A necessary and sufficient condition for global asymptotic stability of time-varying Lotka-Volterra predator-prey systems

Wei Zheng[a], Jitsuro Sugie[*,b]

[a]*School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024 Jilin, People's Republic of China*
[b]*Department of Mathematics, Shimane University, Matsue 690-8504, Japan*

Abstract

The purpose of this paper is to present a necessary and sufficient condition which guarantees that an interior equilibrium of a certain predator-prey system is globally asymptotically stable. This ecological system is a model of Lotka-Volttera type whose prey population receives time-variation of the environment. We assume that the time-varying coefficient is weakly integrally positive and has a weaker property than uniformly continuous. Our necessary and sufficient condition is expressed by an improper double integral on the time-varying coefficient. Our work is inspired by the study of the stability theory for damped linear oscillators.

Key words: Global asymptotic stability; Lotka-Volterra predator-prey model; Weakly integrally positive; Time-varying system
2010 MSC: 34C15; 34D05; 34D23; 37B25; 92D25

1. Introduction

Population ecology is one of major subfields of biology. The main subject is to elucidate population dynamics of species. Expectation of population (or density) in the ecosystem is an extremely important issue for human being society. Knowing the exact trends of population dynamics is essential for environmental protection. For this reason, many mathematicians, statisticians and ecologists have competed fiercely in the study of population dynamics. The basic idea of these studies is to extract a logical mathematical model from the ecosystem.

Let us put aside single species population models, such as the Malthusian growth model, and the logistic model provided by Pierre François Verhulst. When discussing the

*Corresponding author
Email addresses: vividloky@yahoo.co.jp (Wei Zheng), jsugie@riko.shimane-u.ac.jp (Jitsuro Sugie)

relationship between the population of two species, we should quote the famous Lotka-Volterra model

$$N' = (a - bP)N,$$
$$P' = (-c + dN)P, \tag{LV}$$

where $' = d/dt$; N and P represent the prey population (or density) and the predator population (or density) respectively, and a, b, c and d are positive parameters: (i) a is the growth rate of prey; (ii) b is the rate at which predators destroy prey; (iii) c is the death rate of predators; (iv) d is the rate at which predators increase by consuming prey. Alfred J. Lotka has proposed a model describing the quantitative relationship between host and parasite (or prey and predator) for the first time. By taking into account of the change of fish catches of prey and predator in the Adriatic, Vito Volterra derived model (LV) independently of Lotka.

This model has a single interior equilibrium $\mathbf{x}^* = (c/d, a/b)$. Let $(N(t), P(t))$ be any solution of (LV). Then, the first integral of (LV) is given by the expression

$$c \ln N(t) - dN(t) + a \ln P(t) - bP(t) = \alpha,$$

where α is an arbitrary constant. Hence, the interior equilibrium $\mathbf{x}^*$ is a neutrally stable fixed point; namely, it is surrounded by a family of periodic orbits whose amplitudes depend on the initial datum of the prey population and the predator population.

Although model (LV) is simple and easy to handle, it has a weakness. Since the interior equilibrium $\mathbf{x}^*$ is neutrally stable, if the prey population or the predator population change suddenly for some reason, then the population state cannot return to the original state. In this sense, model (LV) is said to be structurally unstable. However, nature is more flexible and keeps harmony. It has been often reported that the population state will return to the original state as time passes. It is said that this model is undesirable from this meaning. Thus, researchers understood that model (LV) was not able to simulate the operation of nature appropriately; in other words, some factors that provide the balance of nature have been ignored in model (LV). Then, researchers have paid various efforts to find the neglected factors.

Crawford S. Holling paid his attention to the capture rate of prey per predator. This rate is called a *functional response* of predator to prey. The orthodox functional responses are generally classified into three types, which are named Holling's type I, II and III (for example, see [13, 16, 17]). There are different kinds of the functional response such as Ivlev type (about the result of Ivlev type, refer to [12]). Afterwards, the idea that the functional response is also influenced by the predator population has arisen. Recently, we can find many papers concerning analysis of ecological models with the ratio-dependent type, Beddington-DeAngelis type, Crowley-Martin type, Hassell-Varley type, Leslie-Gower type and so on.

The viewpoint that environment changes over time was disregarded in model (LV). However, it is safe to say that the time-variation of the environment is an important factor to expect the population of species. First of all, it is thought that the seasonal variation influences population dynamics. It is difficult to assume that the birth rate and the death rate are constant like model (LV). To be precise, both rates receive the seasonal variation and they change.

By assuming that the birth rate and carrying capacity for the prey are particularly sensitive to time-variation of the environment, in this paper, we consider the time-varying Lotka-Volterra predator-prey system

$$\begin{aligned} N' &= (a + ch(t) - dh(t)N - bP)N, \\ P' &= (-c + dN)P, \end{aligned} \tag{E}$$

where $' = d/dt$; N and P represent the prey population (or density) and the predator population (or density), respectively; the function h is nonnegative and locally integrable, and a, b, c and d are positive constants. Model (E) also has a unique interior equilibrium $\mathbf{x}^* = (c/d, a/b)$. Since N and P are populations of two species, we have only to consider model (E) in the region

$$R \stackrel{\text{def}}{=} \{(N, P) \colon N > 0 \text{ and } P > 0\}.$$

Needless to say, the region R is a positive invariant set of (E).

Let $t_0 \geq 0$ be the initial time and let $\mathbf{x}_0 = (N_0, P_0) \in R$ be the initial data; namely, $(N_0, P_0) = (N(t_0), P(t_0))$. For the sake of simplicity, we denote the solution $(N(t), P(t))$ of (E) through $\mathbf{x}_0$ at $t = t_0$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$.

Let $\|\cdot\|$ be any suitable norm. The interior equilibrium $\mathbf{x}^*$ is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0) - \mathbf{x}^*\| < \varepsilon$ for all $t \geq t_0$. The interior equilibrium is said to be *globally attractive* if, for any $t_0 \geq 0$, any $\eta > 0$ and any $\mathbf{x}_0 \in \mathbb{R}^2$, there is a $T(t_0, \eta, \mathbf{x}_0) > 0$ such that $\|\mathbf{x}(t; t_0, \mathbf{x}_0) - \mathbf{x}^*\| < \eta$ for all $t \geq t_0 + T(t_0, \eta, \mathbf{x}_0)$. The interior equilibrium is *globally asymptotically stable* if it is stable and globally attractive. About the definitions of stability and attractivity, refer to the books [1, 2, 8, 9, 20] for example.

