{t_1}^{t_2} \big(y^2(t)\big)' dt \\ &= -2\int_{t_1}^{t_2} \big(\omega x(t)y(t) + \omega^{q-2} h(t)\phi_q(y(t))y(t)\big) dt \le 2\,\omega \int_{t_1}^{t_2} |x(t)y(t)| dt. \end{aligned}$$

It follows from (2.5) that

$$|x(t)y(t)| \le \frac{1}{2}\big(x^2(t) + y^2(t)\big) < \frac{\rho^2}{2}$$

for $t_0 \le t \le t_0 + T$. Consequently, we obtain

$$\frac{\mu}{2\rho^2\omega} < t_2 - t_1. \tag{2.8}$$

Using the estimations given in the preceding step, we examine the loss of the total energy $v(t)$.

Step 4: From (2.7) and (2.8) it turns out that

$$\begin{aligned} v(t_2) - v(t_1) &= \int_{t_1}^{t_2} v'(t) dt = -\,\omega^{q-2} \int_{t_1}^{t_2} h(t)|y(t)|^q dt \\ &\le -\,\omega^{q-2} \Big(\frac{\mu}{2}\Big)^{q/2} \int_{t_1}^{t_2} h(t) dt \\ &\le -\,\omega^{q-2} \Big(\frac{\mu}{2}\Big)^{q/2} \int_{t_1}^{t_1+\mu/(2\rho^2\omega)} h(t) dt. \end{aligned}$$

Hence, by (2.3) we have

$$v(t_2) - v(t_1) \le -\frac{\rho^2 \nu}{2}.$$

Since $v'(t) = -\omega^{q-2}h(t)|y(t)|^q \le 0$ for $t \ge t_0$, it is clear that

$$v(t_1) - v(t_0+\tau_3) \le 0 \quad \text{and} \quad v(t_0+\tau_1+\tau_2+\tau_3+1) - v(t_2) \le 0.$$

We therefore conclude that

$$\begin{aligned}\int_{J_1} v'(t)dt &= v(t_0+\tau_1+\tau_2+\tau_3+1) - v(t_2) + v(t_2) - v(t_1) + v(t_1) - v(t_0+\tau_3) \\ &\le -\frac{\rho^2 \nu}{2}.\end{aligned}$$

Repeating the same process as in the proof of Step 3, we can estimate that

$$\int_{J_i} v'(t)dt \le -\frac{\rho^2 \nu}{2}$$

for $i = 2, 3, \ldots, [1/\nu]+1$. This means that the loss of the total energy $v(t)$ in each interval J_i is at least $\rho^2\nu/2$. Hence, we obtain

$$v(t_0+T) - v(t_0+\tau_3) = \sum_{i=1}^{[1/\nu]+1} \int_{J_i} v'(t)dt \le -\frac{\rho^2\nu}{2}\left(\left[\frac{1}{\nu}\right]+1\right) < -\frac{\rho^2}{2},$$

and therefore, by (2.5) we have

$$v(t_0+T) < v(t_0+\tau_3) - \frac{\rho^2}{2} < 0.$$

This contradicts the fact that $v(t) \ge 0$ for $t \ge t_0$. Thus, inequality (2.4) was proved.

Part (d): For any $\rho > 0$, let

$$B(\rho) = \rho.$$

Since $v'(t) = -\omega^{q-2}h(t)|y(t)|^q \le 0$ for $t \ge t_0$, if $\|\mathbf{x_0}\| < \rho$, then

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| = \sqrt{2v(t)} \le \sqrt{2v(t_0)} = \|\mathbf{x_0}\| < \rho = B$$

for all $t \ge t_0$. Thus, the solutions of (2.1) are uniformly bounded.

The proof of Theorem 1.1 is now complete. □

3. On the uniform divergence condition (1.7) in Theorem 1.1

Let q^* be the conjugate number of q; namely,

$$\frac{1}{q} + \frac{1}{q^*} = 1.$$

Since $q \ge 2$, it follows that $1 < q^* \le 2$. Note that ϕ_{q^*} is the inverse function of ϕ_q.

When is condition (1.7) satisfied? Conversely, when is condition (1.7) not satisfied? In this section, we will answer these questions.

Proposition 3.1. *Suppose that there exists an* $\overline{h} > 0$ *such that*

$$0 < h(t) \le \overline{h} \quad \textit{for} \;\; t > 0. \tag{3.1}$$

Then condition (1.7) *is satisfied*.

Proof. Consider the curve C defined by

$$u = -\,\phi_{q^*}\!\left(\frac{1}{h(t)}\right).$$

It follows from (3.1) that $\phi_{q^*}(1/h(t)) \ge \phi_{q^*}(1/\overline{h})$ for $t > 0$. This means that the curve C is located in the region

$$\left\{(t,u): t > 0 \;\text{ and }\; u \le -\,\phi_{q^*}\!\left(\frac{1}{\overline{h}}\right)\right\}.$$

Let σ be any fixed nonnegative number and let us pay attention to the behavior of the solution $u(t;\sigma)$ of (1.8). Note that $u(\sigma;\sigma) = 0$. Taking into account that

$$u'(t;\sigma) = -1 - h(t)\phi_q(u(t;\sigma)) < 0 \tag{3.2}$$

as long as the solution curve $u = u(t;\sigma)$ is over the curve C, we see that the solution curve arrives at the straight line $u = -\,\phi_{q^*}(1/(2\overline{h}))$. Let t_1 be the arrival time. Then,

$$-\,\phi_{q^*}\!\left(\frac{1}{2\overline{h}}\right) < u(t;\sigma) < 0; \tag{3.3}$$

namely,

$$0 < \phi_q(-u(t;\sigma)) < \frac{1}{2\overline{h}}$$

for $\sigma < t < t_1$. Hence, we have

$$\begin{aligned} -1 < u'(t;\sigma) &= -1 + h(t)\phi_q(-u(t;\sigma)) \\ &< -1 + \overline{h}\,\frac{1}{2\overline{h}} = -\frac{1}{2} \end{aligned}$$

for $\sigma < t < t_1$. From this estimation it turns out that

$$\phi_{q^*}\!\left(\frac{1}{2\overline{h}}\right) < t_1 - \sigma < 2\,\phi_{q^*}\!\left(\frac{1}{2\overline{h}}\right). \tag{3.4}$$

It also turns out from (3.2) that the solution curve cannot return to the region

$$\left\{(t,u): t \ge t_1 \;\text{ and }\; -\,\phi_{q^*}\!\left(\frac{1}{2\overline{h}}\right) \le u \le 0\right\}.$$

In other words, $u(t;\sigma) < -\phi_{q^*}(1/(2\bar{h}))$ for $t \geq t_1$. Hence, by (3.3) and (3.4) we have

$$\int_{\sigma}^{t+\sigma} u(s;\sigma)ds < \int_{t_1}^{t+\sigma} u(s;\sigma)ds$$
$$< -\phi_{q^*}\left(\frac{1}{2\bar{h}}\right)(t+\sigma-t_1) < -\phi_{q^*}\left(\frac{1}{2\bar{h}}\right)\left(t-2\,\phi_{q^*}\left(\frac{1}{2\bar{h}}\right)\right) \tag{3.5}$$

for t sufficiently large.

