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$(Z; \lambda, \mu)$-Absolutely Summing Operators

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We extend two kinds of concepts, (λ, μ)-absolutely summing operators and (Z; p)-absolutely summing operators, to $(Z; \lambda, \mu)$-absolutely summing operators.

Introduction

Pietsch [11] introduced the concept of absolutely p-summing operators in normed spaces. This concept was extended in Ramanujan [12] to absolutely λ-summing operators by using the symmetric sequence space λ. Also, Mityagin and Pelczyński [8] introduced the concept of (p, r)-absolutely summing operators in Banach spaces and this was extended in Miyazaki [9] to $(p, q; r)$-absolutely summing operators by using the sequence spaces $l_{p,q}$ and l_r. We extended the above concepts to (λ, μ)-absolutely summing operators in normed spaces by the aid of abstract sequence spaces λ and μ. On the other hand, the first concept was recently extended in Ceĭtlin [2] to (Z, p)-absolutely summing operators by making use of the Banach space Z.

The main object of this paper is to extend these two kinds of concepts, (λ, μ)-absolutely summing operators and (Z, p)-absolutely summing operators, to $(Z; \lambda, \mu)$-absolutely summing operators by using the above λ, μ and the Banach space Z and to develop a theory of such operators. We also investigate $(Z; \lambda, \mu)$-quasi-integrable operators which extend the concept of (λ, μ)-absolutely summing operators.

In Section 1, we define the $(Z; \lambda, \mu)$-quasi-integrable operators and the $(Z; \lambda, \mu)$-absolutely summing operators. In Section 2, we state some general properties of $(Z; \lambda, \mu)$-quasi-integrable operators and $(Z; \lambda, \mu)$-absolutely summing operators. We investigate in Section 3 some inclusion relations between the spaces of the above operators. Section 4 is devoted to studying the composition theorem of two $(Z; \lambda, \mu)$-absolutely summing operators. In Section 5, we investigate $(Z; \lambda, \mu)$-quasi-integrable operators and $(Z; \lambda, \mu)$-absolutely summing operators, when their domain and range are particular normed spaces.

§1. Notations and Definitions

For a sequence space λ the α-dual is denoted by $\lambda^{\times}$. If $\lambda^{\times\times} = \lambda$, then λ is said to be a Köthe space. We start with the sequence space c_0 of all scalar sequences converging to zero and the sequence space ω of all scalar sequences in which an extended quasi-norm p and an extended norm q are given respectively. We shall then define the sequence space $\lambda\subset c_0$ (resp. $\mu\subset\omega$) to be the space consisting of all $x\in c_0$ (resp. $x\in\omega$) such that $p(x)<\infty$ (resp. $q(x)<\infty$). We shall denote the extended quasi-norm p (resp. the extended norm q) by $\|\cdot\|_\lambda$ (resp. $\|\cdot\|_\mu$).

We assume that λ and μ satisfy the following conditions:

(a) *If for any* $x=(x_1,\dots, x_n,\dots)\in c_0$ *and* $y=(y_1,\dots, y_n,\dots)\in\omega$ *we set* $x^i=(x_1,\dots, x_i, 0,\dots)$ *and* $y^i=(y_1,\dots, y_i, 0,\dots)$ *for* $i=1, 2,\dots$, *then* $p(x^i)\to p(x)$ *and* $q(y^i)\to q(y)$.

(b) *p and q are both absolutely monotone.*

(c) *λ and μ are both the K-symmetric spaces. That is, if x_π is the sequence which is obtained as a rearrangement of the sequence x corresponding to a permutation π of the positive integers, then $p(x)=p(x_\pi)$ for each $x\in\lambda$ and each π, and $q(y)=q(y_\pi)$ for each $y\in\mu$ and each π.*

(d) *μ is a Köthe space.*

(e) *The topology given by the norm q on μ is the Mackey topology of the dual pair $(\mu, \mu^\times)$ so that $\mu^\times=(\mu, q)'$.*

(f) *λ and μ have the norm preservation property. That is, if $x=(x_i)$ is such that $x_i=0$ for all $i\neq n$, then $p(x)=|x_n|$ and $q(x)=|x_n|$.*

We say the above λ and l_∞ to be spaces of type Λ and say the above μ to be a space of type M.

If μ is of type M, then we have $l_1\subset\mu\subset l_\infty$ and either $\mu\subset c_0$ or $\mu=l_\infty$.

We showed in [6] that the Lorentz space $l_{p,q}$ $(1\leq p,\ q\leq\infty)$ is of type Λ and the space l_p $(1\leq p\leq\infty)$ is of type M.

Next we start with two normed spaces $(E, \|\ \|)$ and $(F, \|\ \|)$ and a Banach space $(Z, \|\ \|)$. Let μ be of type M. Then we denoted in [6] by $\mu(E)$ the vector sequences $x=(x_n)$, $x_n\in E$, which are weakly contained in μ in the sense that for each $a\in E'$ the sequence $(<x_n, a>)$ of scalars is in μ. Here we denote by $S(E, Z)$ the unit sphere in the space $L(E, Z)$ of all continuous linear operators of E to Z. Then we shall denote by $\mu_K^Z(E)$ (resp. $\mu^Z(E)$) the vector sequences $x=(x_n)$, $x_n\in E$ such that for each $A\in K\subset S(E, Z)$ with K compact in the simple convergence topology (resp. for each $A\in L(E, Z)$), the sequence $(\|Ax_n\|)$ of scalars is in μ.

If $x=(x_n)$ belongs to $\mu(E)$, we showed in [6] that $\sup_{\|a\|\leq 1} q((|<x_n, a>|))<\infty$. We denoted by ε_μ the functional defined on $\mu(E)$ by $\varepsilon_\mu(x)=\sup_{\|a\|\leq 1} q((|<x_n, a>|))$. We shall denote by $\varepsilon_{\mu K}^Z$ (resp. ε_μ^Z) the functional defined on $\mu_K^Z(E)$ (resp. $\mu^Z(E)$) by $\varepsilon_{\mu K}^Z(x)=\sup_{A\in K} q((\|Ax_n\|))$ (resp. $\varepsilon_\mu^Z(x)=\sup_{A\in S(E,Z)} q((\|Ax_n\|))$). Here ε_μ, $\varepsilon_{\mu K}^Z$ and ε_μ^Z can easily be verified to be semi-norms.