To describe a result concerning the global asymptotic stability of (E), we introduce a family of functions. We say that the function h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$ if

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n < \sigma_n < \tau_{n+1}$,

$$\liminf_{n\to\infty}(\sigma_n - \tau_n) > 0 \quad \text{and} \quad \limsup_{n\to\infty}(\tau_{n+1} - \sigma_n) < \infty.$$

The concept of the weak integral positivity was first published in Hatvani [3]. It is clear that if the function h has a positive lower bound, then h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. There is a possibility that h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$ even if $\liminf_{t\to\infty} h(t) = 0$. For example, $1/(1 + t) \in \mathcal{F}_{[\mathrm{WIP}]}$ and $\sin^2 t/(1 + t) \in \mathcal{F}_{[\mathrm{WIP}]}$ (for the proof, see [15, Proposition 2.1]). Sugie et al. [19] obtained the following result (see also [10, 18]).

Theorem A. *Suppose that there exists an* $\overline{h}$ *such that* $0 \le h(t) \le \overline{h}$ *for* $t \ge 0$*. If the function* h *belongs to* $\mathcal{F}_{[\mathrm{WIP}]}$*, then the interior equilibrium* $(c/d, a/b)$ *of* (E) *is globally asymptotically stable.*

Model (E) approaches model (LV) as time t passes in the case that $\liminf_{t\to\infty} h(t) = 0$; namely, the limiting system of (E) is model (LV) which is structurally unstable. From Theorem A, we see that if $h \in \mathcal{F}_{[\mathrm{WIP}]}$, then the interior equilibrium of (E) can be stabilized even in this case. However, Theorem A gives only sufficient conditions which guarantee that the interior equilibrium of (E) is globally asymptotically stable. Then, what condition will be necessary? We give an answer to this question and present a necessary and sufficient condition under weak assumptions.

Theorem 1. *Suppose that there exist an* $\varepsilon_0 > 0$ *and a* $\delta_0 > 0$ *such that* $|h(t) - h(s)| < \varepsilon_0$ *for all* $t \ge 0$ *and* $s \ge 0$ *with* $|t - s| < \delta_0$ *and suppose that the function* h *belongs to* $\mathcal{F}_{[\mathrm{WIP}]}$*. Then the interior equilibrium* $(c/d, a/b)$ *of* (E) *is globally asymptotically stable if and only if*

$$\int_0^\infty \frac{\int_0^t e^{H(s)} ds}{e^{H(t)}} dt = \infty, \tag{1}$$

where

$$H(t) = \int_0^t h(s) ds.$$

Remark 1. To show that the interior equilibrium $(c/d, a/b)$ of (E) is stable (to be precise, uniformly stable), it is enough only to assume that the function h is nonnegative (for the proof, see [19, Proposition 2]). Hence, we may say that Theorem 1 gives a necessary and sufficient condition for the interior equilibrium to be globally attractive.

Remark 2. Suppose that there exists an $\overline{h} > 0$ such that $0 \le h(t) \le \overline{h}$ for $t \ge 0$. Then, $|h(t) - h(s)| \le |h(t)| + |h(s)| \le 2\overline{h}$ for all $t \ge 0$ and $s \ge 0$. Hence, in Theorem 1, the first assumption of $h(t)$ is satisfied with respect to $\varepsilon_0 = 2\overline{h}$ and any $\delta_0 > 0$. Clearly, the converse is not always true. The first assumption may be satisfied even if h is a discontinuous function. For example, if the function h is a step function such as

$$h(t) = \begin{cases} 1 & \text{if } 2n-2 \le t < 2n-1, \\ 1/2 & \text{if } 2n-1 \le t < 2n \end{cases}$$

for $n \in \mathbb{N}$, then the first assumption holds. Of course, $h(t)$ belongs to $\mathcal{F}_{[\mathrm{WIP}]}$.

2. Related research

Many attempts have been made to find good conditions for judging whether the origin $(0, 0)$ of the damped linear oscillator

$$x'' + h(t)x' + \omega^2 x = 0$$

is asymptotically stable or not. The research of this theme originated from the works of Levin and Nohel [7] and Smith [11]. Progress of the history of this research is briefly summarized in Hatvani [4, Section 1] and Sugie [14, Section 1]. The damped linear oscillator is equivalent to the system

$$\begin{aligned} x' &= \omega y, \\ y' &= -\omega x - h(t)y. \end{aligned} \tag{2}$$

Sugie [14] discussed the problem about the asymptotic stability of nonlinear systems including (2). By applying his result to system (2), we can derive the following necessary and sufficient condition.

Theorem B. *Suppose that the function h is uniformly continuous and nonnegative, and it belongs to* $\mathcal{F}_{[\mathrm{WIP}]}$. *Then the zero solution of* (2) *is asymptotically stable if and only if condition* (1) *holds.*

Theorem B covers many previous researches for system (2). It is well known that if the zero solution of (2) is asymptotically stable, then it is globally asymptotically stable. Since $h(t) \geq 0$ for $t \geq 0$, the integral $H(t)$ is increasing for $t \geq 0$ (needless to say, it is not necessarily strictly increasing). Define

$$H^{-1}(r) = \min\{t \in \mathbb{R}\colon H(t) \geq r\}.$$

Then, the inverse function $H^{-1}(r)$ is strictly increasing for $r \geq 0$ (it may be discontinuous). Hatvani et al. [5] showed that condition (1) is equivalent to

$$\sum_{n=1}^{\infty} \left(H^{-1}(\kappa n) - H^{-1}(\kappa(n-1)) \right)^2 = \infty \tag{3}$$

for any $\kappa > 0$, provided that $H(t)$ tends to ∞ as $t \to \infty$. Note that $h \in \mathcal{F}_{[\mathrm{WIP}]}$ implies

$$\lim_{t \to \infty} H(t) = \infty. \tag{4}$$

Using their method, we can prove the following equivalence relation.

Lemma 2. *Under assumption* (4), *condition* (1) *holds if and only if*

$$\int_0^{\infty} \frac{\int_0^t e^{\rho H(s)} ds}{e^{\rho H(t)}} dt = \infty \tag{5}$$

for any $\rho > 0$.

From the equivalence of conditions (1) and (3), in order to prove Lemma 2 we have only to show that condition (3) is equivalent to condition (5) (for the proof, see Appendix).

Let

$$x = -\ln(bP/a) \quad \text{and} \quad y = -\ln(dN/c).$$

Then model (E) becomes the system

$$\begin{aligned} x' &= c(1 - e^{-y}), \\ y' &= -a(1 - e^{-x}) - ch(t)(1 - e^{-y}). \end{aligned} \tag{6}$$

System (6) has the zero solution $(x(t), y(t)) \equiv (0, 0)$, which corresponds to the interior equilibrium $(c/d, a/b)$ of (E). This transformation is a one-to-one correspondence from the region R to the whole real plane $\{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$. Hence, the interior equilibrium $(c/d, a/b)$ of (E) is globally attractive if and only if every solution $(x(t), y(t))$ of (6) tends to $(0, 0)$ as $t \to \infty$.

