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Linear Space Topologies on Riesz Spaces

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In the Lebesgue integration theory, Dominated Convergence theorem plays a very important role. It combines an order and a topology. So we abstract it on Riesz spaces and father define another linear space topology and call it a semi-Lebesgue topology. We discuss some relations between this and another linear space topologies on Riesz spaces.

At first, we explain some notations and terminologies. A Riesz space is a vector lattice over the field of real numbers. Let E be a Riesz space. $B\uparrow b$ means that B is a directed upwards subset of E and has a supremum b. $\mathfrak{F}(B\uparrow)$ is a filter on E generated by the set

$$\{x: x\in B \quad \text{and} \quad x\geq b \quad \text{for some point} \quad b \quad \text{in} \quad B\}$$

and call it a section filter of B. $B\downarrow b$ and $\mathfrak{F}(B\downarrow)$ are defined dually, that is, by replacing upwards, supremum and $\geq$ by downwards, infimum and $\leq$ respectively. A set B $(\subset E)$ is a solid set whenever $x\in B$ and $|y|\leq|x|$ implies $y\in B$. A set B $(\subset E)$ is an upwards-Dedekind complete set whenever for any x in B, there is a directed upwards subset C of B such that $C\uparrow x$. A downwards-Dedekind complete set is defined similarly. E is Archimedean if x and y in E and $nx\leq y$ for all natural number n implies $x\leq 0$.

I do not assume that a linear space topology is locally convex. A linear space topology on E will be called locally solid if there is a neighborhood basis of 0 consisting of the solid sets. A linear space topology on E will be called a Lebesgue topology if whenever B is a non-empty subset of E and $B\downarrow 0$, then 0 belongs to the closure of B.

DEFINITION. (1) *A locally order complete topology on E is a locally solid topology on E and has a neighborhood basis of* 0 *consisting of the upwards-Dedekind complete subsets of E.*

(2) *A linear space topology on E will be called semi-Lebesgue if, whenever for any non-empty subset B of E which is bounded above and directed upwards, then the section filter* $\mathfrak{F}(B\uparrow)$ *is Cauchy.*

We first give some characterizations of the semi-Lebesgue topology.

PROPOSITION 1. *Let E be a Riesz space.* $\mathfrak{I}$ *is a linear space topology on E.*

It is semi-Lebesgue if and only if every directed upwards and bounded above sequence in E is Cauchy.

PROOF. Necessity. We assume that a sequence $\{x_n\}$ is directed upwards and bounded above. As $\mathfrak{J}$ is semi-Lebesgue, the section filter $\mathfrak{F}(x_n \uparrow)$ becomes Cauchy. So for any circled neighbourhood V of 0, there is some integer n_0 such that

$$x_n - x_{n_0} \in V \qquad \text{for any} \quad n \geq n_0 .$$

For any n, $m \geq n_0$,

$$x_n - x_m = x_n - x_{n_0} + x_{n_0} - x_m \in 2V .$$

Sufficiency. We suppose that a linear space topology $\mathfrak{J}$ is not semi-Lebesgue. Then there is a bounded above and directed upwards subset A of E and its section filter $\mathfrak{F}(A \uparrow)$ is not Cauchy. Thus there exists a neighbourhood V of 0 and a directed upwards sequence $\{x_n\}$ in A such that

$$x_{n+1} - x_n \notin V \qquad \text{for any} \quad n .$$

So the sequence $\{x_n\}$ is not Cauchy. But by the property of $\{x_n\}$, the sequence must be Cauchy. Therefore we have a contradiction.

The following corollary is easily proved.

COROLLARY. *Let E be a Riesz space. $\mathfrak{J}$ is a linear space topology on E. Then $\mathfrak{J}$ is semi-Lebesgue if and only if every sequence $\{x_n\}$ in E, which has the following property (O), converges to* 0.

The property (O): there is a point x in E such that

$$\sum_{i=1}^{n} |x_i| \leq |x| \qquad \textit{for any} \quad n .$$

We next consider the Dieudoné topology $|w|(E^\sim, E)$ on $E^\sim$, where $(E, \mathfrak{J})$ is a semi-Lebesgue Riesz space. $E^\sim$ is the set of all real linear maps on E which is bounded on every order bounded subset of E. $|w|(E^\sim, E)$ is a topology on $E^\sim$ defined by the seminorms

$$f \longrightarrow \langle |x|, |f| \rangle \quad \text{as } x \text{ runs through} \quad E.$$

For any subset A of E, A° is an absolute polar of A which is the set

$$\{f \in E^\sim : |\langle x, f \rangle| \leq 1 \qquad \text{for all} \quad x \in A\} .$$

COROLLARY. *E is a Riesz space and $\mathfrak{J}$ is a semi-Lebesgue topology on E. If V is a neighbourhood of* 0, *then V° is $|w|(E^\sim, E)$-bounded.*

PROOF. Suppose that V° is not $|w|(E^\sim, E)$-bounded. Then there is a point x in E and a sequence $\{f_n\}$ in V° such that

$$<|x|, |f_n|> > 2^n \qquad \text{for any} \quad n.$$

As $<2^{-n}|x|, |f_n|> > 1$ and f_n in V°, the sequence $\{2^{-n}|x|\}$ does not belongs to V. But

$$\sum_{n=1}^{k} 2^{-n}|x| \leq |x| \qquad \text{for any} \quad k,$$

$2^{-m}|x|$ must belong to 0-neighbourhood V for sufficiently large positive integer m by the above corollary. This is a contradiction.

