Monatsh Math manuscript No.
(will be inserted by the editor)

Parameter diagram for global asymptotic stability of damped half-linear oscillators

Wei Zheng · **Jitsuro Sugie**

Received: 13 September 2014 / Accepted: date

Abstract We consider the half-linear differential equation with an unbounded damped term,

$$(\phi_p(x'))' + h(t)\phi_p(x') + \omega^p\phi_p(x) = 0,$$

where $\omega > 0$ and $\phi_p(z) = |z|^{p-2}z$ with $p > 1$. The divergence speed of the damping coefficient $h(t)$ is assumed to be determined by some parameters. By using the relations between the index number p and the parameters, we describe some criteria judging whether the equilibrium of this equation is globally asymptotically stable or not. We also present parameter diagrams to clarify the relations between them.

Keywords Time-varying differential equation · Unbounded damping · Growth condition · Improper integral · Limit comparison test

Mathematics Subject Classification (2010) 34D05 · 34D23 · 37C75 · 93D20

1 Introduction

We consider the half-linear differential equation with a damping term,

$$(\phi_p(x'))' + h(t)\phi_p(x') + \omega^p\phi_p(x) = 0. \tag{1.1}$$

Here, the prime denotes d/dt, the damping coefficient $h(t)$ is continuous and nonnegative for $t \in [a,\infty)$, the number ω is positive, and the function $\phi_p(z)$ is defined by

$$\phi_p(z) = \begin{cases} |z|^{p-2}z & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases} \quad z \in \mathbb{R}$$

W. Zheng
School of Mathematics and Statistics,
Northeast Normal University, Changchun 130024, P. R. China
e-mail: vividloky@yahoo.co.jp

J. Sugie
Department of Mathematics and Computer Science,
Shimane University, Matsue 690-8504, Japan
e-mail: jsugie@riko.shimane-u.ac.jp

with $p > 1$. Note that $\phi_2(z) = z$ for $z \in \mathbb{R}$. Equation (1.1) is a special case of the self-adjoint type

$$(r(t)\phi_p(x'))' + c(t)\phi_p(x) = 0. \tag{1.2}$$

In fact, equation (1.2) coincides with equation (1.1) in the case that $r(t) = e^{H(t)}$ and $c(t) = \omega^p e^{H(t)}$. As known well, the solution space of linear differential equations has two characteristics; namely, (i) the sum of two solutions is another solution; (ii) any constant multiple of a solution is also a solution. On the other hand, the solution space of (1.2) has only the characteristic (ii). This is the reason that equation (1.2) is called 'half-linear'.

The global existence and uniqueness of solutions of (1.1) (or (1.2)) are guaranteed for the initial value problem. For details, see Došlý [5, p. 170] or Došlý and Řehák [8, pp. 8–10]. Studies on equation (1.2) concentrate on the oscillation theory. We can find the results that were published up to 2005 in these books [1, 5, 8] and the references cited therein. Even after that, the research is continuing actively. For example, see [6, 7, 9, 18, 19, 22, 31].

The purpose of this paper is to clarify the growth rate of the damping coefficient $h(t)$ deciding whether the equilibrium is globally asymptotically stable or not. Equation (1.1) has the unique equilibrium $(x(t), x'(t)) \equiv (0, 0)$. Let $t_0 \geq a$ be the initial time and let $\mathbf{x}_0 \in \mathbb{R}^2$ be the initial value; namely, $\mathbf{x}_0 = (x(t_0), x'(t_0))$. For the sake of simplicity, we denote the solution of (1.1) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$.

Using any suitable norm $\|\cdot\|$, we can define as follows. The equilibrium is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq a$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The equilibrium is said to be *attractive* if, for any $t_0 \geq a$, there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \to 0$ as $t \to \infty$. The equilibrium is said to be *globally attractive* if, for any $t_0 \geq a$, any $\eta > 0$ and any $\mathbf{x}_0 \in \mathbb{R}^2$, there is a $T(t_0, \eta, \mathbf{x}_0) > a$ such that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T(t_0, \eta, \mathbf{x}_0)$. The equilibrium is *asymptotically stable* if it is stable and attractive. The equilibrium is *globally asymptotically stable* if it is stable and globally attractive. About the definitions of stability and attractivity, refer to the books [2, 4, 10, 17, 23, 33] for example.