By means of the above-mentioned transformation, we can rewrite Theorem 1 as follows.

Proposition 3. *Suppose that there exist an* $\varepsilon_0 > 0$ *and a* $\delta_0 > 0$ *such that* $|h(t) - h(s)| < \varepsilon_0$ *for all* $t \geq 0$ *and* $s \geq 0$ *with* $|t - s| < \delta_0$ *and suppose that the function h belongs to* $\mathcal{F}_{[\mathrm{WIP}]}$. *Then the zero solution of* (6) *is globally asymptotically stable if and only if condition* (1) *holds.*

Taking into account the fact that $1 - e^{-x} \approx x$ and $1 - e^{-y} \approx y$ for $|x|$ and $|y|$ sufficient small, we notice that systems (2) and (6) are very similar. From this viewpoint, we will prove the main theorem. However, since Proposition 3 provides us with the global property of the solutions of (6), we cannot give the proof of the main theorem only considering system (2) as the linear approximation of (6).

Obviously, conditions of Theorem B require more of function h than those of Proposition 3.

3. Proof of the main result

Before proving Proposition 3 which is equivalent to Theorem 1, it is useful to examine some properties of the function

$$f(z) = e^{-z} + z - 1$$

and its derivative

$$g(z) \stackrel{\text{def}}{=} \frac{d}{dz} f(z) = 1 - e^{-z}$$

for $z \in \mathbb{R}$. It is clear that $f(z)$ is strictly increasing for $z \geq 0$ and strictly decreasing for $z \leq 0$, and $f(0) = 0$. Let

$$w = \hat{f}(z) \stackrel{\text{def}}{=} f(z) \operatorname{sgn} z,$$

and $\hat{f}^{-1}(w)$ be the inverse function of $\hat{f}(z)$. Needless to say, $\hat{f}^{-1}(w)$ is strictly increasing for $w \in \mathbb{R}$ and $\hat{f}^{-1}(0) = 0$. Since

$$\frac{d}{dz}(f(z) - f(-z)) = g(z) + g(-z) = 2 - (e^{z} + e^{-z}) \leq 0$$

for $z \in \mathbb{R}$, with equality if and only if $z = 0$, we see that

$$f(z_2) - f(z_1) < f(-z_2) - f(-z_1) \quad \text{for } z_1 < z_2 \tag{7}$$

and

$$f(z) < f(-z) \quad \text{for } z > 0.$$

From the second inequality it follows that

$$0 \le f(z) \le f(-\alpha) \quad \text{for } |z| \le \alpha \tag{8}$$

with α positive and

$$0 < -\hat{f}^{-1}(-w) < \hat{f}^{-1}(w) \quad \text{for } w > 0. \tag{9}$$

It is clear that $g(z)$ is strictly increasing for $z \in \mathbb{R}$ with $g(0) = 0$, $\lim_{z\to\infty} g(z) = 1$, $\lim_{z\to-\infty} g(z) = -\infty$. Since $g(z) + g(-z) \le 0$ for $z \in \mathbb{R}$, with equality if and only if $z = 0$, we see that

$$g^2(z) \ge g^2(\alpha) \quad \text{for } |z| \ge \alpha \tag{10}$$

and

$$|g(z)| \le |g(-\alpha)| \quad \text{for } |z| \le \alpha \tag{11}$$

with α positive.

We are now ready to prove Proposition 3.

Proof of Proposition 3. Let $(x(t), y(t))$ be any solution of (6) with the initial time $t_0 \ge 0$ and let $(x_0, y_0) = (x(t_0), y(t_0))$. As mentioned in Remark 1, the interior equilibrium $(c/d, a/b)$ of (E) is stable. Since the zero solution of (6) corresponds to the interior equilibrium $(c/d, a/b)$ of (E), the zero solution is also stable; that is, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $|x_0| + |y_0| < \delta$ implies $|x(t)| + |y(t)| < \varepsilon$ for all $t \ge t_0$. Hence, to prove Proposition 3, we have only to check whether every solution of (6) tends to the origin or not.

Necessity. We will show that there exists a solution of (6) which does not approach the origin provided that

$$\int_0^\infty \frac{\int_0^t e^{H(s)} ds}{e^{H(t)}}\, dt < \infty.$$

From Lemma 2 with $\rho = c$, we see that

$$\int_0^\infty \frac{\int_0^t e^{cH(s)} ds}{e^{cH(t)}}\, dt < \infty.$$

Hence, we can choose a $T \ge 0$ so large that

$$\int_T^\infty \frac{\int_0^t e^{cH(s)} ds}{e^{cH(t)}}\, dt < \frac{1}{2ac(e-1)}. \tag{12}$$

Let $\delta^* = \delta(1, T)/2$ and consider the solution $(\tilde{x}(t), \tilde{y}(t))$ of (6) that passes through $(\delta^*, 0)$ at $t = T$. Then

$$|\tilde{x}(t)| + |\tilde{y}(t)| < 1 \quad \text{for} \;\; t \ge T.$$

Since $\tilde{x}'(T) = 0$ and $\tilde{y}'(T) < 0$, it turns out that $(\tilde{x}(t), \tilde{y}(t))$ enters the fourth quadrant

$$Q_4 \stackrel{\text{def}}{=} \{(x, y) \colon x > 0 \text{ and } y < 0\}$$

in a right-hand neighborhood of $t = T$. Taking account of the vector field of (6) on the positive x-axis, we see that $(\tilde{x}(t), \tilde{y}(t))$ does not move to the first quadrant

$$Q_1 \stackrel{\text{def}}{=} \{(x, y) \colon x > 0 \text{ and } y > 0\}$$

from Q_4 directly as t increases.

Suppose that there exists a $T^* > T$ such that $\tilde{x}(T^*) = \delta^*/2$ and $\delta^*/2 < \tilde{x}(t) \le \delta^*$ for $T \le t < T^*$. Then, from the above-mentioned property of $(\tilde{x}(t), \tilde{y}(t))$ it follows that $-1 < \tilde{y}(t) \le 0$ for $T \le t < T^*$. Note that

$$1 - e^{-x} \le x \quad \text{for} \;\; x \in \mathbb{R}$$

and

$$(e-1)y \le 1 - e^{-y} \le y \quad \text{for} \;\; -1 < y \le 0. \tag{13}$$

Since

$$\begin{aligned} \tilde{y}'(t) + ch(t)\tilde{y}(t) &\ge \tilde{y}'(t) + ch(t)\left(1 - e^{-\tilde{y}(t)}\right) = -a\left(1 - e^{-\tilde{x}(t)}\right) \\ &\ge -a\tilde{x}(t) \ge -a\delta^* \end{aligned}$$

for $T \le t < T^*$, we obtain

$$(e^{cH(t)}\tilde{y}(t))' \ge -a\delta^* e^{cH(t)} \quad \text{for} \;\; T \le t < T^*.$$

