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Note on Algebraic K3-Surfaces

Hiromichi MATSUNAGA
Department of Mathematics, Shimane University, Matsue, Japan
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In this note the author analyzes candidates for a moduli space of algebraic K3-surfaces which have appeared in literatures. In §1, Griffith's period matrices and Šapiro-Šafarevič's space are treated. It is revealed in §2 that the complex analytic family of all non-singular surfaces of degree four in the complex projective three space is a local moduli space for algebraic K3-surfaces.

§1. An universal space

Let V be a Kähler K3-surface and ω be its Kähler form. Concerning to the Hodge decomposition with complex coefficient C,

$$H^2(V, C)=H^{2,0}(V, C)+H^{1,1}(V, C)+H^{0,2}(V, C),$$

we have

$$\dim H^2(V, C)=22, \quad \dim H^{2,0}(V, C)=\dim H^{0,2}(V, C)=1,$$

$$\dim H^{1,1}(V, C)=20.$$

Let $L\colon H^2(V, C)\to H^4(V, C)$ be the homomorphism defined by $\eta\to\omega\eta$. Denote by $H^{p,q}(V, C)_0$ the kernel of the restricted homomorphism $L\colon H^{p,q}(V, C)\to H^4(V, C)$, then we have

$$\dim H^{1,1}(V, C)_0=19, \quad H^{2,0}(V, C)_0=H^{2,0}(V, C),$$

$$H^{0,2}(V, C)_0=H^{0,2}(V, C), \text{ p. 35, [1]}.$$

Let $V=\{V_t\}\to N$ be an analytic family of Kähler K3-surfaces. The family admits a differentiable local triviality

$$\begin{array}{ccc} V_{t_0}\times U & \xrightarrow{\phi} & V|U \\ \downarrow & & \downarrow \\ U & = & U, \end{array}$$

where U is a neighborhood of t_0 and ϕ is a fiber preserving C^∞-isomorphism, and it induces an isomorphism $\phi^*\colon H^2(V_t, C)\to H^2(V_{t_0}, C)$. Define a mapping $\Omega\colon U\to D$

by $\Omega(t)=\phi^*(H^{2,0}(V_t, C))$, where D is the set of all lines which satisfies the relations

$$Q(\xi, \eta)=0,\ Q(\xi, \bar{\xi})>0, \quad \text{and} \quad Q(\xi, \eta)=\int_{V_{t_0}} \xi \wedge \eta,$$

and we identify the image $\phi^*(H^{2,0}(V_t, C)$ as an element of twenty one dimensional complex projective space CP^{21}. The space D is a twenty dimensional open complex manifold and a submanifold of CP^{21}, and it is called the period matrix space [2]. $H^{2,0}(V_t, C)$ is generated by some (2, 0)-form ψ_t. We let $\Gamma_1,\dots, \Gamma_{22}$ be an integral basis of $H_2(V_{t_0}, Z)$. The period $\left(\int\Gamma_1\psi_t,\dots, \int\Gamma_{22}\psi_t\right)$ is considered as an element of CP^{21}. Then we have

PROPOSITION 1. *The period and the period matrix is the same one with respect to the moduli problem.*

PROOF. The period is a holomorphic function of t [7], and the period matrix is also a holomorphic function of t [2]. By Korollar 1, § 2 [4] the proposition is obtained.

Now we let ω be an exterior form associated with a Hodge metric on an algebraic K3-surface V. Then by p. 40 [6], there exists a positive integer h such that $h\omega \in c(F)$, where $c(F)$ is the Chern class of a line bundle F. The divisor corresponding to F is ample. In fact let ω_0 be a fundamental form on the complex projective m-space CP^m and $\mathfrak{D}_0$ be the Poincaré duality, and f: $V\subset CP^m$ be a complex submanifold and ω be its fundamental form, $\mathfrak{D}$ be the Poincaré duality, since

$$H^2_{(1,1)}(CP^m, Z) \overset{\mathfrak{D}_0}{\approx} H^{(1,1)}_{2m-2}(CP^m, Z) \approx H_{2m-2}(CP^m, Z), \quad \text{and} \quad \mathfrak{D}_0(\omega_0)=[CP^{m-1}],$$

then

$$f_*(\mathfrak{D}(\omega))=f_*(\omega \cap V)=f_*(f^*\omega_0 \cap M)=\omega_0 \cap f_*(V)=(\mathfrak{D}_0\omega_0)\circ(f_*V),$$

where $\circ$ denotes the intersection of homology classes. Hence $\mathfrak{D}(\omega)$ is ample. Thus we have

PROPOSITION 2. *The moduli space* $\Omega(l)$, p. 548 [8] *and the Griffith's space D is the same one.*

PROOF. We take the Hodge metric corresponding to $h\omega$ and l to be an element corresponding to $\xi=\{E\}$, where E is the divisor which corresponds to F. By I. 1, I [2] we have the proposition.