Judging from the importance of pure mathematical theories and of applications to other sciences, it is no exaggeration to say that the study of the global asymptotic stability is one of the major themes in the qualitative theory of differential equations. Since equation (1.1) is a natural generalization of the damped linear oscillator

$$x'' + h(t)x' + \omega^2 x = 0, \tag{1.3}$$

we will call equation (1.1) the *damped half-linear oscillator*. It is well known that if the equilibrium of (1.3) is asymptotically stable, then it is globally asymptotically stable. There are a good many articles about the asymptotic stability for the damped linear oscillator (1.3) and its generalization equations. Historical advancement of this research has been concisely summarized in Hatvani [13, Section 1] and Sugie [27, Section 1]. Among them, we should mention specially about Smith's result [24, Theorems 1 and 2].

Theorem A *Suppose that there exists an $\underline{h} > 0$ such that $h(t) \geq \underline{h}$ for $t \geq a$. Then the equilibrium of* (1.3) *is asymptotically stable if and only if*

$$\int_a^\infty \frac{\int_a^t e^{H(s)} ds}{e^{H(t)}}\, dt = \infty, \tag{1.4}$$

where

$$H(t) = \int_a^t h(s) ds.$$

The criterion (1.4) is the so-called growth condition on $h(t)$. Thereafter, to remove the lower bound $\underline{h}$ of the assumption of $h(t)$, a lot of efforts were poured. For example, Matrosov [16] introduced a family of functions as follows (see also [11]). The damping coefficient $h(t)$ is said to be *integrally positive* if

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n + \lambda < \sigma_n \le \tau_{n+1}$ for some $\lambda > 0$. It is known that $h(t)$ is integrally positive if and only if

$$\liminf_{t\to\infty} \int_t^{t+\gamma} h(s)ds > 0$$

for every $\gamma > 0$. If $h(t)$ has the positive lower bound $\underline{h}$, then $h(t)$ is integrally positive. It is clear that $\sin^2 t$ is an integrally positive function. In this case, Theorem A is inapplicable because the lower bound of $h(t)$ is zero. Using the concept of the integral positivity, Hatvani [12] studied the asymptotic stability for a two-dimensional linear nonautonomous differential system including the damped linear oscillator (1.3). Applying his results to equation (1.3), we obtain the following result (see [12, Corollary 4.3]).

Theorem B *Suppose that $h(t)$ is integrally positive and condition* (1.4) *holds. Then the equilibrium of* (1.3) *is asymptotically stable.*

As to other efforts, refer to [3, 13, 15, 20, 21, 25, 26, 28, 29, 32]. Now, let us come back to (1.1) that is the research subject of this paper. Let p^* be the conjugate number of p; namely,

$$\frac{1}{p} + \frac{1}{p^*} = 1. \tag{1.5}$$

Since the index number p is greater than 1, the conjugate number p^* is also greater than 1. It is easy to check that $p = 2$ if and only if $p^* = 2$, and ϕ_{p^*} is the inverse function of ϕ_p.

Recently, Sugie [27, Theorem 3.5] has presented the following criterion which can judge whether the equilibrium of (1.1) is globally asymptotically stable (in his original result, the lower limit of integration is 0 because the left edge of the domain of $h(t)$ is 0 instead of a).

Theorem C *Suppose that $h(t)$ is integrally positive. Then the equilibrium of* (1.1) *is globally asymptotically stable if and only if*

$$\int_a^{\infty} \phi_{p^*}\left(\frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}\right)dt = \infty. \tag{1.6}$$

Theorem C is a perfect generalization of Theorems A and B.