Integrate both sides of this inequality from T to $t < T^*$ to obtain

$$e^{cH(t)}\tilde{y}(t) \ge e^{cH(T)}\tilde{y}(T) - a\delta^* \int_T^t e^{cH(s)} ds = -a\delta^* \int_T^t e^{cH(s)} ds.$$

Hence, by (13) we have

$$\tilde{x}'(t) = c\left(1 - e^{-\tilde{y}(t)}\right) \ge c(e-1)\tilde{y}(t) \ge -ac(e-1)\delta^* \frac{\int_T^t e^{cH(s)} ds}{e^{cH(t)}}$$

for $T \le t < T^*$. From this estimation and (12) it follows that

$$\begin{aligned} \tilde{x}(T^*) &\ge \tilde{x}(T) - ac(e-1)\delta^* \int_T^{T^*} \frac{\int_T^t e^{cH(s)} ds}{e^{cH(t)}} dt \\ &\ge \delta^* - ac(e-1)\delta^* \int_T^{\infty} \frac{\int_T^t e^{cH(s)} ds}{e^{cH(t)}} dt \\ &\ge \delta^* - ac(e-1)\delta^* \int_T^{\infty} \frac{\int_0^t e^{cH(s)} ds}{e^{cH(t)}} dt > \frac{\delta^*}{2}. \end{aligned}$$

This contradicts the assumption that $\tilde{x}(T^*) = \delta^*/2$. We therefore conclude that $\tilde{x}(t) > \delta^*/2$ for $t \geq T$. This fact means that the solution $(\tilde{x}(t), \tilde{y}(t))$ of (6) does not tend to the origin as $t \to \infty$. Thus, the zero solution of (6) is not globally attractive.

Sufficiency. Let

$$v(t) = af(x(t)) + cf(y(t)) \tag{14}$$

for any solution $(x(t), y(t))$ of (6). Then, we have

$$v'(t) = ag(x(t))x'(t) + cg(y(t))y'(t) = -c^2h(t)g^2(y(t)) \leq 0$$

for $t \geq t_0$. Hence, $v(t)$ is decreasing for $t \geq t_0$. Since $v(t) \geq 0$ for $t \geq t_0$, there exists a limiting value $v^* \geq 0$ of $v(t)$. If $v^* = 0$, then it follows from (14) that the solution $(x(t), y(t))$ of (6) tends to $(0, 0)$ as $t \to \infty$. This is our desired conclusion. Thus, we have only to show that the case in which $v^* > 0$ does not occur.

By way of contradiction, we suppose that v^* is positive. Then, there exists a $T_1 \geq t_0$ such that

$$0 < v^* \leq v(t) \leq 2v^* \quad \text{for} \;\; t \geq T_1. \tag{15}$$

Hence, it follows from (14) that

$$\hat{f}^{-1}(-2v^*/a) \leq x(t) \leq \hat{f}^{-1}(2v^*/a) \quad \text{and} \quad \hat{f}^{-1}(-2v^*/c) \leq y(t) \leq \hat{f}^{-1}(2v^*/c)$$

for $t \geq T_1$. From (9) it turns out that

$$|x(t)| \leq \hat{f}^{-1}(2v^*/a) \quad \text{and} \quad |y(t)| \leq \hat{f}^{-1}(2v^*/c) \tag{16}$$

for $t \geq T_1$.

Hereafter, we will complete the proof of sufficiency in two steps. In the first step, we show that $y(t)$ approaches zero as $t \to \infty$. If $\lim_{t\to\infty} y(t) = 0$, then from (14) we see that $\lim_{t\to\infty} x(t) = \hat{f}^{-1}(v^*/a) \overset{\text{def}}{=} \beta_1 > 0$ or $\lim_{t\to\infty} x(t) = \hat{f}^{-1}(-v^*/a) \overset{\text{def}}{=} -\beta_2 < 0$. In the second step, we will show that $\beta_1 = \beta_2 = 0$. This is a contradiction.

Since $|y(t)|$ is bounded, it has an inferior limit and a superior limit. In the first step, we show that the inferior limit is zero, and then show that the superior limit is also zero. In the second step, we examine the movement of $(x(t), y(t))$ in the whole x-y plane in details.

Step (1): We first suppose that $\liminf_{t\to\infty} |y(t)| > 0$. Then, we can choose a $\gamma > 0$ and a $T_2 \geq t_0$ such that $|y(t)| > \gamma$ for $t \geq T_2$. From (10), we see that

$$v'(t) = -c^2h(t)g^2(y(t)) \leq -c^2g^2(\gamma)h(t)$$

for $t \geq T_2$. Integrating this inequality from t_0 to t, we can evaluate as follow:

$$v^* - v(t_0) \leq v(t) - v(t_0) = \int_{t_0}^{t} v'(s)ds \leq -c^2g^2(\gamma)\int_{T_2}^{t} h(s)ds.$$

Recall that $h \in \mathcal{F}_{[\text{WIP}]}$ implies condition (4) holds. Then, we see that the evaluation above is not right. Thus, we conclude that $\liminf_{t\to\infty} |y(t)| = 0$.

Next, we suppose that $\limsup_{t\to\infty} |y(t)| \stackrel{\text{def}}{=} \lambda > 0$. Let ε be so small enough as to satisfy the inequalities $0 < \varepsilon < \min\{\lambda/2, -\hat{f}^{-1}(-v^*/c)/2\}$,

$$\frac{4\varepsilon}{a\delta_0} + \frac{c(1+2\varepsilon_0)}{a}|g(-2\varepsilon)| < 1 - \exp\left(-f^{-1}\left(\frac{v^* - cf(-2\varepsilon)}{a}\right)\right) \tag{17}$$

and

$$\frac{4\varepsilon}{a\delta_0} + \frac{c(1+2\varepsilon_0)}{a}|g(-2\varepsilon)| < \exp\left(-f^{-1}\left(\frac{cf(-2\varepsilon) - v^*}{a}\right)\right) - 1. \tag{18}$$

Note that $f(-2\varepsilon) > 0$ and $g(-2\varepsilon) < 0$ for $\varepsilon > 0$, and $f(-2\varepsilon)$ and $g(-2\varepsilon)$ approach zero as $\varepsilon \to 0$. Hence, we can find a positive number ε which satisfies (17) and (18).