§ 2. A local moduli space

In this section we follow the notational conventions in [5]. The mapping

$$(\sum_{\alpha}\theta_\alpha\partial/\partial_{z_\alpha})\otimes\Big(\frac{1}{2}\sum_{\alpha,\beta=1}^{2}\psi_{\alpha\beta}dz_\alpha\wedge dz_\beta\Big)\longrightarrow\sum\theta_\alpha\psi_{\alpha\beta}dz_\beta$$

induces an isomorphism $\Theta=\Omega^1$ and a pairing $\Theta\times\Omega^2\to\Omega$. The pairing gives a product

$$H^1(V,\Theta)\otimes H^{2,0}(V)\longrightarrow H^{1,1}(V)$$

and $\theta\in H^1(V,\Theta)$ defines an element $\hat\theta\in\mathrm{Hom}(H^{2,0},H^{1,1})$.

The differential $d\Omega$ of Ω in § 1 takes the form

$$T_{t_0}(N)\longrightarrow T_{\Omega(t_0)}(CP^{21})\approx\mathrm{Hom}(H^{2,0},H^{1,1}+H^{0,2}).$$

By Proposition 1.20 [2] $(d\Omega)(\lambda)=\widehat{\rho(\lambda)}$, where $\lambda\in T_{t_0}(N)$ and $\rho(\lambda)\in H^1(V,\Theta)$ is the Kodaira Spencer class. Now $\Phi\colon\mathfrak{B}\to CP^3$ is a complex analytic family of non singular surfaces of CP^3, § 12 [5]. Let ω_0 be the fundamental form of CP^3 and $\mathfrak{L}\to\mathfrak{B}$ be the line bundle such that $c(\mathfrak{L})\ni\omega_0$. Then the family $\Phi\colon\mathfrak{B}\to CP^3$ is a polarized family of algebraic surfaces, (b) [2]. In this case the differential admits the following decomposition

$$\begin{array}{ccc} T_{t_0}(N)\longrightarrow \mathrm{Hom}(H^{2,0},H_0^{1,1})\approx T_{\Omega(t_0)}(D)=\mathrm{Hom}(H^{2,0},H_0^{1,1}+H^{0,2}) \\ \rho\big\downarrow\quad\nearrow\kappa \\ H^1_{CP^3}(V_{t_0},\Theta), \end{array}$$

where κ is a homomorphism which is defined by $\kappa(\theta)=\hat\theta$, and $H^1_{CP^3}(V_{t_0},\Theta)$ denotes the group in Definition 12.2 [5].

PROPOSITION 3. *κ is an isomorphism.*

At first we give a lemma, from which the proposition is obtained immediately. Let $V_{2,4}$ be the complex analytic family of all non singular hypersurfaces of degree four on CP^3. Its base space is an open submanifold of CP^{34} [5], which we denote by M.

LEMMA. *The family $V_{2,4}\to M$ is a complex analytic family of K3-surfaces.*

PROOF. Let V_t be a non singular surface defined by $t(\zeta)=\Sigma t_\nu\zeta_0^{h_0}\cdots\zeta_3^{h_3}$, $h_0+h_1+h_2+h_3=4$, and $E\to CP^3$ be the line bundle defined by $\{\zeta_\nu/\zeta_\lambda\}$. Then the divisor V_t in CP^3 determines the line bundle $[V_t]=4E$. Let T, T_t be holomorphic tangent bundles of CP^3, V_t respectively. Then we have an exact sequence

$$0\longrightarrow[V_t]\longrightarrow T\longrightarrow T_t\longrightarrow 0,$$

and their determinant bundles satisfy the relation

$$\det|T|=\det|T_t|\cdot\det|[V_t]|.$$

Denote by K, K_t the canonical bundles of CP^3, V_t respectively. Then we have

$$K^{-1}=K_t^{-1}\cdot[V_t], \quad \text{and} \quad K_t=K\cdot[V_t].$$

Since $K=-4E$, K_t is a trivial bundle. By 18.2.1, p. 138 [3], if a line bundle F^{-1} is positive then $H^i(V, F)=0$ for $i\neq\dim V$. Since the Poincaré dual of $4E$ is $4[CP^2]$, $c(4E)$ contains a Hodge metric and $4E$ is positive. Thus $H^2(CP^3, \Omega(-4E))=0$. By the exact sequence

$$H^1(CP^3, \Omega) \longrightarrow H^1(V_t, \Omega) \longrightarrow H^2(CP^3, \Omega(-4E)),$$

we have $q=\dim H^1(V_t, \Omega)=0$. Hence V_t is a K3-surface for each $t\in M$ and the lemma is proved.

Now the bundle $[V_t]=4E$ is positive and the dimension of the complex projective linear group $PL(4, C)$ is 15 and of the base M is 34, then by (12.19) [5] $\dim H^1_{CP^3}(V_t, \Theta_t)=19$ for each $t\in M$. By Theorem 2 Chapter IX [7] $d\Omega$ is an isomorphism, so κ is an isomorphism.

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