It is reported that the overdamping phenomenon happens when the damping coefficient $h(t)$ increases rapidly. The phenomenon of overdamping is that a solution converging to a non-zero value exists (for example, see [30, Section 6]). If the overdamping phenomenon occurs, naturally the equilibrium of (1.1) is not globally asymptotically stable. Condition (1.6) as well as (1.4) prohibits too fast growth of the damping coefficient $h(t)$. It is easy to verify that if $h(t)$ is bounded from above, conditions (1.4) and (1.6) are satisfied. Hatvani et al. [14, Theorem 1.1] obtained another necessary and sufficient condition for the asymptotic stability of the equilibrium of (1.3). It may be easier to check their condition than the growth condition (1.4) given by Smith [24]. Their condition shows that if $h(t) = t$, then condition (1.4) holds; if $h(t) = t^2$, then condition (1.4) fails to hold. Then, the following question arises. How much is the upper limit of the growth rate which can guarantee that condition (1.6) (or (1.4)) is satisfied?

In this paper, we would like to answer this question. In Section 3, we will examine three cases that the damping coefficient $h(t)$ is unbounded: (i) $h(t) = t^\ell$, where $\ell > 0$; (ii) $h(t) = t^n + c_1 t^{n-1} + \cdots + c_{n-1} t + c_n$ for t sufficiently large, where n is any integer and $c_i \in \mathbb{R}$ for $1 \le i \le n$; (iii) $h(t) = t^\ell (\log(1+t))^m$, where $\ell > 0$ and $m \in \mathbb{R}$. Needless to say, whether the growth condition (1.6) holds or not is depending on the value of p. We draw the parameter diagram which shows the relation between the index number p and exponents ℓ, m, and n in each case.

2 Convergence and divergence of improper integrals

In general, it is difficult to calculate the improper integral (1.6) directly. For this reason, we need some kind of means to judge whether the growth condition (1.6) is satisfied or not. We use the following results which are well known as 'limit comparison test'.

Lemma 2.1 *Suppose that $f(t) \ge 0$ for $t \ge a$ and there exist numbers $\lambda > 0$ and $\nu > 0$ such that*

$$\nu = \lim_{t\to\infty} t^\lambda f(t). \tag{2.1}$$

Then,

(i) if $\lambda \le 1$, then $\displaystyle\int_a^\infty f(t)dt = \infty$;

(ii) if $\lambda > 1$, then $\displaystyle\int_a^\infty f(t)dt < \infty$.

Lemma 2.2 *Suppose that $f(t) \ge 0$ for $t \ge a$ and there exist numbers $\lambda > 0$, $\mu \in \mathbb{R}$ and $\nu > 0$ such that*

$$\nu = \lim_{t\to\infty} t^\lambda (\log(1+t))^\mu f(t). \tag{2.2}$$

Then,

(i) if $\lambda < 1$, then $\displaystyle\int_a^\infty f(t)dt = \infty$;

(ii) if $\mu \le \lambda = 1$, then $\displaystyle\int_a^\infty f(t)dt = \infty$;

(iii) if $\mu > \lambda = 1$, then $\displaystyle\int_a^\infty f(t)dt < \infty$;

(iv) if $\lambda > 1$, then $\displaystyle\int_a^\infty f(t)dt < \infty$.

Because we can find results of such type in many books of elementary mathematical analysis, we omit the proof of Lemmas 2.1 and 2.2.

3 Main results

To give some answers to our question which was raised in Section 1, we consider three cases that equation (1.1) has an unbounded damping coefficient. In the three cases, the damping coefficients are a power function, a polynomial and another function, respectively.

Let

$$f(t) = \phi_{p^*}\left(\frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}\right).$$

Then we have the following result.

Theorem 3.1 *Consider equation* (1.1) *with* $h(t) = t^\ell$ *and* $\ell > 0$. *Then the equilibrium is globally asymptotically stable if and only if* $\ell \le p-1$.