We can choose three sequences $\{s_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ with $T_1 < \tau_n < s_n < \sigma_n \le \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$ such that $|y(s_n)| = 2\varepsilon$, $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$ and

$$|y(t)| \ge \varepsilon \quad \text{for} \ \ \tau_n < t < \sigma_n, \tag{19}$$

$$|y(t)| \le 2\varepsilon \quad \text{for} \ \ \sigma_n < t < \tau_{n+1}, \tag{20}$$

$$\varepsilon < |y(t)| < 2\varepsilon \quad \text{for} \ \ \tau_n < t < s_n. \tag{21}$$

In fact, since the inferior limit of $|y(t)|$ is zero, there exists a $t_* > T_1$ such that $|y(t_*)| < \varepsilon$. Because $\limsup_{t\to\infty} |y(t)| = \lambda > 2\varepsilon$, we can choose numbers s_1, τ_1 and σ_1 such that

$$s_1 = \inf\{t > t_* : |y(t)| > 2\varepsilon\}, \quad \tau_1 = \sup\{t < s_1 : |y(t)| < \varepsilon\}$$

and

$$\sigma_1 = \inf\{t > s_1 : |y(t)| < \varepsilon\}.$$

It is clear that $|y(s_1)| = 2\varepsilon$, $|y(\tau_1)| = |y(\sigma_1)| = \varepsilon$ and $|y(t)| \ge \varepsilon$ for $\tau_1 < t < \sigma_1$. Using σ_1 instead of t_*, we define s_2, τ_2 and σ_2 similarly to s_1, τ_1 and σ_1, and so on. Then, we obtain three sequences $\{s_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ with $n \in \mathbb{N}$ such that

$$s_n = \inf\{t > \sigma_{n-1} : |y(t)| > 2\varepsilon\}, \quad \tau_n = \sup\{t < s_n : |y(t)| < \varepsilon\}$$

and

$$\sigma_n = \inf\{t > s_n : |y(t)| < \varepsilon\}.$$

It is also clear that $|y(s_n)| = 2\varepsilon$, $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$,

$$|y(t)| \ge \varepsilon \quad \text{for} \ \ \tau_n < t < \sigma_n,$$

$$|y(t)| \le 2\varepsilon \quad \text{for} \ \ \sigma_n < t < \tau_{n+1}$$

and

$$\varepsilon < |y(t)| < 2\varepsilon \quad \text{for} \ \ \tau_n < t < s_n.$$

Hence, the inequalities (19)–(21) are satisfied.

Let us evaluate the distance between τ_n and s_n for $n \in \mathbb{N}$. From (21) and the continuity of $y(t)$, we see that there are two cases to be considered: (i) $y(\tau_n) = \varepsilon < y(t) < 2\varepsilon = y(s_n)$ for $\tau_n < t < s_n$ and (ii) $y(s_n) = -2\varepsilon < y(t) < -\varepsilon = y(\tau_n)$ for $\tau_n < t < s_n$. It is clear that

$$f(|y(s_n)|) - f(|y(\tau_n)|) = f(y(s_n)) - f(y(\tau_n))$$

in case (i). It follows from (7) that

$$\begin{aligned} f(|y(s_n)|) - f(|y(\tau_n)|) &= f(-y(s_n)) - f(-y(\tau_n)) \\ &< f(y(s_n)) - f(y(\tau_n)) \end{aligned}$$

in case (ii). Hence, in either case, we get

$$\begin{aligned} f(|y(s_n)|) - f(|y(\tau_n)|) &\le f(y(s_n)) - f(y(\tau_n)) \\ &= \int_{y(\tau_n)}^{y(s_n)} g(\eta)d\eta = \int_{\tau_n}^{s_n} g(y(t))y'(t)dt \\ &= -a\int_{\tau_n}^{s_n} g(x(t))g(y(t))dt - c\int_{\tau_n}^{s_n} h(t)g^2(y(t))dt \\ &\le a\int_{\tau_n}^{s_n} |g(x(t))||g(y(t))|dt. \end{aligned}$$

From (11), (16) and (21) it turns out that

$$|g(x(t))| \le \left|g\Big(-\hat{f}^{-1}(2v^*/a)\Big)\right| = -g\Big(-\hat{f}^{-1}(2v^*/a)\Big)$$

for $t \ge T_1$ and

$$|g(y(t))| \le |g(-2\varepsilon)| = -g(-2\varepsilon)$$

for $\tau_n \le t \le s_n$. Hence, we obtain

$$\begin{aligned} 0 &< f(2\varepsilon) - f(\varepsilon) = f(|y(s_n)|) - f(|y(\tau_n)|) \\ &\le a\int_{\tau_n}^{s_n} g\Big(-\hat{f}^{-1}(2v^*/a)\Big)g(-2\varepsilon)dt = a\,g\Big(-\hat{f}^{-1}(2v^*/a)\Big)g(-2\varepsilon)(s_n - \tau_n); \end{aligned}$$

namely,

$$s_n - \tau_n \ge \frac{f(2\varepsilon) - f(\varepsilon)}{a\,g\Big(-\hat{f}^{-1}(2v^*/a)\Big)g(-2\varepsilon)} \stackrel{\text{def}}{=} m > 0$$

for each $n \in \mathbb{N}$. It is clear that the positive number m is independent of $n \in \mathbb{N}$. Since $[\tau_n, s_n] \subsetneq [\tau_n, \sigma_n]$, we see that $\liminf_{n\to\infty}(\sigma_n - \tau_n) \ge m > 0$.

From the assumption of h it follows that

$$|h(t) - h(\sigma_n)| < \varepsilon_0 \quad \text{for } \sigma_n - \delta_0 < t < \sigma_n + \delta_0. \tag{22}$$

Let us examine the value of h at $t = \sigma_n$ for each $n \in \mathbb{N}$. Define

$$S = \{n \in \mathbb{N}\colon h(\sigma_n) \ge 1 + \varepsilon_0\}.$$

We will show that the number of elements in the set S is finite. Suppose that it is not right. Let card S denote the cardinal number of the set S. As shown above, $\tau_n + m < \sigma_n$ for each $n \in \mathbb{N}$. Let $\ell = \min\{\delta_0, m\}$. Then, from (19) and (22) it follows that

$$|y(t)| \geq \varepsilon \quad \text{for} \ \ \sigma_n - \ell \leq t \leq \sigma_n,$$

and that $n \in S$ implies

$$h(t) \geq 1 \quad \text{for} \ \ \sigma_n - \ell \leq t \leq \sigma_n.$$

Hence, we obtain

$$\int_{\sigma_n-\ell}^{\sigma_n} h(t)g^2(y(t))dt \geq \ell\, g^2(\varepsilon) \quad \text{if} \ \ n \in S.$$

Using this inequality, we get

$$\begin{aligned} v^* - v(t_0) &= \int_{t_0}^{\infty} v'(t)dt = -c^2\int_{t_0}^{\infty} h(t)g^2(y(t))dt \\ &\leq -c^2\sum_{n\in S}\int_{\sigma_n-\ell}^{\sigma_n} h(t)g^2(y(t))dt = -c^2\ell g^2(\varepsilon)\text{card}\, S = -\infty. \end{aligned}$$

This is a contradiction.