LEMMA 1. *Let E be a normed space, let Z be a Banach space and let μ be of type M. Then we have the following properties:*

(1) *Let a set* $K \subset S(E, Z)$ *be compact in the simple convergence topology. Then if* $x=(x_n)\in \mu_K^Z(E)$, $\varepsilon_{\mu K}^Z(x)=\sup_{A\in K}\|(\|Ax_n\|)\|_\mu<\infty$.

(2) *If* $x=(x_n)\in \mu^Z(E)$, $\varepsilon_\mu^Z(x)=\sup_{A\in S(E,Z)}\|(\|Ax_n\|)\|_\mu<\infty$.

PROOF. We shall prove (1). The proof of (2) is similar. Let $B_{\mu\times}$ be the unit sphere in $\mu^\times$ as the dual space and let $x=(x_n)\in \mu_K^Z(E)$. Then we set

$$W((x_n), B_{\mu\times})=\{A\in \bigcup_{n=1}^\infty n\Gamma(K) | \sum_{n=1}^\infty |\alpha_n|\,\|Ax_n\|\le 1 \quad \text{for any} \quad (\alpha_n)\in B_{\mu\times}\}.$$

We can easily prove that $W((x_n), B_{\mu\times})$ is absolutely convex and closed in the simple convergence topology. Here $W((x_n), B_{\mu\times})$ is absorbing in $\bigcup_{n=1}^\infty n\Gamma(K)$. In fact, let A be an arbitrary element in $\bigcup_{n=1}^\infty n\Gamma(K)$. Since $B_{\mu\times}$ is a bounded set in $\mu^\times$, it follows that

$$\sum_{n=1}^\infty |\alpha_n|\,\|Ax_n\|\le\rho \qquad \text{for any} \quad (\alpha_n)\in B_{\mu\times},$$

that is, $A\in\rho W((x_n), B_{\mu\times})$. Consequently $W((x_n), B_{\mu\times})$ is a barrel in $\bigcup_{n=1}^\infty n\Gamma(K)$ in the simple convergence topology. Since K is compact in the simple convergence topology, it follows that

$$\sup_{A\in K}\sum_{n=1}^\infty |\alpha_n|\,\|Ax_n\|\le\rho_1 \qquad \text{for all} \quad (\alpha_n)\in B_{\mu\times}.$$

Therefore we have $\sup_{A\in K}\|(\|Ax_n\|)\|_\mu<\infty$. The proof is complete.

Let λ be of type Λ. Then we define $\lambda[F]$ as the space of all vector sequences $y=(y_n)$, $y_n\in F$, such that the sequence $(\|y_n\|)\in\lambda$. We denote by α_λ the functional defined on $\lambda[F]$ by $\alpha_\lambda(y)=p((\|y_n\|))$ which is also denoted by $\|(y_n)\|_{\lambda[F]}$. Thus $\lambda[F]$ is topologised in a natural way by quasi-norm $\alpha_\lambda(y)$. We can easily show that $\mu_K^Z(E)\supset\mu^Z(E)\supset\mu[E]$ for any μ of type M.

We now recall the definition of (λ,μ)-absolutely summing operators [6].

DEFINITION 1. *Let E and F be normed spaces, let λ and μ be of type Λ and of type M respectively and let T be a linear operator of E to F. Then the operator T is said to be (λ,μ)-absolutely summing if there exists a number $\rho>0$ such that*

$$\|(Tx_i)\|_{\lambda[F]}\le\rho\sup_{\|a\|\le1}\|(|<x_i, a>|)\|_\mu$$

for all finite sets of elements $x_1,\dots,x_n$ in E.

For each (λ,μ)-absolutely summing operator $T: E\to F$ we put

$$\pi_{\lambda,\mu}(T)=\inf\rho,$$

where the infimum is taken over all ρ *with the properties indicated. We denote by* $\pi_{\lambda,\mu}(E, F)$ *the collection of all* (λ, μ)*-absolutely summing operators of E to F.*

An operator $T \in L(E, F)$ is called finite if its image space is finite dimensional and the collection of all finite operators is denoted by $A(E, F)$. Then we give the following definitions.

DEFINITION 2. *Let E and F be normed spaces, let Z be a Banach space, let* λ *and* μ *be of type Λ and of type M respectively and let T be a linear operator of E to F. Then the operator T is said to be* $(Z; \lambda, \mu)$*-quasi-integrable* (resp. *finitely* $(Z; \lambda, \mu)$*-quasi-integrable) if there exists a set* $K \subset S(E, Z)$ (resp. $K \subset S(E, Z) \cap A(E, Z)$), *compact in the simple convergence topology, and* $\rho > 0$ *such that*

$$\|(Tx_i)\|_{\lambda[F]} \leq \rho \sup_{A \in K} \|(\|Ax_i\|)\|_\mu$$

for each finite set of elements $x_1, \dots, x_n$ *in E. Let* ρ_K *be the infimum of such* ρ *for fixed K and let* $\sigma^Z_{\lambda,\mu}$ *be the infimum of the number* ρ_K *over all K. We denote by* $QI^Z_{\lambda,\mu}(E, F)$ (resp. $FQI^Z_{\lambda,\mu}$) *the collection of all* $(Z; \lambda, \mu)$*-quasi-integrable operators* (resp. *finitely* $(Z; \lambda, \mu)$*-quasi-integrable operators) of E to F.*

DEFINITION 3. *Let E and F be normed space, let Z be a Banach space, let* λ *and* μ *be of type Λ and of type M respectively and let T be a linear operator of E to F. Then the operator T is said to be* $(Z; \lambda, \mu)$*-absolutely summing if there exists a number* $\rho > 0$ *such that*

$$\|(Tx_i)\|_{\lambda[F]} \leq \rho \sup_{A \in S(E, Z)} \|(\|Ax_i\|)\|_\mu$$

for each finite set of elements $x_1, \dots, x_n$ *in E.*

For each $(Z; \lambda, \mu)$*-absolutely summing operator* $T: E \to F$ *we put*

$$\pi^Z_{\lambda,\mu}(T) = \inf \rho ,$$

where the infimum is taken over all ρ *with the properties indicated. We denote by* $\pi^Z_{\lambda,\mu}(E, F)$ *the collection of all* $(Z; \lambda, \mu)$*-absolutely summing operators of E to F.*

REMARK. $\|(Tx_i)\|_{\lambda[F]}$ appearing above is to be interpreted as the quasi-norm of the element $(Tx_1, \dots, Tx_n, 0, \dots)$ in the vector sequence space $\lambda[F]$, and $\|(\|Ax_i\|)\|_\mu$ is similarly interpreted.