Proof Let $\lambda = \ell(p^*-1)$. Then we have

$$t^\lambda = (t^\ell)^{p^*-1} = \phi_{p^*}(t^\ell).$$

Using l'Hôpital's rule twice, we obtain

$$\begin{aligned}
\lim_{t\to\infty} t^\lambda f(t) &= \lim_{t\to\infty} \phi_{p^*}\left(\frac{t^\ell \int_0^t e^{\frac{1}{\ell+1}s^{\ell+1}}ds}{e^{\frac{1}{\ell+1}t^{\ell+1}}}\right) = \phi_{p^*}\left(\lim_{t\to\infty} \frac{t^\ell \int_0^t e^{\frac{1}{\ell+1}s^{\ell+1}}ds}{e^{\frac{1}{\ell+1}t^{\ell+1}}}\right) \\
&= \phi_{p^*}\left(\lim_{t\to\infty} \frac{\ell t^{\ell-1}\int_0^t e^{\frac{1}{\ell+1}s^{\ell+1}}ds + t^\ell e^{\frac{1}{\ell+1}t^{\ell+1}}}{t^\ell e^{\frac{1}{\ell+1}t^{\ell+1}}}\right) \\
&= \phi_{p^*}\left(\lim_{t\to\infty} \frac{\ell \int_0^t e^{\frac{1}{\ell+1}s^{\ell+1}}ds}{t\, e^{\frac{1}{\ell+1}t^{\ell+1}}} + 1\right) \\
&= \phi_{p^*}\left(\lim_{t\to\infty} \frac{\ell e^{\frac{1}{\ell+1}t^{\ell+1}}}{e^{\frac{1}{\ell+1}t^{\ell+1}} + t^{\ell+1}e^{\frac{1}{\ell+1}t^{\ell+1}}} + 1\right) \\
&= \phi_{p^*}\left(\lim_{t\to\infty} \frac{\ell}{1+t^{\ell+1}} + 1\right) = \phi_{p^*}(1) = 1.
\end{aligned}$$

Hence, condition (2.1) is satisfied with $\lambda = \ell(p^*-1)$ and $\nu = 1$.

The damping coefficient t^ℓ is integrally positive. Taking into consideration of (1.5), we see that $\ell(p^*-1) \le 1$ if and only if $\ell \le p-1$. Hence, if $\ell \le p-1$, then it follows from Lemma 2.1 (i) that

$$\int_a^\infty f(t)dt = \int_a^\infty \phi_{p^*}\left(\frac{\int_a^t e^{\frac{1}{\ell+1}s^{\ell+1}}ds}{e^{\frac{1}{\ell+1}t^{\ell+1}}}\right)dt = \infty.$$

We therefore conclude that the equilibrium of

$$(\phi_p(x'))' + t^\ell \phi_p(x') + \omega^p \phi_p(x) = 0 \tag{3.1}$$

is globally asymptotically stable, by using Theorem C. Conversely, if $\ell > p-1$, then it follows from Lemma 2.1 (ii) that

$$\int_a^\infty f(t)dt = \int_a^\infty \phi_{p^*}\left(\frac{\int_a^t e^{\frac{1}{\ell+1}s^{\ell+1}}ds}{e^{\frac{1}{\ell+1}t^{\ell+1}}}\right)dt < \infty.$$

By Theorem C again, we see that the equilibrium of (3.1) is not globally asymptotically stable. □

In Figure 1, we draw the parameter diagram for equation (3.1). If (ℓ, p) is included in the domain where a shadow was attached, then the equilibrium of (3.1) is globally asymptotically stable. If (ℓ, p) is on the line $p = \ell + 1$, then the equilibrium of (3.1) is globally

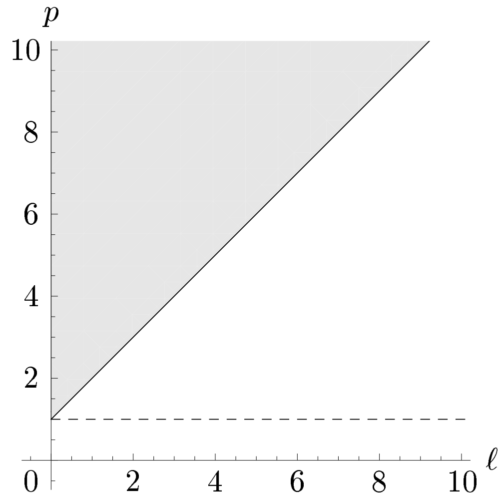

Fig. 1 The parameter diagram of (3.1)

asymptotically stable. If (ℓ, p) is in the other domain where $p > 1$, the equilibrium of (3.1) is not globally asymptotically stable.