For any $K > 0$, let

$$T = T(K) = 2\,\phi_{q^*}\left(\frac{1}{2\bar{h}}\right) + \frac{K}{\phi_{q^*}(1/(2\bar{h}))}.$$

Then, by (3.5) we obtain

$$\int_{\sigma}^{t+\sigma} u(s;\sigma)ds < -K \quad \text{for } t \geq T.$$

Hence, the integral from σ to $t+\sigma$ of $u(t;\sigma)$ diverges to negative infinity as $t \to \infty$ uniformly with respect to σ; namely, condition (1.7) holds. □

Proposition 3.2. *Let $h(t)$ is a positive differentiable function on $(0,\infty)$. Suppose that*

$$\lim_{t\to\infty} h(t) = \infty \tag{3.6}$$

and there exist constants $\kappa > 0$ and $\tau > 0$ such that

$$\frac{h'(t)}{(h(t))^{q^*}} \leq \kappa \quad \text{for } t \geq \tau, \tag{3.7}$$

where $\kappa < 1$ if $q = 2$ and κ may be any positive number if $q > 2$. Then condition (1.7) is not satisfied.

Remark 3.1. In the proposition mentioned above, it is not necessarily assumed that the damping coefficient $h(t)$ does not necessarily need to be increasing. Hence, the curve $C\colon u = -\phi_{q^*}(1/h(t))$ does not also necessarily increase monotonously. It follows from (3.6) that the curve C is asymptotic to the t-axis.

To prove Proposition 3.2, we prepare the following lemma.

Lemma 3.3. *Let $f(t)$ be a negative continuous function on $(0,\infty)$. If $f(t)$ increases and approaches zero as $t \to \infty$, then the integral from σ to $t+\sigma$ of $f(t)$ does not diverge to negative infinity as $t \to \infty$ uniformly with respect to σ.*

Proof. By way of contradiction, we suppose that

$$\lim_{t\to\infty}\int_{\sigma}^{t+\sigma} f(s)ds = -\infty \quad \text{uniformly with respect to } \sigma \geq 0.$$

Then, for any $K > 0$ there exists a $T(K) > 0$ such that

$$\int_{\sigma}^{T+\sigma} f(s)ds < -K \quad \text{for any } \sigma \geq 0.$$

Since $f(t)$ is negative and increasing, we see that

$$-K > \int_{\sigma}^{T+\sigma} f(s)ds \geq f(\sigma)T;$$

namely, $f(\sigma) < -K/T$ for all $\sigma \geq 0$. This is a contradiction, because $f(t)$ tends to 0 as $t \to \infty$. □

Proof of Proposition 3.2. By (3.6), we can define

$$g(t) = -\max_{t\leq s}\phi_{q^*}\left(\frac{1}{h(s)}\right) < 0$$

for $t > 0$. The function $g(t)$ approaches zero as $t \to \infty$. Since

$$\min_{\tau_1\leq s} h(s) \leq \min_{\tau_2\leq s} h(s)$$

for any τ_1 and τ_2 satisfying $0 < \tau_1 < \tau_2$, we see that

$$\begin{aligned} g(\tau_1) &= -\max_{\tau_1\leq s}\phi_{q^*}\left(\frac{1}{h(s)}\right) = -\phi_{q^*}\left(\frac{1}{\min_{\tau_1\leq s} h(s)}\right) \\ &\leq -\phi_{q^*}\left(\frac{1}{\min_{\tau_2\leq s} h(s)}\right) = -\max_{\tau_2\leq s}\phi_{q^*}\left(\frac{1}{h(s)}\right) = g(\tau_2); \end{aligned}$$

that is, $g(t)$ is increasing. Since $h(t)$ is differentiable for $t > 0$, we also see that $g(t)$ is right differentiable for $t > 0$.

We divide the interval $[\tau, \infty)$ into

$$I = \left\{ t \geq \tau : g(t) = -\phi_{q^*}\left(\frac{1}{h(t)}\right)\right\},$$

$$J = \left\{ t \geq \tau : g(t) < -\phi_{q^*}\left(\frac{1}{h(t)}\right)\right\},$$

where τ is the constant given in (3.7). Let $t \in J$. Then,

$$\min_{t\leq s}\left(-\phi_{q^*}\left(\frac{1}{h(s)}\right)\right) = g(t) < -\phi_{q^*}\left(\frac{1}{h(t)}\right).$$

Hence, there exists a $\hat{t} > t$ such that

$$g(t) = -\phi_{q^*}\left(\frac{1}{h(\hat{t})}\right).$$

This means that

$$-\phi_{q^*}\left(\frac{1}{h(\hat{t})}\right) < -\phi_{q^*}\left(\frac{1}{h(\sigma)}\right) \quad \text{for } t \le \sigma < \hat{t}$$

and

$$-\phi_{q^*}\left(\frac{1}{h(\hat{t})}\right) \le -\phi_{q^*}\left(\frac{1}{h(\sigma)}\right) \quad \text{for } \sigma \ge \hat{t}.$$