PROPOSITION 1. *Let E and F be normed spaces, let Z be a Banach space and let* λ *and* μ *be of type Λ and of type M respectively. Then we have the following properties:*

(1) $\pi_{\lambda,\mu}(E, F) \subset FQI^Z_{\lambda,\mu}(E, F) \subset QI^Z_{\lambda,\mu}(E, F) \subset \pi^Z_{\lambda,\mu}(E, F)$.

(2) *If Z is finite dimensional, we have*

$$\pi_{\lambda,\mu}(E,\, F)=FQI^Z_{\lambda,\mu}(E,\, F)=QI^Z_{\lambda,\mu}(E,\, F)=\pi^Z_{\lambda,\mu}(E,\, F).$$

PROOF. We shall prove (2). If Z is finite dimensional, the identity operator I: $Z \to Z$ is $(\mu,\, \mu)$-absolutely summing. Therefore if $T \in \pi^Z_{\lambda,\mu}(E,\, F)$, we have the following inequalities for each finite set of elements $x_1,\dots,\, x_n$ in E:

$$\begin{aligned}\|(\|Tx_i\|)\|_\lambda &\le \rho \sup_{A\in S(E,Z)} \|(\|Ax_i\|)\|_\mu \\ &\le \rho\rho' \sup_{A\in S(E,Z)} \sup_{\|\xi\|\le 1} \|(<Ax_i,\, \xi>)\|_\mu \\ &\le \rho\rho' \sup_{A\in S(E,Z)} \sup_{\|\xi\|\le 1} \|(<x_i,\, A'\xi>)\|_\mu \\ &\le \rho\rho' \sup_{\|a\|\le 1} \|(<x_i,\, a>)\|_\mu .\end{aligned}$$

Therefore we have $T \in \pi_{\lambda,\mu}(E,\, F)$. The proof of (2) is complete. We omit the proof of (1).

§2. General properties

PROPOSITION 2. *Let $L(E,\, F)$ be the normed space of all bounded linear operators with the norm $\|T\| = \sup_{\|x\|\le 1} \|Tx\|$, let λ be of type Λ and let μ be of type M. Then*:

(1) $QI^Z_{\lambda,\mu}(E,\, F) \subset L(E,\, F)$ and $\|T\| \le \sigma^Z_{\lambda,\mu}(T)$ for every $T \in QI^Z_{\lambda,\mu}(E,\, F)$.

(2) $\pi^Z_{\lambda,\mu}(E,\, F) \subset L(E,\, F)$ and $\|T\| \le \pi^Z_{\lambda,\mu}(T)$ for every $T \in \pi^Z_{\lambda,\mu}(E,\, F)$.

PROOF. If $T \in QI^Z_{\lambda,\mu}(E, F)$, for any $\varepsilon > 0$ there exists $\rho_K < \sigma^Z_{\lambda,\mu}(T) + \varepsilon$. Therefore we have

$$\begin{aligned}\|(\|Tx\|,\, 0,\dots)\|_\lambda &\le \rho_K(T) \sup_{A\in K} \|(\|Ax\|,\, 0,\dots)\|_\mu \\ &\le \rho_K(T) \sup_{A\in K} \|Ax\| \\ &\le (\sigma^Z_{\lambda,\mu}(T)+\varepsilon)\, \|x\| .\end{aligned}$$

Consequently we have $\|T\| \le \sigma^Z_{\lambda,\mu}(T)$. Thus (1) has been proved. (2) can be similarly proved.

THEOREM 1. *Let E and F be normed spaces, let Z be a Banach space and let λ and μ be of type Λ and of type M respectively. Then*:

(a) *Let λ be of type M. Then if a set $K \subset S(E,\, F)$ is compact in the simple convergence topology, the following properties of T: $E \to F$ are equivalent*:

(1) $\|(\|Tx_i\|)\|_\lambda \le \rho \sup_{A\in K} \|(\|Ax_i\|)\|_\mu$ *for each finite set $\{x_1,\dots,\, x_n\}$ in E.*

(2) *If $x=(x_i) \in \mu^Z_K(E)$, then $\hat{T}x=(Tx_i) \in \lambda[F]$.*

(b) *Let us consider the following properties of* $T: E \to F$.

(i) T *is a* $(Z; \lambda, \mu)$*-absolutely summing operator.*

(ii) *If* $x=(x_i) \in \mu^Z(E) \cap c_0(E)$, $\hat{T}x=(Tx_i) \in \lambda[F]$.

(iii) *If* $x=(x_i) \in \mu^Z(E)$, *then* $\hat{T}x=(Tx_i) \in \lambda[F]$.

Then

(1) (i) *and* (ii) *are equivalent.*

(2) *If* λ *is of type* M, (i), (ii) *and* (iii) *are equivalent.*

(3) *Let* λ *be of type* M. *Then even if* λ *and* μ *do not satisfy the condition* (f), (i) *and* (iii) *are equivalent.*

PROOF. We shall prove (a). The proof of (b) is similar.

(1)⇒(2): Let (1) be valid and let $x=(x_i) \in \mu_K^Z(E)$. Then for each fixed i, we consider $x^i=(x_1,\dots, x_i, 0,\dots)$ and obtain

$$\|(\|Tx_1\|,\dots, \|Tx_i\|, 0,\dots)\|_\lambda \le \rho \sup_{A \in K} \|(\|Ax_1\|,\dots, \|Ax_i\|, 0,\dots)\|_\mu$$

and since the norm $\|\cdot\|_\mu$ is absolutely monotone, the above expression is $\le \rho \varepsilon_{\mu_K}^Z(x)$. Since λ satisfies the condition (a), we have $\|(\|Tx_i\|)\|_\lambda < \infty$. Consequently $\hat{T}x \in \lambda[F]$. Thus (1)⇒(2) is proved.

(2)⇒(1): Let (2) be valid and let (1) be not valid. Then for any positive integer j there exists a finite set $\{x_i^j\}_{1 \le i \le n(j)}$ in E satisfying $\sup_{A \in K} \|(\|Ax_i^j\|)\|_\mu \le 1$ and $\|(\|Tx_i^j\|)\|_\lambda > j2^j$. By our assumption it follows that the sequence x of vectors

$$x_1^1/2,\dots, x_{n(1)}^1/2,\ x_1^2/2^2,\dots, x_{n(2)}^2/2^2,\dots, x_1^j/2^j,\dots, x_{n(j)}^j/2^j,\dots$$

is in $\mu_K^Z(E)$. Also since the norm defining the topology λ is absolutely monotone, it follows that $\hat{T}x \notin \lambda[F]$. This is a contradiction. The proof is complete.