We next consider some relations between a semi-Lebesgue and a Lebesgue topologies.

PROPOSITION 2. *Let E be a Riesz space. If $\mathfrak{J}$ is a locally solid, Hausdorff, and Lebesgue linear space topology on E, it is semi-Lebesgue.*

PROOF. Let A be a directed upwards and bounded above subset of E and B be the set of all elements which is an upper bound of A. Since $\mathfrak{J}$ is locally solid and Hausdorff, the set $E^+ = \{x: x \geq 0\}$ is closed.

So E is an Archimeden Riesz space. Therefore by (3, Theorem 22.5), we can conclude that $B - A \downarrow 0$. As $\mathfrak{J}$ is Lebesgue, 0 belongs to the closure of $B - A$. Therefore for any solid neighbourhood V of 0, there is a point b in B and a point a in A such that $b - a \in V$. For any point x $(\geq a)$ in A, we can say that

$$x - a \leq b - a \in V.$$

By the solidness of V, $x - a$ also belongs to V. So that we can say that the section filter $\mathfrak{F}(A\uparrow)$ is Cauchy, $\mathfrak{J}$ is semi-Lebesgue.

We next consider that for what conditions are added to the semi-Lebesgue topology, it becomes Lebesgue.

THEOREM 1. *Let E be a Riesz space, $\mathfrak{J}$ a locally solid, Hausdorff and semi-Lebesgue topology on E. If $(E, \mathfrak{J})$ is a topologically complete Riesz space, then $\mathfrak{J}$ is Lebesgue.*

PROOF. Let A be a non-empty subset of E and $A \downarrow 0$. Then $-A$ becomes upwards directed and its supremum is 0. As $\mathfrak{J}$ is semi-Lebesgue, the section filter $\mathfrak{F}(-A\uparrow)$ is Cauchy. By the completeness of $(E, \mathfrak{J})$, the section filter $\mathfrak{F}(-A\uparrow)$ converges to some point x_0 in E. So x_0 is an accumulation point of the section filter $\mathfrak{F}(-A\uparrow)$, the point x_0 belongs to the closure of the set

$$\{x: x \geq -a \qquad \text{for any} \quad a \in A\}.$$

But the set

$$\{x: x \geq -a\} = E^+ + \{-a\}$$

is closed. Therefore for any point a in A, $x_0 \geq -a$ and $-x_0$ is a lower bound of A. Let u be any lower bound of A, the set

$$\{x: x \leq -u\}$$

belongs to the section filter $\mathfrak{F}(-A\uparrow)$. The point x_0 belongs to the set

$$\{x: x \leq -u\}^- = \{x: x \leq -u\}.$$

Therefore $-x_0$ is an infimum of A. The section filter $\mathfrak{F}(-A\uparrow)$ converges to 0. So for any neighbourhood U of 0, there is a solid neighbourhood V of 0 such that

$$V \subset U \quad \text{and} \quad V \in \mathfrak{F}(-A\uparrow).$$

There is a point a in A such that $-a \in V$. By the solidness of V, the point a also belongs to $V \subset U$. Therefore 0 belongs to the closure of A.

Even if we replace the topologically completeness with some order condition in the above theorem, we can get the same conclusion.

THEOREM 2. *Let E be a Riesz space, $\mathfrak{I}$ a Hausdorff, semi-Lebesgue and locally order complete linear space topology on E. Then $\mathfrak{I}$ is Lebesgue.*

PROOF. Let A be a non-empty subset of E and $A\downarrow 0$. Then the section filter $\mathfrak{F}(A\downarrow)$ is Cauchy. For any neighbourhood U of 0, there is a neighbourhood V of 0 such that the set V is an upwards-Dedekind complete subset of U by the definition of a locally order complete topology. For this set V, we can fined a point a in A such that

$$a - x \in V \qquad \text{for any} \quad x \in A \quad \text{and} \quad x \leq a.$$

The set

$$\{a - x: x \in A \quad \text{and} \quad x \leq a\}$$

is a directed upwards subset of V and has a supremum a. Therefore the point a belongs to the set $V \subset U$. So we can say that 0 belongs to the closure of A.

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