From these inequalities it turns out that

$$g(s) = \min_{s \le \sigma}\left(-\phi_{q^*}\left(\frac{1}{h(\sigma)}\right)\right) = -\phi_{q^*}\left(\frac{1}{h(\hat{t})}\right) = g(t)$$

for any $s \in (t, \hat{t})$. Hence, we have

$$g'_+(t) = \lim_{s \to t+0} \frac{g(s) - g(t)}{s - t} = 0.$$

Let $t \in I$. Then,

$$\min_{t \le s} h(s) = h(t).$$

This means that $t \le s$ implies $h(t) \le h(s)$. From the differentiability of $h(t)$ it follows that

$$h'(t) = h'_+(t) = \lim_{s \to t+0} \frac{h(s) - h(t)}{s - t} \ge 0.$$

Hence, by (3.7) and the assumption that $1 < q^* \le 2$, we have

$$0 \le g'_+(t) = (q^* - 1)\frac{h'(t)}{(h(t))^{q^*}} \le \frac{h'(t)}{(h(t))^{q^*}} \le \kappa$$

for $t \in I$. Thus, we see that

$$0 \le g'_+(t) \le \kappa \quad \text{for } t \ge \tau. \tag{3.8}$$

Let n be any integer. Consider the curve defined by $u = n g(t)$. Let us name this curve C_n. Since $g(t)$ is increasing, it turns out that for each $n \in \mathbb{N}$, the curve C_n does not fall through the region $\{(t,u) : t > 0 \text{ and } u < 0\}$ and approaches the t-axis. Hence, the curve C_n and the straight line $u = -t$ meet only once. Let s_n be the intersecting time. Then, the time s_n satisfies that

$$s_n = n \max_{s_n \le s} \phi_{q^*}\left(\frac{1}{h(s)}\right) = n\,\phi_{q^*}\left(\frac{1}{\min_{s_n \le s} h(s)}\right).$$

For any integers n_1 and n_2 satisfying $n_1 < n_2$, the curve C_{n_2} is under the curve C_{n_1}, because $g(t) < 0$ for $t > 0$. Hence, the sequence $\{s_n\}$ is strictly increasing. It also turns out that

$$s_n \to \infty \quad \text{as} \ \ n \to \infty.$$

Since $\{s_n\}$ diverges to infinity as $n \to \infty$, we can find a sufficiently large $m \in \mathbb{N}$ which satisfies that $s_m > \tau$ and

$$\phi_q(m) > 1 + \kappa m. \tag{3.9}$$

Recall that $0 < \kappa < 1$ if $q = 2$ and κ is any positive number if $q > 2$.

Let $\sigma \geq 0$ be fixed arbitrarily and let us pay attention to the behavior of the solution $u(t;\sigma)$ of (1.8). We will show that the solution curve $u = u(t;\sigma)$ does not meet the curve C_m. By way of contradiction, we suppose that there exists a $t_1 > \sigma$ such that $u(t_1;\sigma) = mg(t_1)$ and $u(t;\sigma) > mg(t)$ for $\sigma \leq t < t_1$. Since $u'(t;\sigma) = -1 - h(t)\phi_q(u(t;\sigma)) > -1$ for $t > \sigma$, the solution curve $u(t;\sigma)$ is located over the curve C_m. Hence, it is obvious that $t_1 > s_m > \tau$.

From the definition of t_1 it follows that

$$u'(t_1;\sigma) \leq mg'_+(t_1). \tag{3.10}$$

On the other hand, it turns out from (3.9) that the differential coefficient of $u(t;\sigma)$ at $t = t_1$ is larger than κm, because

$$\begin{aligned}
u'(t_1;\sigma) &= -1 - h(t_1)\phi_q(mg(t_1)) = -1 - h(t_1)\phi_q(m)\phi_q(g(t_1)) \\
&= -1 + h(t_1)\phi_q(m)\phi_q\left(\phi_{q^*}\left(\frac{1}{\min_{t_1 \leq s} h(s)}\right)\right) \\
&= -1 + \phi_q(m)\frac{h(t_1)}{\min_{t_1 \leq s} h(s)} \\
&\geq -1 + \phi_q(m) > -1 + 1 + \kappa m = \kappa m.
\end{aligned}$$

Hence, by (3.8) we have

$$u'(t_1;\sigma) > \kappa m \geq mg'_+(t_1).$$

This contradicts (3.10). Thus, we conclude that

$$mg(t) < u(t;\sigma) < 0 \tag{3.11}$$

for $t \geq \sigma$.

Suppose that condition (1.7) is satisfied. Then, for any $K > 0$ there exists a $T(K) > 0$ such that

$$\int_\sigma^{t+\sigma} u(s;\sigma)ds < -K \quad \text{for} \ \ t \geq T.$$

Hence, from (3.11) it follows that

$$\int_\sigma^{t+\sigma} g(s)ds < -\frac{K}{m} \quad \text{for} \ \ t \geq T;$$

namely,

$$\lim_{t\to\infty}\int_{\sigma}^{t+\sigma} g(s)ds = -\infty \quad \text{uniformly with respect to } \sigma \geq 0.$$

On the other hand, by means of Lemma 3.3, we conclude that the integral from σ to $t+\sigma$ of $g(t)$ does not diverge to negative infinity as $t \to \infty$ uniformly with respect to σ, because $g(t)$ is a negative continuous increasing function on $(0,\infty)$ and approaches zero as $t \to \infty$. This is a contradiction. Thus, condition (1.7) is not satisfied. □

Remark 3.2. Although it seems that the inequality (3.7) in Proposition 3.2 is a technical condition for the proof, this inequality is quite reasonable. The function $h'(t)/(h(t))^{q^*}$ satisfies that

$$\liminf_{t\to\infty} \frac{h'(t)}{(h(t))^{q^*}} = 0. \tag{3.12}$$

Indeed, if there exist constants $\gamma > 0$ and $\tau > 0$ such that

$$\frac{h'(t)}{(h(t))^{q^*}} \geq \gamma \quad \text{for } t \geq \tau,$$

then we have

$$\frac{d}{dt}\phi_{q^*}\left(\frac{1}{h(t)}\right) = -(q^*-1)\frac{h'(t)}{(h(t))^{q^*}} \leq -(q^*-1)\gamma$$

for $t \geq \tau$. Integrate this inequality to obtain

$$\phi_{q^*}\left(\frac{1}{h(t)}\right) \leq \phi_{q^*}\left(\frac{1}{h(\tau)}\right) - (q^*-1)\gamma(t-\tau)$$

for $t \geq \tau$. The left-hand side is positive and the right-hand side tends to $-\infty$ as $t \to \infty$. This is a contradiction. Thus, if the function $h'(t)/(h(t))^{q^*}$ is strictly decreasing for t sufficiently large, then it follows from (3.12) that condition (3.7) is satisfied with $\kappa = 0$. In addition, it turns out that the function $h'(t)/(h(t))^{q^*}$ never increases monotonously.