Remark 3.1 If $\ell \le 0$, then $h(t) = t^{\ell} \le 1$ for $t \ge 1 = a$. Hence, we can estimate that

$$\begin{aligned}
\int_a^{\infty} \phi_{p^*}\left(\frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}\right)dt &= \int_1^{\infty} \phi_{p^*}\left(\int_1^t e^{-(H(t)-H(s))}ds\right)dt \\
&\ge \int_1^{\infty} \phi_{p^*}\left(\int_1^t e^{-(t-s)}ds\right)dt = \int_1^{\infty} \phi_{p^*}\left(\frac{e^t - e}{e^t}\right)dt \\
&\ge \int_1^{1+\log 2} \phi_{p^*}\left(\frac{e^t - e}{e^t}\right)dt + \int_{1+\log 2}^{\infty} \phi_{p^*}\left(\frac{1}{2}\right)dt = \infty.
\end{aligned}$$

However, in the case that $\ell < 0$, the function t^{ℓ} is not integrally positive. For this reason, we cannot apply Theorem C to this case.

In order to confirm conditions (1.6), the damping coefficient $h(t)$ does not need to be a monomial, like a power function. We next deal with the case where $h(t)$ is a polynomial which diverges to ∞ as $t \to \infty$.

Theorem 3.2 *Let* $g(t) = t^n + c_1 t^{n-1} + \cdots + c_{n-1}t + c_n$, *with* c_i $(1 \le i \le n)$ *real number, and let* b *be a number satisfying*

$$g(t) \ge 0 \quad \text{for } t \ge b \ge a.$$

Then the equilibrium of (1.1) *with*

$$h(t) = \begin{cases} g(t) & \text{if } t \ge b, \\ |g(t)| & \text{if } a \le t < b \end{cases}$$

is globally asymptotically stable if and only if $n \le p - 1$.

Proof Let $\lambda = n(p^*-1)$. Then we have

$$t^\lambda = (t^n)^{p^*-1} = \phi_{p^*}(t^n).$$

Using l'Hôpital's rule, we obtain

$$\begin{aligned}
\lim_{t\to\infty} t^\lambda f(t) &= \lim_{t\to\infty} \phi_{p^*}\left(\frac{t^n\int_0^t e^{H(s)}ds}{e^{H(t)}}\right) = \phi_{p^*}\left(\lim_{t\to\infty}\frac{t^n\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)\\
&= \phi_{p^*}\left(\lim_{t\to\infty}\frac{nt^{n-1}\int_0^t e^{H(s)}ds + t^n e^{H(t)}}{h(t)e^{H(t)}}\right)\\
&= \phi_{p^*}\left(\lim_{t\to\infty}\frac{nt^{n-1}\int_0^t e^{H(s)}ds}{h(t)e^{H(t)}} + \lim_{t\to\infty}\frac{t^n}{h(t)}\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{t\to\infty}\frac{nt^{n-1}\int_0^t e^{H(s)}ds}{h(t)e^{H(t)}} = \lim_{t\to\infty}\frac{n\int_0^t e^{H(s)}ds}{\left(t+c_1+\dfrac{c_2}{t}+\cdots+\dfrac{c_n}{t^{n-1}}\right)e^{H(t)}}\\
&= \lim_{t\to\infty}\frac{n}{1-\dfrac{c_2}{t^2}-\dfrac{2c_3}{t^3}-\cdots-\dfrac{(n-1)c_n}{t^n}+\left(t+c_1+\dfrac{c_2}{t}+\cdots+\dfrac{c_n}{t^{n-1}}\right)h(t)} = 0
\end{aligned}$$

and

$$\lim_{t\to\infty}\frac{t^n}{h(t)} = \lim_{t\to\infty}\frac{1}{1+\dfrac{c_1}{t}+\dfrac{c_2}{t^2}+\cdots+\dfrac{c_n}{t^n}} = 1,$$

we obtain

$$\lim_{t\to\infty} t^\lambda f(t) = \phi_{p^*}(1) = 1.$$

Hence, condition (2.1) is satisfied with $\lambda = n(p^*-1)$ and $\nu = 1$.

The rest of the proof is carried out in the same way as the proof of Theorem 3.1. We leave the detailed analysis to the reader. □

The following result gives us more detailed information on the divergence speed of the damping coefficient $h(t)$ that satisfies condition (1.6).