THEOREM 2. *Let E and F be normed spaces, let Z be a Banach space and let λ and μ be of type M. Then we have the following properties:*

(1) *The space* $QI_{\lambda,\mu}^Z(E, F)$ *is a normed space with the norm* $\sigma_{\lambda,\mu}^Z(T)$ *and if* F *is a Banach space,* $QI_{\lambda,\mu}^Z(E, F)$ *is complete.*

(2) *The space* $\pi_{\lambda,\mu}^Z(E, F)$ *is a normed space with the norm* $\pi_{\lambda,\mu}^Z(T)$ *and if* F *is a Banach space,* $\pi_{\lambda,\mu}^Z(E, F)$ *is complete.*

PROOF. We shall only prove (1), since the proof of (2) is similar. First we prove that $\sigma_{\lambda,\mu}^Z(T)$ is a norm. If $T \in QI_{\lambda,\mu}^Z(E, F)$, for any $\varepsilon > 0$ there exists $\rho_K(T) < \sigma_{\lambda,\mu}^Z(T) + \varepsilon$. Therefore for each finite set of elements $x_1,\dots, x_n$ in E we have the following inequality

$$\|(\|Tx_i\|)\|_\lambda \le \rho_K(T) \sup_{A \in K} \|(\|Ax_i\|)\|_\mu.$$

Hence we have

$$\|(\|aTx_i\|)\|_\lambda \le |a| \rho_K(T) \sup_{A \in K} \|(\|Ax_i\|)\|_\mu.$$

Consequently we have

$$\sigma^Z_{\lambda,\mu}(aT)\leq\rho_K(aT)\leq|a|\rho_K(T)\leq|a|\,(\sigma^Z_{\lambda,\mu}(T)+\varepsilon).$$

Hence $\sigma^Z_{\lambda,\mu}(aT)\leq|a|\sigma^Z_{\lambda,\mu}(T)$. In the same way we have $|a|\sigma^Z_{\lambda,\mu}(T)\leq\sigma^Z_{\lambda,\mu}(aT)$. Therefore we have

$$|a|\sigma^Z_{\lambda,\mu}(T)=\sigma^Z_{\lambda,\mu}(aT).$$

Next we show the following inequality:

$$\sigma^Z_{\lambda,\mu}(S+T)\leq\sigma^Z_{\lambda,\mu}(S)+\sigma^Z_{\lambda,\mu}(T)\qquad\text{for any}\quad S,\ T\in QI^Z_{\lambda,\mu}(E,F).$$

For any $\varepsilon>0$ there exist $\rho_K(S)$ and $\rho_{K'}(T)$ such that

$$\sigma^Z_{\lambda,\mu}(S)+\varepsilon>\rho_K(T)\quad\text{and}\quad\sigma^Z_{\lambda,\mu}(T)+\varepsilon>\rho_{K'}(T).$$

Therefore for each finite set $\{x_1,\dots,x_n\}$ in E, we have the following inequalities:

$$\|(\|Sx_i\|)\|_\lambda\leq\rho_K(S)\sup_{A\in K}\|(\|Ax_i\|)\|_\mu$$

and

$$\|(\|Tx_i\|)\|_\lambda\leq\rho_{K'}(T)\sup_{A\in K'}\|(\|Ax_i\|)\|_\mu.$$

Consequently we have

$$\|(\|(S+T)x_i\|)\|_\lambda\leq(\rho_K(S)+\rho_{K'}(T))\sup_{A\in K\cup K'}\|(\|Ax_i\|)\|_\mu.$$

This implies that $\rho_K(S)+\rho_{K'}(T)\geq\rho_{K\cup K'}(S+T)$. Therefore we have

$$\sigma^Z_{\lambda,\mu}(S)+\sigma^Z_{\lambda,\mu}(T)\geq\sigma^Z_{\lambda,\mu}(S+T).$$

This proves that $\sigma^Z_{\lambda,\mu}$ is a norm.

Secondly, assuming that F is a Banach space, we prove that $QI^Z_{\lambda,\mu}(E,F)$ is complete. Let $\{T_n\}$ be a Cauchy sequence in $QI^Z_{\lambda,\mu}(E,F)$. Then for given $\varepsilon>0$ the inequality $\|T_n-T_m\|\leq\sigma^Z_{\lambda,\mu}(T_n-T_m)<\varepsilon$ holds for $n,\ m>N$. Thus $\{T_n\}$ is a Cauchy sequence in the Banach space $L(E,F)$ and therefore there exists a $T\in L(E,F)$ such that $\lim_{n\to\infty}\|T_n-T\|=0$. Since $\sigma^Z_{\lambda,\mu}(T_n-T_m)<\varepsilon$ for $n,\ m>N$, for $n,\ m>N$ and for each finite set $\{x_i\}_{1\leq i\leq n}$ in E there exists a set $K\subset S(E,Z)$, compact in the simple convergence topology, and we get

$$\|(\|T_nx_i-T_mx_i\|)\|_\lambda\leq\varepsilon\sup_{A\in K}\|(\|Ax_i\|)\|_\mu.$$

This implies

$$\sigma^Z_{\lambda,\mu}(T_n-T)\leq\rho_K(T_n-T)\leq\varepsilon\qquad\text{for any}\quad n>N.$$

The proof is complete.

PROPOSITION 3. *Let E and F be normed spaces, let Z be a Banach space and let λ and μ be of type Λ and of type M respectively. Then:*

(1) *If $\mu \cap c_0 \not\subset \lambda$, then $\pi^Z_{\lambda,\mu}(E, F)=\{0\}$.*

(2) *$FQI^Z_{l_\infty,\mu}(E, F)=L(E, F)$.*

PROOF. (1) If possible, let $T(\neq 0)\in \pi^Z_{\lambda,\mu}(E, F)$ and let $(a_n)\in \mu \cap c_0\backslash\lambda$. Here a_i may be assumed to be positive for $i=1, 2,\dots$. Let x_0 be an element in E such that $\|x_0\|=1$ and $\|Tx_0\|=v\ (\neq 0)$. Then we have $(\|T(a_i/v)x_0\|)=(a_i)\in \mu \cap c_0\backslash\lambda$ but $(\|(a_i/v)x_0\|)=(a_i/v)\in \mu \cap c_0$. For any $A\in L(E, Z)$ we have

$$\|(\|A(a_i/v)x_0\|)\|_\mu \leq \|(\|A\|\ \|(a_i/v)x_0\|)\|_\mu < \infty.$$

Therefore for any $A\in L(E, Z)$, we have $(\|A(a_i/v)x_0\|)\in\mu$. Consequently $((a_i/v)x_0)\in \mu^Z(E)\cap c_0(E)$. This contradicts $T\in \pi^Z_{\lambda,\mu}(E, F)$, which proves (1).