As was mentioned in Section 1, the equilibrium of the damped linear oscillator (1.5) is not uniformly asymptotically stable. As a matter of fact, the following fact is derived from Proposition 3.2.

Example 3.3. If $h(t) = t$ and $q = 2$, then condition (1.7) is not satisfied.

It is clear that t is positive and differentiable for $t > 0$. Condition (3.6) holds when $h(t) = t$. Since $h'(t)/(h(t))^2 = 1/t^2$, condition (3.7) is satisfied with $q = 2$, $\kappa = 1/4 < 1$ and $\tau = 2$. Thus, from Proposition 3.2. it turns out that condition (1.7) is not satisfied.

Remark 3.4. By Remark 3.2 we also conclude that condition (3.7) is satisfied with $\kappa = 0$. In fact, if $h(t) = t$ and $q = q^* = 2$, then

$$h''(t)h(t) - q^*(h'(t))^2 = -2 < 0 \quad \text{for } t > 0.$$

This means that $h'(t)/(h(t))^{q^*}$ is strictly decreasing.

Although a complicated expression will be used, we can loosen condition (3.7) in Proposition 3.2 a little. To this end, we define a family of functions. Let $\psi(r)$ be differentiable and increasing for $r > 0$, and satisfies that

$$\psi(r) \to \infty \quad \text{as } \ r \to \infty \tag{3.13}$$

and there exists an $R > 0$ such that

$$0 < \psi(r) \leq r \quad \text{for } \ r \geq R. \tag{3.14}$$

We denote the derivative of $\psi(r)$ by

$$\psi_r(r) = \frac{d}{dr}\psi(r).$$

Then, we have the following result.

Proposition 3.4. *Let $h(t)$ is a positive differentiable function on $(0,\infty)$. Suppose that condition (3.6) holds and there exist constants $\kappa > 0$ and $\tau > 0$ such that*

$$\frac{h'(t)\psi_r(h(t))}{h(t)\phi_{q^*}(\psi(h(t)))} \leq \kappa \quad \textit{for } \ t \geq \tau, \tag{3.15}$$

where $\kappa < 1$ if $q = 2$ and κ may be any positive number if $q > 2$. Then condition (1.7) is not satisfied.

Because the proof is performed by the same method as the proof of Proposition 3.2, we merely explain only the outline of the proof, in order to focus on the difference.

Outline of the proof of Proposition 3.4. Define

$$g(t) = -\max_{t\leq s}\phi_{q^*}\left(\frac{1}{\psi(h(s))}\right) = -\,\phi_{q^*}\left(\frac{1}{\psi(\min_{t\leq s}h(s))}\right) < 0$$

for $t > 0$. Then, from (3.6), (3.13) and the differentiability of $h(t)$ it turns out that $g(t)$ increases and approaches zero as $t \to \infty$, and it is right differentiable for $t > 0$. By (3.6) again, there exists a $\bar{\tau} > \tau$ such that $h(t) \geq R$ for $t \geq \bar{\tau}$.

We divide the interval $[\bar{\tau},\infty)$ into

$$I = \left\{t \geq \bar{\tau} \colon g(t) = -\,\phi_{q^*}\left(\frac{1}{\psi(h(s))}\right)\right\},$$

$$J = \left\{t \geq \bar{\tau} \colon g(t) < -\,\phi_{q^*}\left(\frac{1}{\psi(h(s))}\right)\right\}.$$

Then, as in the proof of Proposition 3.2, we see that $g'_+(t) = 0$ for $t \in J$. Since $1 < q^* \leq 2$ and $h'(t) \geq 0$ for $t \in I$, by (3.14) and (3.15) we have

$$0 \leq g'_+(t) = (q^*-1)\frac{h'(t)\psi_r(h(t))}{(\psi(h(t)))^{q^*}} \leq \frac{\kappa h(t)}{\psi(h(t))}$$

for $t \in I$. To sum up, we obtain

$$0 \leq g'_+(t) \leq \frac{\kappa h(t)}{\psi(h(t))} \quad \text{for } t \geq \bar{\tau}. \tag{3.16}$$

Consider the curve defined by $u = ng(t)$ for each $n \in \mathbb{N}$ and name this curve C_n. Let s_n be the intersecting time of the curve C_n and the straight line $u = -t$. Then, it turns out that the sequence $\{s_n\}$ is strictly increasing and diverges to infinity as $n \to \infty$. Hence, we can find a sufficiently large $m \in \mathbb{N}$ which satisfies that $s_m > \tau$ and $\phi_q(m) > 1 + \kappa m$.

Let $\sigma \geq 0$ be fixed arbitrarily. By the same way that we used in Proposition 3.2, we can show that the solution curve $u = u(t;\sigma)$ of (1.8) does not meet the curve C_m. In fact, if there exists a $t_1 > \sigma$ such that $u(t_1;\sigma) = mg(t_1)$ and $u(t;\sigma) > mg(t)$ for $\sigma \leq t < t_1$, then $t_1 > s_m > \bar{\tau}$. Hence, by (3.14) we have

$$\begin{aligned} mg'_+(t_1) &\geq u'(t_1;\sigma) = -1 - h(t_1)\phi_q(mg(t_1)) = -1 - h(t_1)\phi_q(m)\phi_q(g(t_1)) \\ &= -1 + h(t_1)\phi_q(m)\phi_q\left(\phi_{q^*}\left(\frac{1}{\psi(\min_{t_1 \leq s} h(s))}\right)\right) \\ &= -1 + \phi_q(m)\frac{h(t_1)}{\psi(\min_{t_1 \leq s} h(s))} \geq -1 + \phi_q(m)\frac{h(t_1)}{\psi(h(t_1))} \\ &> -1 + (1+\kappa m)\frac{h(t_1)}{\psi(h(t_1))} \geq -1 + 1 + m\frac{\kappa h(t_1)}{\psi(h(t_1))}, \end{aligned}$$

which contradicts (3.16) at $t = t_1$. Thus, we conclude that

$$mg(t) < u(t;\sigma) < 0 \tag{3.17}$$

for $t \geq \sigma$.