Since the number of elements in the set S is finite, we can find an $N \in \mathbb{N}$ such that

$$h(\sigma_n) < 1 + \varepsilon_0 \quad \text{for} \ \ n \geq N. \tag{23}$$

We next show that $\tau_{n+1} - \sigma_n \leq \delta_0$ for $n \geq N$. Suppose that it is not right. Then, there exists an $n_0 \geq N$ such that

$$\sigma_{n_0} + \delta_0 < \tau_{n_0+1}.$$

From (8), (14), (15) and (20), we obtain

$$af(x(t)) = v(t) - cf(y(t)) \geq v^* - cf(-2\varepsilon) \stackrel{\text{def}}{=} w^*$$

for $\sigma_{n_0} \leq t \leq \tau_{n_0+1}$. Note that w^* is positive because $0 < \varepsilon < -\hat{f}^{-1}(-v^*/c)/2$. We proceed the proof by dividing into two cases: (a) $x(t) \geq \hat{f}^{-1}(w^*/a) > 0$ for $\sigma_{n_0} \leq t \leq \tau_{n_0+1}$; (b) $x(t) \leq \hat{f}^{-1}(-w^*/a) < 0$ for $\sigma_{n_0} \leq t \leq \tau_{n_0+1}$. Note that

$$h(t) < \varepsilon_0 + h(\sigma_{n_0}) < 1 + 2\,\varepsilon_0 \quad \text{for} \ \ \sigma_{n_0} \leq t \leq \tau_{n_0+1}$$

because of (22) and (23). In the former case, using the second equation in system (6) with (11), (17) and (20), we get

$$\begin{aligned} y'(t) &= -a\left(1 - e^{-x(t)}\right) - ch(t)\left(1 - e^{-y(t)}\right) \\ &\leq -a\left(1 - \exp\left(-\hat{f}^{-1}(w^*/a)\right)\right) + ch(t)|g(y(t))| \\ &\leq -a\left(1 - \exp\left(-\hat{f}^{-1}(w^*/a)\right)\right) + c(1 + 2\varepsilon_0)|g(-2\varepsilon)| \\ &< -\frac{4\varepsilon}{\delta_0} \end{aligned}$$

for $\sigma_{n_0} \le t \le \sigma_{n_0} + \delta_0$. In the latter case, by using (18) instead of (17), we get

$$\begin{aligned} y'(t) &= -a\left(1 - e^{-x(t)}\right) - ch(t)\left(1 - e^{-y(t)}\right) \\ &\ge a\left(\exp\left(-\hat{f}^{-1}(-w^*/a)\right) - 1\right) - ch(t)|g(y(t))| \\ &\ge a\left(\exp\left(-\hat{f}^{-1}(-w^*/a)\right) - 1\right) - c(1 + 2\varepsilon_0)|g(-2\varepsilon)| \\ &> \frac{4\varepsilon}{\delta_0} \end{aligned}$$

for $\sigma_{n_0} \le t \le \sigma_{n_0} + \delta_0$. Thus, in either case, we have

$$|y'(t)| > \frac{4\varepsilon}{\delta_0} \quad \text{for} \quad \sigma_{n_0} \le t \le \sigma_{n_0} + \delta_0.$$

Integrate this inequality from σ_{n_0} to $\sigma_{n_0} + \delta_0$ to obtain

$$|y(\sigma_{n_0} + \delta_0)| + |y(\sigma_{n_0})| \ge \left| \int_{\sigma_{n_0}}^{\sigma_{n_0}+\delta_0} y'(t)dt \right| = \int_{\sigma_{n_0}}^{\sigma_{n_0}+\delta_0} |y'(t)|dt > 4\varepsilon.$$

However, it follows from (20) that

$$|y(\sigma_{n_0} + \delta_0)| + |y(\sigma_{n_0})| \le 4\varepsilon.$$

This is a contradiction. We therefore conclude that $\limsup_{n\to\infty}(\tau_{n+1} - \sigma_n) \le \delta_0 < \infty$.

From how to choose sequences $\{\tau_n\}$ and $\{\sigma_n\}$, it is clear that

$$\tau_n < \sigma_n < \tau_{n+1}.$$

Recall that $\liminf_{n\to\infty}(\sigma_n - \tau_n) > 0$. Then

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty \tag{24}$$

because the function h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$.

On the other hand, from (10) and (19) it turns out that

$$g^2(y(t)) \ge g^2(\varepsilon) > 0 \quad \text{for} \quad \tau_n \le t \le \sigma_n.$$

Hence, we have

$$\int_{t_0}^{\infty} v'(t)dt = -c^2 \int_{t_0}^{\infty} h(t)g^2(y(t))dt \le -c^2 g^2(\varepsilon) \sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt.$$

Since

$$\int_{t_0}^{\infty} v'(t)dt = \lim_{t\to\infty} v(t) - v(t_0) = v^* - v(t_0) < 0,$$

we obtain

$$\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} h(t)dt \le \frac{v(t_0)-v^*}{c^2g^2(\varepsilon)} < \infty.$$

This contradicts (24). Thus, we conclude that $\limsup_{t\to\infty}|y(t)| = \lambda = 0$. The proof of Step (i) is now complete.

Step (2): As already mentioned, since $y(t)$ tends zero as $t \to \infty$, we see that $x(t)$ tends to $\beta_1 = \hat{f}^{-1}(v^*/a)$ or to $-\beta_2 = \hat{f}^{-1}(-v^*/a)$. From the assumption that $v^* > 0$, the numbers β_1 and β_2 have to be positive. Hence, the solution $(x(t), y(t))$ approaches the point $(\beta_1, 0)$ on the positive x-axis or the point $(-\beta_2, 0)$ on the negative x-axis. Note that

$$2y < 1-e^{-y} < y \quad \text{for} \quad -1 < y < 0 \tag{25}$$

and

$$\frac{y}{2} < 1-e^{-y} < y \quad \text{for} \;\; 0 < y < 1. \tag{26}$$

Taking into account of the vector field of (6), we see that there exists a $T_3 \ge t_0$ such that

$$x(t) > \beta_1 \quad \text{and} \quad y(t) < 0 \quad \text{for} \;\; t \ge T_3 \tag{27}$$

or

$$x(t) < -\beta_2 \quad \text{and} \quad y(t) > 0 \quad \text{for} \;\; t \ge T_3. \tag{28}$$