Theorem 3.3 *Consider equation* (1.1) *with* $h(t) = t^\ell(\log(1+t))^m$, $\ell > 0$ *and* $m \in \mathbb{R}$. *Then the equilibrium is globally asymptotically stable if and only if either*

$$\ell < p-1$$

or

$$m \le \ell = p-1.$$

Proof Let $\lambda = \ell(p^*-1)$ and $\mu = m(p^*-1)$. Then we have

$$t^\lambda(\log(1+t))^\mu = (t^\ell(\log(1+t))^m)^{p^*-1} = \phi_{p^*}(t^\ell(\log(1+t))^m).$$

Using l'Hôpital's rule, we obtain

$$\begin{aligned}\lim_{t\to\infty} t^{\lambda}(\log(1+t))^{\mu} f(t) &= \lim_{t\to\infty}\phi_{p^*}\left(\frac{t^{\ell}(\log(1+t))^m\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)\\ &= \phi_{p^*}\left(\lim_{t\to\infty}\frac{t^{\ell}(\log(1+t))^m\int_0^t e^{H(s)}ds}{e^{H(t)}}\right)\\ &= \phi_{p^*}\left(\lim_{t\to\infty}\frac{\ell\int_0^t e^{H(s)}ds}{t\,e^{H(t)}}+\lim_{t\to\infty}\frac{m\int_0^t e^{H(s)}ds}{(1+t)\log(1+t)e^{H(t)}}+1\right).\end{aligned}$$

Since

$$\begin{aligned}\lim_{t\to\infty}\frac{\ell\int_0^t e^{H(s)}ds}{t\,e^{H(t)}} &= \lim_{t\to\infty}\frac{\ell\, e^{H(t)}}{e^{H(t)}+t^{\ell+1}(\log(1+t))^m e^{H(t)}}\\ &= \lim_{t\to\infty}\frac{\ell}{1+t^{\ell+1}(\log(1+t))^m}=0\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty}\frac{m\int_0^t e^{H(s)}ds}{(1+t)\log(1+t)e^{H(t)}} &= \lim_{t\to\infty}\frac{m}{\log(1+t)+1+t^{\ell}(1+t)(\log(1+t))^{m+1}}\\ &= 0,\end{aligned}$$

we obtain

$$\lim_{t\to\infty} t^{\lambda}(\log(1+t))^{\mu} f(t)=\phi_{p^*}(1)=1.$$

Hence, condition (2.2) is satisfied with $\nu=1$, $\lambda=\ell(p^*-1)$ and $\mu=m(p^*-1)$.

The damping coefficient $t^{\ell}(\log(1+t))^m$ is integrally positive. Taking into consideration of (1.5), we see that $\ell(p^*-1)<1$ if and only if $\ell<p-1$. Hence, if $\ell<p-1$, then it follows from Lemma 2.2 (i) that

$$\int_a^{\infty} f(t)dt=\int_a^{\infty}\phi_{p^*}\left(\frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}\right)dt=\infty.$$

Therefore, using Theorem C, we conclude that the equilibrium of

$$(\phi_p(x'))'+t^{\ell}(\log(1+t))^m\phi_p(x')+\omega^p\phi_p(x)=0. \tag{3.2}$$

is globally asymptotically stable. Also, we see that $m(p^*-1)\le\ell(p^*-1)=1$ if and only if $m\le\ell=p-1$. Hence, if $m\le\ell=p-1$, then it follows from Lemma 2.2 (ii) that

$$\int_a^{\infty} f(t)dt=\int_a^{\infty}\phi_{p^*}\left(\frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}\right)dt=\infty,$$

and therefore, the equilibrium of (3.2) is globally asymptotically stable. The negation of $\ell<p-1$ or $m\le\ell=p-1$ is $\ell\ge p-1$, and $m>\ell$ or $\ell\ne p-1$. From these it turns out that the negation is $\ell>p-1$ or $m>\ell=p-1$. If $\ell>p-1$, then $\lambda=\ell(p^*-1)>1$. Hence, it follows from Lemma 2.2 (iv) that

$$\int_a^{\infty} f(t)dt=\int_a^{\infty}\phi_{p^*}\left(\frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}\right)dt<\infty.$$