Suppose that condition (1.7) is satisfied; namely, for any $K > 0$ there exists a $T(K) > 0$ such that

$$\int_\sigma^{t+\sigma} u(s;\sigma)ds < -K \quad \text{for } t \geq T.$$

Then, it follows from (3.17) that

$$\lim_{t \to \infty} \int_\sigma^{t+\sigma} g(s)ds = -\infty \quad \text{uniformly with respect to } \sigma \geq 0.$$

However, since $g(t)$ is a negative continuous increasing function on $(0,\infty)$ and approaches zero as $t \to \infty$, Lemma 3.3 insists that the integral from σ to $t + \sigma$ of $g(t)$ does not diverge to negative infinity as $t \to \infty$ uniformly with respect to σ. This is a contradiction. Thus, condition (1.7) is not satisfied. □

Remark 3.5. When $\psi(r) = r$, condition (3.15) coincides with condition (3.7). Conditions (3.7) and (3.15) are inevitably satisfied for t in which $h'(t) \leq 0$. On the othe hand, if $\psi(r) = \sqrt{r}$, then

$$\frac{h'(t)\psi_r(h(t))}{h(t)\phi_{q^*}(\psi(h(t)))} = \frac{h'(t)}{2(h(t))^{q^*/2+1}} \leq \frac{h'(t)}{2(h(t))^{q^*}} \leq \frac{h'(t)}{(h(t))^{q^*}}$$

for t in which $h'(t) \geq 0$; if $\psi(r) = \log 1 + r$, then

$$\frac{h'(t)\psi_r(h(t))}{h(t)\phi_{q^*}(\psi(h(t)))} = \frac{h'(t)}{h(t)(1+h(t)))(\log(1+h(t)))^{q^*-1}}$$
$$\leq \frac{h'(t)}{h(t)(1+h(t)))} \leq \frac{h'(t)}{(h(t))^2} \leq \frac{h'(t)}{(h(t))^{q^*}}$$

for t in which $h'(t) \geq 0$. Thus, condition (3.7) implies (3.15).

4. Corollaries

In this section, without using the characteristic equation (1.8), we give some sufficient conditions which guarantee that the equilibrium of (1.1) is uniformly globally asymptotically stable. First of all, we answer the question presented in Section 1 as follows.

Corollary 4.1. *Suppose that $h(t)$ is integrally positive. If condition* (1.3) *is satisfied, then the equilibrium of* (1.1) *is uniformly globally asymptotically stable.*

Proof. The proof is performed by the similar method to that of Theorem 1.1. Especially, the proof excluding case (ii) of part (c) is completely the same. We give only the proof of the portion.

Case (ii): Suppose that there exists an interval $[\alpha_1, \beta_1] \subset [t_0, t_0+T]$ with $\beta_1 - \alpha_1 \geq \tau_1$ such that $x(t) \geq \eta/2$ and $-\sqrt{\mu} \leq y(t) \leq 0$ for $\alpha_1 \leq t \leq \beta_1$. Taking into account that $\varepsilon_0 = \min\{1, \omega^2\eta/2\}$ and $\phi_q(\varepsilon_0) \leq \varepsilon_0$, we obtain

$$\left(\frac{\omega y(t)}{\varepsilon_0}\right)' \leq -1 - h(t)\phi_q\left(\frac{\omega y(t)}{\varepsilon_0}\right)$$

for $\alpha_1 \leq t \leq \beta_1$. Since $\mu \leq \varepsilon_0^2/\omega^2$, we see that

$$0 \geq \frac{\omega y(t)}{\varepsilon_0} \geq -\frac{\omega\sqrt{\mu}}{\varepsilon_0} \geq -1.$$

Hence, we conclude that

$$\left(\frac{\omega y(t)}{\varepsilon_0}\right)' \leq -1 - h(t)\frac{\omega y(t)}{\varepsilon_0}$$

for $\alpha_1 \leq t \leq \beta_1$, because $q \geq 2$. Define $\xi(t) = \omega y(t)/\varepsilon_0$ and let $f(t,u) = -1 - h(t)u$. Then,

$$\xi'(t) \leq f(t, \xi(t))$$

for $\alpha_1 \leq t \leq \beta_1$. We compare $\xi(t)$ with the solution $u(t;\alpha_1)$ of (1.6) satisfying $u(\alpha_1;\alpha_1) = 0$. By using (1.3) instead of (1.7), we obtain the estimation (2.2). The rest of the proof is carried out in the same way as the proof of Theorem 1.1. We leave the detailed analysis to the reader. □

We give a result of not using the characteristic equation (1.8).

Corollary 4.2. *Suppose that $h(t)$ is integrally positive and that there exists a right differentiable function $k(t)$ such that*

$$0 \le h(t) \le k(t) \quad \textit{for } t \ge 0.$$

Suppose also that $1/k(t)$ and $(1/k(t))'_+$ are bounded from above, where $(1/k(t))'_+$ is the right-hand derivative of $1/k(t)$. If

$$\lim_{t\to\infty} \int_\sigma^{t+\sigma} \phi_{q^*}\left(\frac{1}{k(s)}\right) ds = \infty \quad \textit{uniformly with respect to } \sigma \ge 0,$$

then the equilibrium of (1.1) *is uniformly asymptotically stable.*

Remark 4.1. Although the upper function $k(t)$ has to be right differentiable, the damping coefficient $h(t)$ does not necessarily need to be right differentiable.

Proof of Corollary 4.2. By assumption, there exist numbers $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{1}{k(t)} \le c_1 \quad \text{and} \quad \left(\frac{1}{k(t)}\right)'_+ \le c_2$$

for $t \ge 0$. Define

$$g(t) = -\phi_{q^*}\left(\frac{1}{k(t)}\right) \quad \text{for } t \ge 0.$$

Then, it is clear that

$$-\phi_{q^*}(c_1) \le g(t) < 0 \quad \text{and} \quad g'_+(t) \ge -(q^*-1)\, c_1^{q^*-2} c_2 \tag{4.1}$$

for $t \ge 0$.