In fact, it follows from (6) that $y'(t) < 0$ on the positive x-axis and $y'(t) > 0$ on the negative x-axis. This means that the solution $(x(t), y(t))$ does not enter Q_1 (resp., the second quadrant Q_3) from Q_4 (resp., the third quadrant Q_2). Recall that there are two possible cases: (i) $(x(t), y(t)) \to (\beta_1, 0)$ as $t \to \infty$; (ii) $(x(t), y(t)) \to (-\beta_2, 0)$ as $t \to \infty$. In the former, the solution $(x(t), y(t))$ has to approach the point $(\beta_1, 0)$ through either Q_1 or Q_4. Suppose that it approaches the point through Q_1. Then, there exists a $\widetilde{T}_3 \ge t_0$ such that

$$\frac{\beta_1}{2} \le x(t) \quad \text{and} \quad y(t) > 0 \quad \text{for} \;\; t \ge \widetilde{T}_3.$$

Hence, we have

$$y'(t) \le -a(1-e^{-x(t)}) \le -a(1-e^{-\beta_1/2})$$

for $t \ge \widetilde{T}_3$, and therefore,

$$y(t) \le y(\widetilde{T}_3) - a(1-e^{-\beta_1/2})(t-\widetilde{T}_3),$$

which diverges to $-\infty$. This contradicts that $\lim_{t\to\infty} y(t) = 0$. Thus, we see that the solution approaches the point through Q_4. We therefore conclude that (27) holds. In the latter, using a similar way to the above, we see that the solution $x(t), y(t)$ has to approach the point $(-\beta_2, 0)$ through Q_2, and hence (28) holds.

Since $\lim_{t\to\infty} y(t) = 0$, there exists a $T_4 \ge T_3$ such that

$$|y(t)| < 1 \quad \text{for} \;\; t \ge T_4. \tag{29}$$

We consider only the case where (27) holds, because the case where (28) holds is carried out in the same way by using (26) and (28) instead of (25) and (27). In the former, by (25), (27) and (29) we have

$$
\begin{aligned}
y'(t) + 2ch(t)y(t) &\le y'(t) + ch(t)\Big(1 - e^{-y(t)}\Big) \\
&= -a\Big(1 - e^{-x(t)}\Big) < -a\Big(1 - e^{-\beta_1}\Big) < 0
\end{aligned}
$$

for $t \ge T_4$. Hence, by (27) we get

$$
y(t) < y(t) - e^{2c(H(T_4)-H(t))}y(T_4) < -a\Big(1 - e^{-\beta_1}\Big)\frac{\int_{T_4}^{t} e^{2cH(s)}ds}{e^{2cH(t)}}
$$

for $t \ge T_4$. From this estimation and (25) it follows that

$$
x'(t) = c\Big(1 - e^{-y(t)}\Big) \le cy(t) < -ac\Big(1 - e^{-\beta_1}\Big)\frac{\int_{T_4}^{t} e^{2cH(s)}ds}{e^{2cH(t)}}
$$

for $t \ge T_4$. Integrating this inequality from T_4 to t, we obtain

$$
x(t) < -ac\Big(1 - e^{-\beta_1}\Big)\int_{T_4}^{t}\frac{\int_{T_4}^{s} e^{2cH(\tau)}d\tau}{e^{2cH(s)}}\,ds + x(T_4).
$$

Since $h(t) \ge 0$ for $t \ge 0$, it is clear that

$$
\int_0^{\infty} e^{2cH(t)}dt = \infty.
$$

Hence, there exists a $T_5 \ge T_4$ such that

$$
\int_{T_4}^{s} e^{2cH(\tau)}d\tau > \frac{1}{2}\int_0^{s} e^{2cH(\tau)}d\tau \quad \text{for} \quad s \ge T_5.
$$

Using this inequality, we can evaluate that

$$
\begin{aligned}
\beta_1 < x(t) < &-\frac{ac\Big(1 - e^{-\beta_1}\Big)}{2}\int_{T_5}^{t}\frac{\int_0^{s} e^{2cH(\tau)}d\tau}{e^{2cH(s)}}\,ds \\
&- ac\Big(1 - e^{-\beta_1}\Big)\int_{T_4}^{T_5}\frac{\int_{T_4}^{s} e^{2cH(\tau)}d\tau}{e^{2cH(s)}}\,ds + x(T_4)
\end{aligned}
$$

for $t \ge T_5$. From condition (1) and Lemma 2 with $\rho = 2c$, we see that

$$
\int_0^{\infty}\frac{\int_0^{t} e^{2cH(s)}ds}{e^{2cH(t)}}\,dt = \infty.
$$

This contradicts the evaluation above. The proof of Step (ii) is now complete.

Proposition 3 is thus proved. □

Acknowledgements

The second author's work was supported in part by Grant-in-Aid for Scientific Research (C), No. 25400165 from the Japan Society for the Promotion of Science. The authors thank two anonymous reviewers for their valuable comments.

Appendix

In the Appendix, we give a short proof for Lemma 2. The proof is essentially the same as that of Theorem 1.1 in Hatvani [5].

Proof of Lemma 2. Let $t_0 = 0$ and define

$$t_n = H^{-1}(\kappa n), \qquad a_n = t_n - t_{n-1}$$

and

$$\Delta_n = \{(t,s)\colon t \geq 0,\ s \geq 0,\ \kappa(n-1) \leq H(t) - H(s) < \kappa n\}$$

for $n \in \mathbb{N}$. Then, as can be seen from Figure 1 below, we obtain the following inequalities:

$$\begin{aligned} \iint_{\Delta_n} ds\,dt &\leq a_1(a_n + a_{n+1}) + a_2(a_{n+1} + a_{n+2}) + \cdots\cdots \\ &\leq \frac{a_1^2 + a_n^2}{2} + \frac{a_1^2 + a_{n+1}^2}{2} + \frac{a_2^2 + a_{n+1}^2}{2} + \frac{a_2^2 + a_{n+2}^2}{2} + \cdots\cdots \\ &\leq 2\sum_{i=1}^{\infty} a_i^2; \end{aligned} \tag{30}$$

$$\iint_{\Delta_1} ds\,dt \geq \frac{1}{2}\left(a_1^2 + a_2^2 + \cdots\cdots\right) = \frac{1}{2}\sum_{i=1}^{\infty} a_i^2. \tag{31}$$

Suppose that condition (3) holds. Then, from (31) it follows that

$$\begin{aligned} \int_0^\infty \frac{\int_0^t e^{\rho H(s)} ds}{e^{\rho H(t)}} dt &= \int_0^\infty e^{-\rho H(t)} \int_0^t e^{\rho H(s)} ds\,dt = \int_0^\infty \int_0^t e^{-\rho(H(t)-H(s))} ds\,dt \\ &= \sum_{n=1}^{\infty} \iint_{\Delta_n} e^{-\rho(H(t)-H(s))} ds\,dt \geq \sum_{n=1}^{\infty} \iint_{\Delta_n} e^{-\kappa\rho n} ds\,dt \\ &\geq e^{-\kappa\rho} \iint_{\Delta_1} ds\,dt \geq \frac{1}{2e^{\kappa\rho}} \sum_{i=1}^{\infty} a_i^2 \\ &= \frac{1}{2e^{\kappa\rho}} \sum_{n=1}^{\infty} \left(H^{-1}(\kappa n) - H^{-1}(\kappa(n-1))\right)^2 = \infty. \end{aligned}$$