(2) By Proposition 1, we have

$$\pi_{l_\infty,\mu}(E, F)\subset FQI^Z_{l_\infty,\mu}(E, F)\subset L(E, F).$$

Also, by [6], we have $\pi_{l_\infty,\mu}(E, F)=L(E, F)$. This implies $FQI^Z_{\lambda,\mu}(E, F)=L(E, F)$. The proof is complete.

THEOREM 3. *Let E, F and G be normed spaces, let Z be a Banach space and let λ and μ be of type Λ and of type M respectively. Then:*

(1) (i) *If $S\in L(E, F)$ and $T\in QI^Z_{\lambda,\mu}(F, G)$, then $TS\in QI^Z_{\lambda,\mu}(E, G)$ and $\sigma^Z_{\lambda,\mu}(TS)\leq \|S\|\sigma^Z_{\lambda,\mu}(T)$.*

(ii) *If $S\in QI^Z_{\lambda,\mu}(E, F)$ and $T\in L(F, G)$, then $TS\in QI^Z_{\lambda,\mu}(E, G)$ and $\sigma^Z_{\lambda,\mu}(TS)\leq \|T\|\sigma^Z_{\lambda,\mu}(S)$.*

(2) (i) *If $S\in L(E, F)$ and $T\in \pi^Z_{\lambda,\mu}(F, G)$, then $TS\in \pi^Z_{\lambda,\mu}(E, G)$ and $\pi^Z_{\lambda,\mu}(TS)\leq \|S\|\pi^Z_{\lambda,\mu}(T)$.*

(ii) *If $S\in \pi^Z_{\lambda,\mu}(E, F)$ and $T\in L(F, G)$, then $TS\in \pi^Z_{\lambda,\mu}(E, G)$ and $\pi^Z_{\lambda,\mu}(TS)\leq \|T\|\pi^Z_{\lambda,\mu}(S)$.*

PROOF. We shall prove (1). The proof of (2) is similar.

(i) For each finite set of elements $x_1,\dots, x_n$ in E, by our assumption the following inequalities are valid:

$$\begin{aligned}\|(\|TSx_i\|)\|_\lambda &\leq \rho_K(T)\sup_{A\in K}\|(\|ASx_i\|)\|_\mu\\ &\leq \rho_K(T)\ \|S\|\sup_{A\in K}\|(\|A(S/\|S\|)x_i\|)\|_\mu\\ &\leq \rho_K(T)\ \|S\|\sup_{Z\in K'}\|(\|Ax_i\|)\|_\mu.\end{aligned}$$

Therefore $TS\in QI^Z_{\lambda,\mu}(E, G)$. For any $\varepsilon>0$ there exists $\rho_K(T)<\sigma^Z_{\lambda,\mu}(T)+\varepsilon$ and by the above inequality the following inequality is valid:

$$\sigma^Z_{\lambda,\mu}(TS) \leq \rho_{K'}(TS) \leq \rho_K(T)\,\|S\| \leq \|S\|\,(\sigma^Z_{\lambda,\mu}(T)+\varepsilon)\,.$$

Consequently we have $\sigma^Z_{\lambda,\mu}(TS) \leq \|S\|\sigma^Z_{\lambda,\mu}(T)$, which proves (i).

(ii) By our assumption there exists a set $K \subset S(F, Z)$, compact in the simple convergence topology, and for each finite set of elements $x_1,\dots, x_n$ in E the following inequality is valid:

$$\|(\|TSx_i\|)\|_\lambda \leq \|T\|\ \|(\|Sx_i\|)\|_\lambda \leq \|T\|\rho_K(S) \sup_{A\in K} \|(\|Ax_i\|)\|_\mu.$$

Therefore we have $TS \in QI^Z_{\lambda,\mu}(E, G)$. For any $\varepsilon>0$ there exists $\rho_K(S)$ such that $\rho_K(S) < \sigma^Z_{\lambda,\mu}(S)+\varepsilon$. Then by the above inequality we have

$$\sigma^Z_{\lambda,\mu}(TS) \leq \|T\|\rho_K(S) \leq \|T\|\,(\sigma^Z_{\lambda,\mu}(S)+\varepsilon)\,.$$

Therefore $\sigma^Z_{\lambda,\mu}(TS) \leq \|T\|\sigma^Z_{\lambda,\mu}(S)$. The proof is complete.

COROLLARY. *Let E be a normed space, let Z be a Banach space and let λ and μ be of type Λ and of type M respectively. Then* $QI^Z_{\lambda,\mu}(E, E)$ (resp. $\pi^Z_{\lambda,\mu}(E, E)$) *is a two side ideal in $L(E, E)$ and for* $S \in QI^Z_{\lambda,\mu}(E, E)$ (resp. $S \in \pi^Z_{\lambda,\mu}(E, E)$) *and $T \in L(E, E)$, the following inequalities hold:* $\sigma^Z_{\lambda,\mu}(ST) \leq \sigma^Z_{\lambda,\mu}(S)\,\|T\|$ (resp. $\pi^Z_{\lambda,\mu}(ST) \leq \pi^Z_{\lambda,\mu}(S)\,\|T\|$) *and* $\sigma^Z_{\lambda,\mu}(TS) \leq \sigma^Z_{\lambda,\mu}(S)\,\|T\|$ (resp. $\pi^Z_{\lambda,\mu}(TS) \leq \pi^Z_{\lambda,\mu}(S)\,\|T\|$).

We use the following result of [6].