If $m > \ell = p-1$, then $\mu > 1 = \lambda$. Hence, it follows from Lemma 2.2 (iii) that

$$\int_a^\infty f(t)dt = \int_a^\infty \phi_{p^*}\left(\frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}\right)dt < \infty.$$

Thus, by Theorem C again, we see that if $\ell > p-1$ or $m > \ell = p-1$, then the equilibrium of (3.2) is not globally asymptotically stable. □

Figure 2 shows the parameter diagram for equation (3.2). We can find two triangle prisms that are joining in Figure 2. If (ℓ, m, p) is included in the triangular prism of the rear side, the equilibrium of (3.2) is globally asymptotically stable. If (ℓ, m, p) is in the planes which attached the mesh, the equilibrium of (3.2) is globally asymptotically stable. If (ℓ, m, p) is included in the triangular prism of this side, the equilibrium of (3.2) is not globally asymptotically stable.

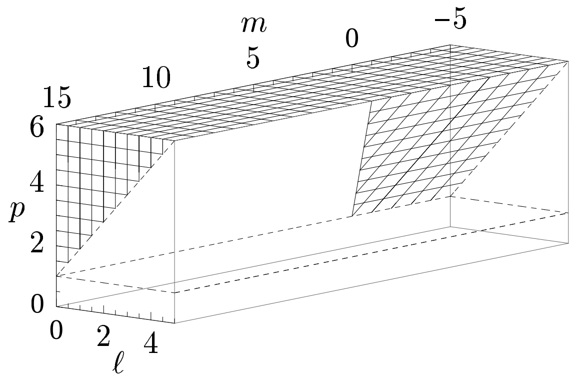

Fig. 2 The parameter diagram of (3.2)

Remark 3.2 Even when $\ell = 0$, the damping coefficient of (3.2) is unbounded if $m > 0$. In this case, it is easy to verify that $(\log(1+t))^m$ is integrally positive and the growth condition (1.6) is satisfied. Hence, by Theorem C, the damping coefficient of (3.2) is globally asymptotically stable.

Acknowledgments This research was supported in part by a Grant-in-Aid for Scientific Research, No. 25400165, from the Japan Society for the Promotion of Science (J. S.).

References

1. Agarwal, R.P., Grace, S.R., O'Regan, D.: Oscillation Theory for Second Order Linear, Half-linear, Superlinear and Sublinear Dynamic Equations. Kluwer Academic Publishers, Dordrecht (2002)
2. Bacciotti, A., Rosier, L.: Liapunov Functions and Stability in Control Theory. 2nd edn, Springer-Verlag, Berlin–Heidelberg–New York (2005)
3. Ballieu, R.J., Peiffer, K.: Attractivity of the origin for the equation $\ddot{x} + f(t,\dot{x},\ddot{x})|\dot{x}|^\alpha \dot{x} + g(x) = 0$. J. Math. Anal. Appl. **65**, 321–332 (1978)
4. Brauer, F., Nohel, J.: The Qualitative Theory of Ordinary Differential Equations. W. A. Benjamin, New York–Amsterdam (1969); (revised) Dover, New York (1989)
5. Došlý, O.: Half-linear differential equations. In: Cañada, A., Drábek, P., Fonda, A. (eds.) Handbook of Differential Equations, Ordinary Differential Equations I, pp. 161–357. Elsevier, Amsterdam (2004)