Consider the characteristic equation (1.8) and let $u(t;\sigma)$ be the solution of (1.8) satisfying the initial condition $u(\sigma;\sigma) = 0$. Then, we see that

$$u(t;\sigma) < 0 \quad \text{for } t > \sigma. \tag{4.2}$$

In fact, since $u(\sigma;\sigma) = 0$ and $u'(\sigma;\sigma) = -1$, we can find a $t_1 > \sigma$ such that $u(t;\sigma) < 0$ for $\sigma < t \le t_1$. Suppose that there exists a $t_2 > t_1$ such that $u(t_2;\sigma) = 0$ and $u(t;\sigma) < 0$ for $\sigma < t < t_2$. Then, since $u'(t_2;\sigma) = -1$, it follows that $u(t;\sigma) > 0$ in a left-hand neighborhood of t_2. This contradicts the definition of t_2.

Let us compare $u(t;\sigma)$ with $g(t)$. Since $g(\sigma) < 0 = u(\sigma;\sigma)$, there are two cases to consider: (i) $g(t) < u(t;\sigma)$ for $t \ge \sigma$ and (ii) there exists a $t^* > \sigma$ such that $g(t^*) = u(t^*;\sigma)$ and $g(t) < u(t;\sigma)$ for $\sigma \le t < t^*$; namely, the graph of $g(t)$ intersects the solution curve $u(t;\sigma)$ at $t = t^*$ for the first time. Hereafter, we will show that there exists a c_3 with $0 < c_3 < 1$ such that

$$c_3 g(t) \ge u(t;\sigma) \quad \text{for } t \ge \sigma + 1 \tag{4.3}$$

in both cases.

Case (i): Since $0 \le h(t) \le k(t)$ for $t \ge 0$, we see that

$$u(t;\sigma) > g(t) = -\phi_{q^*}\left(\frac{1}{k(t)}\right) \ge -\phi_{q^*}\left(\frac{1}{h(t)}\right)$$

for $t \ge \sigma$. Hence, we have

$$u'(t;\sigma) = -1 - h(t)\phi_q(u(t;\sigma)) < 0;$$

that is, $u(t;\sigma)$ is strictly decreasing for $t \ge \sigma$. Let

$$c_3 = \min\left\{\frac{u(\sigma+1;\sigma)}{g(\sigma+1)}, \frac{1}{1+(q^*-1)\,c_1^{q^*-2}c_2}\right\}.$$

Then, $0 < c_3 \le 1/(1+(q^*-1)\,c_1^{q^*-2}c_2) < 1$. For simplicity, let $\zeta(t) = c_3 g(t)$. Then,

$$k(t)\phi_q(\zeta(t)) = -\phi_q(c_3) \ge -c_3 > -1$$

for $t \ge 0$. Hence, it turns out from (4.1) that

$$\zeta_+'(t) = c_3 g'(t) \ge -(q^*-1)\,c_1^{q^*-2}c_2\,c_3 \ge -1+c_3 \ge -1-k(t)\phi_q(\zeta(t))$$

for $t \ge 0$. Let $f(t,u) = -1-h(t)\phi_q(u)$. Taking $\zeta(t) < 0$ for $t \ge 0$ into account, we obtain

$$\zeta_+'(t) \ge -1-k(t)\phi_q(\zeta(t)) \ge -1-h(t)\phi_q(\zeta(t)) = f(t,\zeta(t))$$

for $t \ge 0$. Since $c_3 \le u(\sigma+1;\sigma)/g(\sigma+1)$ and $g(\sigma+1) < 0$, we see that

$$\zeta(\sigma+1) = c_3 g(\sigma+1) \ge u(\sigma+1;\sigma).$$

Consequently, we can get (4.3) by virtue of a standard comparison theorem.

Case (ii): We subdivide this case as follows: (a) $t^* > \sigma+1$ and (b) $\sigma < t^* \le \sigma+1$. If $t^* > \sigma+1$, then $g(t) < u(t;\sigma)$ for $\sigma \le t \le \sigma+1$. Hence, by the same way as the case (i), we can get (4.3). If $\sigma < t^* \le \sigma+1$, then $g(t) < u(t;\sigma)$ for $\sigma \le t < t^*$, and therefore,

$$u(t;\sigma) \ge g(t) = -\phi_{q^*}\left(\frac{1}{k(t)}\right) \ge -\phi_{q^*}\left(\frac{1}{h(t)}\right)$$

for $\sigma \le t \le t^*$. Hence, we have

$$u'(t;\sigma) = -1-h(t)\phi_q(u(t;\sigma)) \le 0 \quad \text{for } \sigma \le t \le t^*.$$

Let $c_3 = 1/(1+(q^*-1)\,c_1^{q^*-2}c_2) < 1$ and $\zeta(t) = c_3 g(t) < 0$. Then, by (4.1) we obtain

$$\begin{aligned}\zeta_+'(t) &= c_3 g_+'(t) \ge -(q^*-1)\,c_1^{q^*-2}c_2\,c_3 = -1+c_3 \\ &\ge -1-k(t)\phi_q(\zeta(t)) \ge -1-h(t)\phi_q(\zeta(t)) = f(t,\zeta(t))\end{aligned}$$

for $t \geq 0$, where $f(t,u)$ is the function given in the case (i). Since $0 < c_3 < 1$ and $u(t^*;\sigma) < 0$, we see that $\zeta(t^*) = c_3 u(t^*;\sigma) > u(t^*;\sigma)$. We therefore conclude that $\zeta(t) \geq u(t;\sigma)$ for $t \geq t^*$. Since $\sigma + 1 \geq t^*$, we get (4.3).

From (4.1)–(4.3) it turns out that for t sufficiently large,

$$\begin{aligned}
\int_{\sigma}^{t+\sigma} u(s;\sigma)ds &= \int_{\sigma}^{\sigma+1} u(s;\sigma)ds + \int_{\sigma+1}^{t+\sigma} u(s;\sigma)ds \\
&\leq \int_{\sigma}^{\sigma+1} u(s;\sigma)ds + \int_{\sigma+1}^{t+\sigma} \zeta(s)ds < \int_{\sigma+1}^{t+\sigma} \zeta(s)ds \\
&= \int_{\sigma}^{\sigma+1} c_3\,\phi_{q^*}\left(\frac{1}{k(s)}\right)ds \;-\; \int_{\sigma}^{t+\sigma} c_3\,\phi_{q^*}\left(\frac{1}{k(s)}\right)ds \\
&\leq c_1^{q^*-1} c_3 \;-\; \int_{\sigma}^{t+\sigma} c_3\,\phi_{q^*}\left(\frac{1}{k(s)}\right)ds.
\end{aligned}$$