LEMMA 2. *Let λ be of type Λ. Then we have $\lambda\otimes E \subset \lambda[E]$.*

Now we denote by $\lambda\otimes_{\alpha_\lambda} F$ the quasi-normed space $\lambda\otimes F$ with the topology induced by the quasi-norm α_λ and also by $\mu\otimes_{\varepsilon^Z_\mu} E$ (resp. $\mu\otimes_{\varepsilon^Z_{\mu_K}} E$) the normed space $\mu\otimes E$ (resp. the semi-normed space $\mu\otimes E$) with the topology induced by the norm ε^Z_μ (resp. the semi-norm $\varepsilon^Z_{\mu_K}$).

PROPOSITION 4. *Let E and F be normed spaces, let Z be a Banach space and let λ and μ be of type Λ and of type M respectively. If $T: E \to F$, we have the following properties:*

(1) *Let λ be of type M and let $QI^Z_{\lambda,\mu}(E, F) \neq \{0\}$. Then T belongs to $QI^Z_{\lambda,\mu}(E, F)$ if and only if there exists a set $K \subset S(E, Z)$, compact in the simple convergence topology, such that $I\otimes T$: $\mu\otimes_{\varepsilon^Z_{\mu_K}} E \to \lambda\otimes_{\alpha_\lambda} F$ is continuous.*

(2) *Let $\mu \neq l_\infty$ and let $\pi^Z_{\lambda,\mu}(E, F) \neq \{0\}$. Then T belongs to $\pi^Z_{\lambda,\mu}(E, F)$ if and only if $I\otimes T$: $\mu\otimes_{\varepsilon^Z_\mu} E \to \lambda\otimes_{\alpha_\lambda} F$ is continuous.*

PROOF. We shall prove (1). The proof of (2) is similar. Assume that there exists a set $K \subset S(E, Z)$, compact in the simple convergence topology, such that $I\otimes T$: $\mu\otimes_{\varepsilon^Z_{\mu_K}} E \to \lambda\otimes_{\alpha_\lambda} F$ is continuous and T does not belong to $QI^Z_{\lambda,\mu}(E, F)$. Then for any positive integer j there exists a finite set $\{x^j_i\}_{1\leq i\leq n(j)}$ in E satisfying $\alpha_\lambda((Tx^j_i)) > j\varepsilon^Z_{\mu_K}((x^j_i))$.

Since $\sum_{i=1}^{n} e_i \otimes x_i = \sum_{i=1}^{n} (0,\dots, 0, x_i, 0,\dots) = (x_1,\dots, x_n, 0,\dots)$, we have

$$\begin{aligned}\alpha_\lambda(I\otimes T(\sum_{i=1}^{n} e_i\otimes x_i^j)) &= \alpha_\lambda(\sum_{i=1}^{n} e_i\otimes Tx_i^j)\\ &= \alpha_\lambda((Tx_i^j)) > j\varepsilon_{\mu_K}^Z((x_i^j))\\ &= j\varepsilon_{\mu_K}^Z(\sum_{i=1}^{n} e_i\otimes x_i^j).\end{aligned}$$

Consequently $I\otimes T\colon \mu\otimes_{\varepsilon_{\mu_k}^Z} E\to\lambda\otimes_{\alpha_\lambda} F$ is not continuous. This is a contradiction. Thus the sufficiency is proved. Conversely, assume that $T\in QI_{\lambda,\mu}^Z(E, F)$. Then there exists a set $K\subset S(E, Z)$, compact in the simple convergence topology, such that T: $\mu_K^Z(E)\to\lambda[F]$ is continuous. Therefore $I\otimes T\colon \mu\otimes_{\varepsilon_{\mu_k}^Z} E\to\lambda\otimes_{\alpha_\lambda} F$ is continuous, for $\mu\otimes_{\varepsilon_{\mu_k}^Z} E\subset\mu_K^Z(E)$ and $\hat{T}$ and $I\otimes T$ have the same values on $\mu\otimes E$. This completes the proof.

§3. Some inclusion relations

Suppose that α and β are sequence spaces. We define $\alpha\cdot\beta=\{(x_n y_n)|(x_n)\in\alpha, (y_n)\in\beta\}$. Here we denote by $D(\beta, \alpha)$ the set of diagonal matrices carrying β into α. We use the following results of Crofts [4].

Lemma 3. $D(\beta, \alpha)\subset(\beta\cdot\alpha^\times)^\times$ and, if α is a Köthe space, $D(\beta, \alpha)=(\beta\cdot\alpha^\times)^\times$.

Proposition 5. *Let E and F be normed spaces, let Z be a Banach space, let λ_1 and λ_2 be of type Λ and let μ_1 and μ_2 be of type M. Then:*

(1) *Let λ_1 and λ_2 be of type M. Then if $\mu_1\supset\mu_2$ and $\lambda_2\supset\lambda_1$, then $QI_{\lambda_1,\mu_1}^Z(E, F)\subset QI_{\lambda_2,\mu_2}^Z(E, F)$.*

(2) *If $\mu_1\supset\mu_2$ and $\lambda_2\supset\lambda_1$, then $\pi_{\lambda_1,\mu_1}^Z(E, F)\subset\pi_{\lambda_2,\mu_2}^Z(E, F)$.*

Theorem 4. *Let E and F be normed spaces, let Z be a Banach space, let λ and $\tilde{\lambda}$ be of type Λ and let μ and $\tilde{\mu}$ be of type M. Then:*

(1) *Let λ and $\tilde{\lambda}$ be of type M. If there exists a sequence space $\nu\subset l_\infty$ satisfying the condition $\nu\cdot\tilde{\mu}\subset\mu$ and $(\nu\cdot\lambda^\times)^\times\subset\tilde{\lambda}$, then we have $QI_{\lambda,\mu}^Z(E, F)\subset QI_{\tilde{\lambda},\tilde{\mu}}^Z(E, F)$.*

(2) *If there exists a sequence space $\nu\subset l_\infty$ satisfying the condition $\nu\cdot\tilde{\mu}\subset\mu$ and $(\nu\cdot\lambda^\times)^\times\subset\tilde{\lambda}$, then we have $\pi_{\lambda,\mu}^Z(E, F)\subset\pi_{\tilde{\lambda},\tilde{\mu}}^Z(E, F)$.*

Proof. We shall only prove (1), since the proof of (2) is similar. Let $T\in QI_{\lambda,\mu}^Z(E, F)$. Then there exists a set $K\subset S(E, Z)$, compact in the simple convergence topology, such that $(x_i)\in\mu_K^Z(E)$ implies $(Tx_i)\in\lambda[F]$. If $(x_i)\in\tilde{\mu}_K^Z(E)$, for any $\alpha=(\alpha_i)\in\nu$ and for any $A\in K$ we have $(\alpha_i\|Ax_i\|)=\alpha(\|Ax_i\|)\in\nu\cdot\tilde{\mu}\subset\mu$. Therefore we have $|\alpha|(\|Tx_i\|)=(\|T(\alpha_i x_i)\|)\in\lambda$. Since λ is solid, $\alpha(\|Tx_i\|)\in\lambda$ and therefore we have

$(\|Tx_i\|)\in D(\nu,\lambda)$. Hence by Lemma 3 $(\|Tx_i\|)\in(\nu\cdot\lambda^{\times})^{\times}\subset\tilde{\lambda}$. Thus T is $(Z;\tilde{\lambda},\tilde{\mu})$-quasi-integrable operator. The proof is complete.