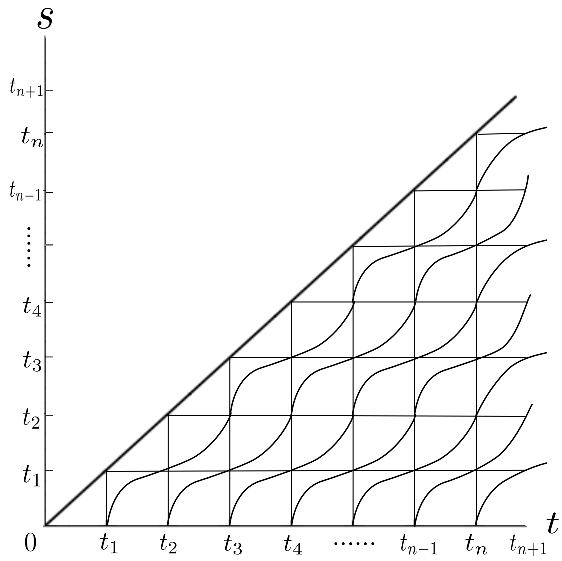

Figure 1: The domain Δ_n and the curve $C_n = \{(t, s): t \geq 0,\ s \geq 0,\ H(t) - H(s) = \kappa n\}$

Hence, condition (5) is satisfied. Conversely, suppose that condition (5) holds. Then, by (30) we have

$$\begin{aligned}\infty &= \int_0^\infty \frac{\int_0^t e^{\rho H(s)} ds}{e^{\rho H(t)}}\, dt = \int_0^\infty e^{-\rho H(t)} \int_0^t e^{\rho H(s)} ds\, dt \\ &= \sum_{n=1}^\infty \iint_{\Delta_n} e^{-\rho(H(t)-H(s))} ds\, dt \leq \sum_{n=1}^\infty e^{-\kappa\rho(n-1)} \iint_{\Delta_n} ds\, dt \\ &\leq 2 \sum_{n=1}^\infty e^{-\kappa\rho(n-1)} \sum_{i=1}^\infty a_i^2 = \frac{2}{1-e^{-\kappa\rho}} \sum_{i=1}^\infty a_i^2 \\ &= \frac{2e^{\kappa\rho}}{e^{\kappa\rho}-1} \sum_{n=1}^\infty \left(H^{-1}(\kappa n) - H^{-1}(\kappa(n-1))\right)^2.\end{aligned}$$

Hence, condition (3) is satisfied. Thus, conditions (3) and (5) are equivalent.

As mentioned in Section 2, it is already proved by Hatvani [5] that conditions (1) and (3) are equivalent. We therefore conclude that condition (1) holds if and only if condition (5) is satisfied with $\rho > 0$ arbitrary. □

References

[1] F. Brauer, J.A. Nohel, The Qualitative Theory of Ordinary Differential Equations. Benjamin, New York, 1969; Dover, New York, 1989.

[2] W.A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heath, Boston, 1965

[3] L. Hatvani, A generalization of the Barbashin-Krasovskij theorems to the partial stability in nonautonomous systems, in: M. Farkas (Ed.), Qualitative Theory of Differential Equations, vol. I, Szeged, 1979, in: Colloq. Math. Soc. János Bolyai, vol. 30, North-Holland, Amsterdam, New York, 1981, pp. 381–409.

[4] L. Hatvani, Integral conditions on the asymptotic stability for the damped linear oscillator with small damping, Proc. Amer. Math. Soc. 124 (1996) 415–422.

[5] L. Hatvani, T. Krisztin, V. Totik, A necessary and sufficient condition for the asymptotic stability of the damped oscillator, J. Differential Equations 119 (1995) 209–223.

[6] A.O. Ignat'ev, On the global asymptotic stability for the equilibrium of the Lotka-Volterra equations in a varying environment, Translation of Differ. Uravn. 50 (2014) 290–295. Differ. Equ. 50 (2014) 286–291.

[7] J.J. Levin, J.A. Nohel, Global asymptotic stability for nonlinear systems of differential equations and applications to reactor dynamics, Arch. Rational Mech. Anal. 5 (1960) 194–211.

[8] A.N. Michel, L. Hou, D. Liu, Stability Dynamical Systems: Continuous, Discontinuous, and Discrete Systems, Birkhäuser, Boston, Basel, Berlin, 2008.

[9] N. Rouche, P. Habets, M. Laloy, Stability Theory by Liapunov's Direct Method, Applied Mathematical Sciences 22, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[10] Y. Saito, J. Sugie and Y.-H. Lee,Global asymptotic stability for predator-prey models with environmental time-variations, Appl. Math. Lett. 24 (2011) 1973–1980.

[11] R.A. Smith, Asymptotic stability of $x'' + a(t)x' + x = 0$, Quart. J. Math. Oxford (2) 12 (1961) 123–126.

[12] J. Sugie, Two-parameter bifurcation in a predator-prey system of Ivlev type, J. Math. Anal. Appl. 217 (1998) 349–371.

[13] J. Sugie, Uniqueness of limit cycles in a predator-prey system of Holling-type functional response, Quart. Appl. Math. 58 (2000) 577–590.

[14] J. Sugie, Global asymptotic stability for damped half-linear oscillators, Nonlinear Anal. 74 (2011) 7151–7167.

[15] J. Sugie, S. Hata, M. Onitsuka, Global asymptotic stability for half-linear differential systems with periodic coefficients, J. Math. Anal. Appl. 371 (2010) 95–112.

[16] J. Sugie and M. Katayama, Global asymptotic stability of a predator-prey system of Holling type, Nonlinear Anal. 38 (1999) 105–121.

[17] J. Sugie, R. Kohno, R. Miyazaki, On a predator-prey system of Holling type, Proc. Amer. Math. Soc. 125 (1997) 2041–2050.

[18] J. Sugie, Y. Saito, Convergence of solutions of nonlinear systems with integrable forcing term and its application to a biological model, Appl. Math. Comput. 219 (2013) 8169–8177.

[19] J. Sugie, Y. Saito, M. Fan, Global asymptotic stability for predator-prey systems whose prey receives time-variation of the environment, Proc. Amer. Math. Soc. 139 (2011) 3475–3483.

[20] T. Yoshizawa, Stability Theory by Liapunov's Second Method, Math. Soc. Japan, Tokyo, 1966.