6. Došlý, O., Fišnarová, S., Mařík, R.: Power comparison theorems in half-linear oscillation theory. J. Math. Anal. Appl. **401**, 611–619 (2013)
7. Došlý, O., Özbekler, A., Šimon Hilscher, R.: Oscillation criterion for half-linear differential equations with periodic coefficients, J. Math. Anal. Appl. **393**, 360–366 (2012)
8. Došlý, O., Řehák, P.: Half-linear Differential Equations. North-Holland Mathematics Studies, vol. 202. Elsevier, Amsterdam (2005)
9. Fišnarová, S., Mařík, R.: Half-linear ODE and modified Riccati equation: comparison theorems, integral characterization of principal solution. Nonlinear Anal. **74**, 6427–6433 (2011)
10. Hale, J.K.: Ordinary Differential Equations. Wiley-Interscience, New York, London and Sydney (1969); (revised) Krieger, Malabar (1980)
11. Hatvani, L.: A generalization of the Barbashin-Krasovskij theorems to the partial stability in nonautonomous systems. In: Farkas, M. (ed.) Qualitative Theory of Differential Equations I, Colloq. Math. Soc. János Bolyai, vol. 30, pp. 381–409. North-Holland, Amsterdam and New York (1981)
12. Hatvani, L.: On the asymptotic stability for a two-dimensional linear nonautonomous differential system. Nonlinear Anal. **25**, 991-1002 (1995)
13. Hatvani, L.: Integral conditions on the asymptotic stability for the damped linear oscillator with small damping. Proc. Amer. Math. Soc. **124**, 415–422 (1996)
14. Hatvani, L., Krisztin, T., Totik, V.: A necessary and sufficient condition for the asymptotic stability of the damped oscillator. J. Differ. Equ. **119**, 209–223 (1995)
15. Hatvani, L., Totik, V.: Asymptotic stability of the equilibrium of the damped oscillator. Differ. Integral Eqn. **6**, 835–848 (1993)
16. Matrosov, V.M.: On the stability of motion. Prikl. Mat. Meh. **26**, 885–895 (1962); translated as J. Appl. Math. Mech. **26**, 1337–1353 (1963)
17. Michel, A.N., Hou, L., Liu, D.: Stability Dynamical Systems: Continuous, Discontinuous, and Discrete Systems. Birkhäuser, Boston–Basel–Berlin (2008)
18. Pašić, M., Wong, J.S.W.: Rectifiable oscillations in second-order half-linear differential equations. Ann. Mat. Pura Appl. (4) **188**, 517–541 (2009)
19. Pátíková, Z.: Asymptotic formulas for nonoscillatory solutions of conditionally oscillatory half-linear equations. Math. Slovaca **60**, 223–236 (2010)
20. Pucci, P., Serrin, J.: Precise damping conditions for global asymptotic stability for nonlinear second order systems. Acta Math. **170**, 275–307 (1993)
21. Pucci, P., Serrin, J.: Asymptotic stability for intermittently controlled nonlinear oscillators. SIAM J. Math. Anal. **25**, 815–835 (1994)
22. Řehák, P.: De Haan type increasing solutions of half-linear differential equations. J. Math. Anal. Appl. **412**, 236–243 (2014)
23. Rouche, N., Habets, P., Laloy, M.: Stability Theory by Liapunov's Direct Method. Applied Mathematical Sciences, vol. 22. Springer-Verlag, New York (1977)
24. Smith, R.A.: Asymptotic stability of $x'' + a(t)x' + x = 0$. Quart. J. Math. Oxford (2) **12**, 123–126 (1961)
25. Sugie, J.: Convergence of solutions of time-varying linear systems with integrable forcing term. Bull. Austral. Math. Soc. **78**, 445–462 (2008)
26. Sugie, J.: Influence of anti-diagonals on the asymptotic stability for linear differential systems. Monatsh. Math. **157**, 163–176 (2009)
27. Sugie, J.: Global asymptotic stability for damped half-linear oscillators. Nonlinear Anal. **74**, 7151–7167 (2011)
28. Sugie, J.: Smith-type criterion for the asymptotic stability of a pendulum with time-dependent damping. Proc. Amer. Math. Soc. **141**, 2419–2427 (2013)
29. Sugie, J., Endo, A.: Attractivity for two-dimensional linear systems whose anti-diagonal coefficients are periodic. Proc. Amer. Math. Soc. **137**, 4117–4127 (2009)
30. Sugie, J., Hata, S.: Global asymptotic stability for half-linear differential systems with generalized almost periodic coefficients. Monatsh. Math. **166**, 255–280 (2012)
31. Sugie, J., Matsumura, K.: A nonoscillation theorem for half-linear differential equations with periodic coefficients. Appl. Math. Comput. **199**, 447–455 (2008)
32. Surkov, A.G.: Asymptotic stability of certain two-dimensional linear systems. Differentsial'nye Uravneniya **20**, 1452–1454 (1984)
33. Yoshizawa, T.: Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions. Applied Mathematical Sciences, vol. 14. Springer-Verlag, New York (1975)