COROLLARY. *Let H be a Hilbert space, let F be a Banach space and let λ and μ be of type A and of type M respectively. Then if $\lambda\supset\mu$ and λ is a Köthe space, we have $\pi^Z_{\lambda,\mu}(H,F)=L(H,F)$.*

PROOF. Set $\nu=(l_1^{\times}\cdot\mu)^{\times}=\mu^{\times}$. Then $\nu\cdot\mu\subset l_1$ and $(\mu^{\times}\cdot l_1^{\times})^{\times}=\mu^{\times\times}\subset\lambda^{\times\times}=\lambda$. By Theorem 4 we have $\pi^Z_{\lambda,\mu}(H,F)\supset\pi^Z_{l_1,l_1}(H,F)$ and by [2] we have $\pi^Z_{l_1,l_1}(H,F)=L(H,F)$. Therefore we have $\pi^Z_{\lambda,\mu}(H,F)=L(H,F)$. The proof is complete.

§4. The composition theorem

DEFINITION 4. *A Banach space F is said to have the extension property if each operator $T_0\in L(E_0,F)$, E_0 being any linear subspace of an arbitrary Banach space E, can be extended to a $T\in L(E,F)$ preserving its norm.*

THEOREM 5. *Let E, F and G be normed spaces, let $1\leq p<\infty$ and $1\leq r_i\leq\infty$ $(i=1,2)$ be real numbers such that $1/p+1/r_1\leq 1/r_2$, let λ_1 and λ_2 be sequence spaces of type A satisfying $\lambda_2\supset\lambda_1\cdot l_p$, and let us assume that Z is a Banach space having the extension property. Then for any $T\in QI^Z_{l_p,l_p}(E,F)$ and $S\in\pi^Z_{\lambda_1,l_{r_1}}(F,G)$ the composition ST belongs to $QI^Z_{\lambda_2,l_{r_2}}(E,G)$ and satisfies $\sigma^Z_{\lambda_2,l_{r_2}}(ST)\leq C\pi^Z_{\lambda_1,l_{r_1}}(S)\cdot\sigma^Z_{l_p,l_p}(T)$, where C is a constant.*

PROOF. It suffices to prove the assertion under assumption $1/r_2=1/p+1/r_1$. Since T is a $(Z;l_p,l_p)$-quasi-integrable operator, by [2] there is a probability measure, that is, a regular positive Borel measure μ with total mass 1 on a set $K\subset S(E,Z)$, compact in the simple convergence topology, such that $\|Tx\|\leq\rho_K(T)\left(\int_K\|Ax\|^p d\mu\right)^{1/p}$ for every $x\in E$. Let $\{x_i\}_{1\leq i\leq n}$ be an arbitrary finite set of elements in E. Put $x_i=x_i^0\xi_i$ where $\xi_i=\left(\int_K\|Ax_i\|^{r_2}d\mu\right)^{1/p}$. Then by our assumption, it follows that

$$\|(\|STx_i\|)\|_{\lambda_2}\leq C\|(\|STx_i^0\|)\|_{\lambda_1}\cdot\|(\|\xi_i\|)\|_{l_p}$$

$$\leq C\pi^Z_{\lambda,{}_1l_{r_1}}(S)\sup_{B\in S(F,Z)}\|(\|BTx_i^0\|)\|_{l_{r_1}}\cdot\left(\sum_{i=1}^n\int_K\|Ax_i\|^{r_2}d\mu\right)^{1/p},$$

where C is a constant. Here there exists a Z-scalarly measurable and Z'-valued derivative g_η such that the terms of form $<BTx,\eta>$, $\eta\in Z'$, can be written as

$$<BTx,\eta>=\int_K<g_\eta,Ax>d\mu$$

with $g_\eta\in L_{Z'W}(K,\mu)$ for all $\eta\in Z'$ satisfying the inequality

$$\left(\int_K\|g_\eta\|^{p'}d\mu\right)^{1/p'}\leq\|B\|\rho_K(T)\|\eta\|,\quad 1/p+1/p'=1.$$

In fact, let $E^p_Z(K, \mu)$ be the subspace of $L^p_Z(K, \mu)$ which is constituted by the rest class $\hat{\phi}_x$ for $\phi_x(A)=Ax\in L(K, Z)$. Then for each $B\in L(F, Z)$, there exists a linear operator β_B on $E^p_Z(K, \mu)$ into Z defined by $<\hat{\phi}_x, \beta_B>=BTx$. It satisfies

$$\|<\hat{\phi}_x, \beta_B>\|\leq\|B\|\,\|Tx\|\leq\rho_K(T)\,\|B\|\Big(\int_K\|Ax\|^p d\mu\Big)^{1/p}.$$

Since Z has the extension property, we obtain the above result by [1]. Hence by Hölder's inequality, we obtain

$$|<BTx, \eta>|\leq\int_K\|Ax\|\,\|g_\eta\|d\mu$$

$$\leq\int_K\|Ax\|^{r_2/p}(\|Ax\|^{r_2}\|g_\eta\|^{p'})^{1/r_1}\|g_\eta\|^{p'/r'_2}d\mu$$

$$\leq\Big(\int_K\|Ax\|^{r_2}d\mu\Big)^{1/p}\Big(\int_K\|Ax\|^{r_2}\|g_\eta\|^{p'}d\mu\Big)^{1/r_1}\Big(\int_K\|g_\eta\|^{p'}d\mu\Big)^{1/r'_2}.$$