Since

$$\lim_{t\to\infty} \int_{\sigma}^{t+\sigma} \phi_{q^*}\left(\frac{1}{k(s)}\right)ds = \infty$$

uniformly with respect to $\sigma \geq 0$, condition (1.7) holds. Thus, by Theorem 1.1, the equilibrium of (1.1) is uniformly globally asymptotically stable. □

5. Example with unbounded damping

Thanks to Theorem 1.1 and Proposition 3.1, we can conclude that the equilibrium of (1.1) is uniformly globally asymptotically stable provided that the damping coefficient $h(t)$ is integrally positive and bounded. On the other hand, from Proposition 3.2 (or Proposition 3.4) it turns out that condition (1.7) does not hold when $h(t)$ satisfies conditions (3.6) and (3.7) (or (3.15)), and therefore, Theorem 1.1 cannot be applied. It is obvious that condition (3.6) implies that $h(t)$ is unbounded. Then, a natural question arise. Does the equilibrium of (1.1) become uniformly globally asymptotically stable even if $h(t)$ is unbounded?

In this section, we will give an affirmative answer to the question above by using Corollary 4.2. For this purpose, we define a sequence of functions as follows: For any $n \in \mathbb{N}$, let

$$I_n = [2n-1, 2n]$$

and $\{f_n(t)\}$ be a sequence of nonnegative and continous functions on $[0,1]$ satisfying $f_n(0) = f_n(1) = 0$, $(f_n)'_+(0) = (f_n)'_-(1) = 0$ and

$$\max_{0\leq t\leq 1} f_n(t) = p > 0 \quad \text{for } n \in \mathbb{N},$$

where $(f_n)'_+(t)$ and $(f_n)'_-(t)$ are the right-hand and left-hand derivatives of $f_n(t)$, respectively.

Consider the damping coefficient $h(t)$ defined by

$$\phi_{q^*}\left(\frac{1}{h(t)}\right) = \begin{cases} p - \dfrac{n}{n+1} f_n(t-2n+1) & \text{if } t \in I_n, \\ p & \text{if } t \notin I_n. \end{cases} \tag{5.1}$$

Then, we can present the desired example.

Example 5.1. If $h(t)$ satisfies (5.1) and $(f_n)'_+(t)$ has the same upper bound for all $n \in \mathbb{N}$, then the equilibrium of (1.1) is uniformly globally asymptotically stable.

Let us constitute a concrete $h(t)$. We arbitrarily choose two constants α and β so that $0 < \alpha < \beta < 1$. Let $R(n)$ denote the n-th random number between α and β. Define $f(t) = \sin^2(\pi t)$ and

$$g_n(t) = \begin{cases} \dfrac{t}{2R(n)} & \text{if } 0 \le t < R(n), \\ \dfrac{t+1-2R(n)}{2(1-R(n))} & \text{if } R(n) \le t \le 1 \end{cases}$$

for each $n \in \mathbb{N}$. Let

$$f_n(t) = f(g_n(t)).$$

Then the sequence of functions $\{f_n(t)\}$ satisfies all the above-mentioned properties with $p = 1$. Since the damping coefficient $h(t)$ is defined by

$$\phi_{q^*}\left(\frac{1}{h(t)}\right) = \begin{cases} 1 - \dfrac{n}{n+1} \sin^2(\pi g_n(t-2n+1)) & \text{if } t \in I_n, \\ 1 & \text{if } t \notin I_n, \end{cases} \tag{5.2}$$

we see that $h(t_n) = \phi_q(n+1)$, where $t_n = 2n-1+R(n)$. Hence, $h(t)$ is unbounded.

For example, we choose $q = 3$ and

$$\begin{aligned} &R(1) = 0.1945927041728601, && R(2) = 0.8353180866862129, \\ &R(3) = 0.7966404885855496, && R(4) = 0.6071706008592361, \\ &R(5) = 0.2478007450224066, && R(6) = 0.2576346233202667, \\ &R(7) = 0.3445768583669053, && R(8) = 0.4184119536296158, \\ &R(9) = 0.7857282689788707, && \cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{5.3}$$

as random numbers between $1/8$ and $7/8$. Then, the graphs of the functions $1/h(t)$ and $h(t)$ are presented in Figures 1 and 2, respectively.

Since $h(t) \ge 1$ for $t \ge 0$, it follows that $h(t)$ is integrally positive. We define $k(t)$ by $h(t)$. Then, it is clear that $k(t)$ is a right differentiable function and $1/k(t) \le 1$ for $t \ge 0$. Taking $0 < \alpha \le \beta < 1$ into account, we obtain

$$0 < (g_n)'_+(t) \le \max\left\{\frac{1}{2R(n)}, \frac{1}{2(1-R(n))}\right\} \le \max\left\{\frac{1}{2\alpha}, \frac{1}{2(1-\beta)}\right\}$$

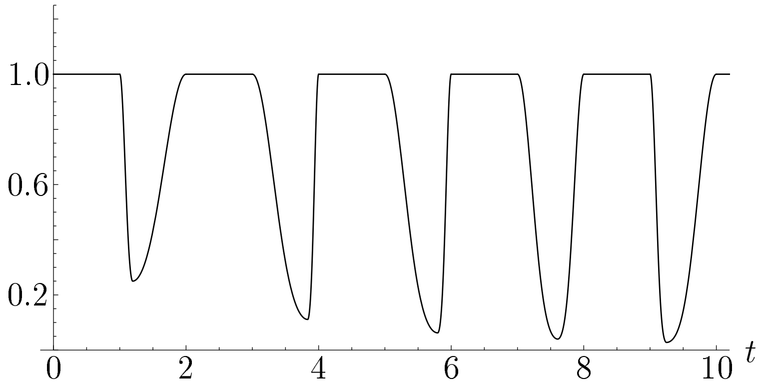

Figure 1: The value of $1/h(t_n)$ approaches zero as $n \to \infty$.

for $t \geq 0$, where $(g_n)'_+(t)$ is the right-hand derivative of $g_n(t)$. Hence, we have

$$\begin{aligned}\left(\frac{1}{k(t)}\right)'_+ &\leq \frac{n\pi}{n+1}|\sin(2\pi g_n(t-2n+1))|(g_n)'_+(t-2n+1)\\ &< \pi \max\left\{\frac{1}{2\alpha}, \frac{1}{2(1-\beta)}\right\}\end{aligned}$$

for $t \geq 0$. For any $n \in \mathbb{N}$, we can estimate that

$$\begin{aligned}\int_{2(n-1)}^{2n} \phi_{q^*}\left(\frac{1}{k(t)}\right) dt &= 2 - \int_{I_n} \frac{n}{n+1} \sin^2(\pi g_n(t-2n+1))\, dt\\ &= 2 - \int_0^1 \frac{n}{n+1} \sin^2(\pi g_n(s))\, ds\\ &= 2 - \int_0^{R(n)} \frac{n}{n+1} \sin^2\left(\frac{\pi s}{2R(n)}\right) ds\\ &\quad - \int_{R(n)}^1 \frac{n}{n+1} \sin^2\left(\frac{\pi(s+1-2R(n))}{2(1-R(n))}\right) ds\\ &> 2 - \frac{1}{2}R(n) - \frac{1}{2}(1-R(n)) = \frac{3}{2}.\end{aligned}$$

For $t > 0$ sufficiently large, there exists an n_1 such that $2(n_1-1) \leq t < 2n_1$. Of course, n_1 is a large integer. Similarly, for any $\sigma \geq 0$ there exists an $n_2 \in \mathbb{N}$ such that $2(n_2-1) \leq \sigma < 2n_2$. Hence, $2n_2 < 2n_1 + 2n_2 - 4 \leq t + \sigma < 2n_1 + 2n_2$ and therefore,

$$\int_\sigma^{t+\sigma} \phi_{q^*}\left(\frac{1}{k(s)}\right) ds > \int_{2n_2}^{2n_1+2n_2-4} \phi_{q^*}\left(\frac{1}{k(s)}\right) ds = 3n_1 - 6 > \frac{3}{2}t - 6. \tag{5.4}$$

For any large number $K > 0$, let

$$T = T(K) > \frac{2}{3}K + 4.$$

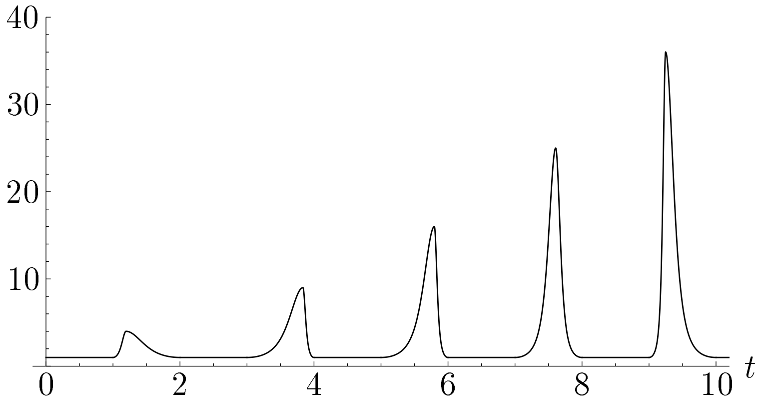

Figure 2: The value of $h(t_n)$ diverges to $+\infty$ as $n \to \infty$.

Then, it turns out from (5.2) that $\sigma \geq 0$ and $t \geq T$ imply

$$\int_{\sigma}^{t+\sigma} \phi_{q^*}\left(\frac{1}{k(s)}\right) ds \geq \int_{\sigma}^{T+\sigma} \phi_{q^*}\left(\frac{1}{k(s)}\right) ds > \frac{3}{2}T - 6 > K.$$

This means that

$$\lim_{t\to\infty} \int_{\sigma}^{t+\sigma} \phi_{q^*}\left(\frac{1}{k(s)}\right) ds = \infty$$

uniformly with respect to $\sigma \geq 0$. Thus, by means of Corollary 4.2, the equilibrium of (1.1) is uniformly globally asymptotically stable.

Finally, we attach a phase portrait of solution curves of

$$x'' + h(t)|x'|x' + x = 0,$$

where $h(t)$ satisfies (5.2) and (5.3). In Figure 3, we draw four solution curves satisfying the initial conditions $(x(t_0), x'(t_0)) = (5,3)$, $(-2,6)$, $(-5,-3)$ and $(2,-6)$, respectively. The initial time t_0 of each solution curve is 0.

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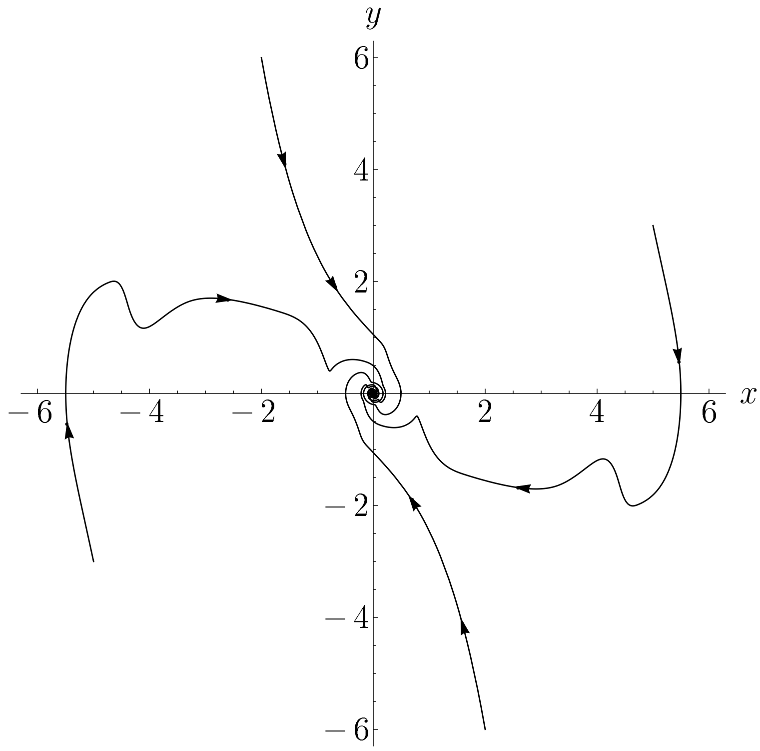

Figure 3: Every solution curve moves round the origin in a clockwise and a counter-clockwise direction by turns and approaches the origin windingly.

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