Replacing x by x_i^0 in the above inequality, we obtain

$$|<BTx_i^0, \eta>|^{r_1}\leq\Big(\int_K\|Ax_i\|^{r_2}\|g_\eta\|^{p'}d\mu\Big)\Big(\int_K\|g_\eta\|^{p'}d\mu\Big)^{r_1/r'_2},$$

$$\|BTx_i^0\|^{r_1}=\sup_{\|\eta\|\leq1}|<BTx_i^0, \eta>|^{r_1}$$

$$\leq\Big(\int_K\|Ax_i\|^{r_2}\,(\sup_{\|\eta\|\leq1}\|g_\eta\|)^{p'}d\mu\Big)\Big(\int_K(\sup_{\|\eta\|\leq1}\|g_\eta\|)^{p'}d\mu\Big)^{r_1/r'_2}.$$

Finally, we get

$$(\sum_{i=1}^n\|BTx_i^0\|^{r_1})^{1/r_1}\leq\Big(\int_K(\sum_{i=1}^n\|Ax_i\|^{r_2})(\sup_{\|\eta\|\leq1}\|g_\eta\|)^{p'}d\mu\Big)^{1/r_1}$$

$$\times\Big(\int_K\sup_{\|\eta\|\leq1}\|g_\eta\|^{p'}d\mu\Big)^{1/r'_2}$$

$$\leq\sup_{A\in K}(\sum_{i=1}^n\|Ax_i\|^{r_2})^{1/r_1}\Big(\sup_{\|\eta\|\leq1}\int_K\|g_\eta\|^{p'}d\mu\Big)^{1/p'}$$

Consequently we have

$$\|(\|STx_i\|)\|_{\lambda_2}\leq C\pi^Z_{\lambda_1,1r_1}(S)\sup_{A\in K}(\sum_{i=1}^n\|Ax_i\|^{r_2})^{1/r_2}\cdot\rho_K(T).$$

The proof is complete.

PROPOSITION 6. *Let E, F and G be normed spaces, let $1\leq p<\infty$, $1\leq r\leq\infty$, $1/p+1/r\leq1$, let λ be of type A satisfying $l_p\cdot\lambda\subset l_1$ and assume that Z is a Banach space having the extension property. Then for any $T\in QI^Z_{l_p,l_p}(E, F)$ and any $S\in\pi^Z_{\lambda,l_r}(F, G)$*

the composition ST belongs to $ST\in QI^Z_{l_1,l_1}(E, G)$.

PROOF. In case of $p=1$, this is clear by Theorem 3. We shall show this in case of $p>1$. Put $1/p+1/p'=1$. Then it satisfies $\lambda\subset l_{p'}$ and $l_r\supset l_{p'}$. By Proposition 5, $S\in\pi^Z_{\lambda,l_r}(F, G)\subset\pi^Z_{l_{p'},l_{p'}}(F, G)$. Hence applying Theorem 5 to S and T, we obtain $ST\in QI^Z_{l_1,l_1}(E, G)$. Thus the proof is complete.

§5. Special cases

LEMMA 4. *Let E be isomorphic to a subspace of* $L_1(\mu)$ *for a measure space* (K, Σ, μ), *let F be any normed space, let Z be a Banach space and let* λ *be of type M. Then* $T\in L(E, F)$ *belongs to* $FQI^Z_{\lambda,l_1}(E, F)$ *if and only if for any* $S\in L(c_0, E)$ *the composition TS belongs to* $FQI^Z_{\lambda,l_1}(c_0, F)$.

PROOF. By virtue of Theorem 3 it is clear that if $T\in FQI^Z_{\lambda,l_1}(E, F)$ and $S\in L(c_0, E)$, then $TS\in FQI^Z_{\lambda,l_1}(c_0, F)$. Conversely, we assume that $T\in L(E, F)$ satisfies the condition $TS\in FQI^Z_{\lambda,l_1}(c_0, F)$ for any $S\in L(c_0, E)$ but $T\notin FQI^Z_{\lambda,l_1}(E, F)$. Then there exists a sequence $\{x_i\}\subset E$ such that $\sum_i x_i$ converges unconditionally and $\|(\|Tx_i\|)\|_\lambda=\infty$. Here we define $S\in L(c_0, E)$ by $S((a_i))=\sum_i a_ix_i$ for each $(a_i)\in c_0$. Then we have $\|(\|TS(e_i)\|)\|_\lambda=\infty$. Since by [3] $L(c_0, Z)=l_1(Z)$, we have $\sum_i\|Ae_i\|<\infty$ for any $A\in L(c_0, Z)\cap A(c_0, Z)$. By virtue of Theorem 1, this is a contradiction and the proof is complete.

THEOREM 6. *Let E and F be normed spaces and let Z be a Banach space. Then*:

(1) *Let* λ_1 *and* λ_2 *be of type M* (resp. Λ). *Then if* $l_2\cdot\lambda_1^\times\supset\lambda_2^\times$ *and* λ_2 *is a Köthe space, we have* $QI^Z_{\lambda_1,l_1}(E, F)\subset QI^Z_{\lambda_2,l_2}(E, F)$ (resp. $\pi^Z_{\lambda_1,l_1}(E, F)\subset\pi^Z_{\lambda_2,l_2}(E, F)$).

(2) *Let* λ_1 *and* λ_2 *be of type M, let E and F be the same spaces as in Lemma* 3 *and assume that Z has the extension property. Then if* $l_2\cdot\lambda_1^\times\subset\lambda_2^\times$ *and* λ_1, λ_2 *are Köthe spaces, we have* $FQI^Z_{\lambda_1,l_1}(E, F)\supset FQI^Z_{\lambda_2,l_2}(E, F)$.

PROOF. (1) Putting $\nu=(l_1^\times\cdot l_2)^\times=l_2$, we have $(l_2\cdot\lambda_1^\times)^\times\subset\lambda_2^{\times\times}=\lambda_2$ and $l_2\cdot l_2\subset l_1$. Therefore by Theorem 4 $QI^Z_{\lambda_1,l_1}(E, F)\subset QI^Z_{\lambda_2,l_2}(E, F)$.

(2) Let $T\in FQI^Z_{\lambda_2,l_2}(E, F)$, $S\in L(c_0, E)$ is always 2-absolutely summing and therefore (Z; l_2, l_2)-quasi-integrable by Proposition 1. Since $(l_2\cdot\lambda_1^\times)\subset\lambda_2^\times$, it follows that $l_2\cdot\lambda_2\subset\lambda_1$. Therefore by Theorem 5 we have $TS\in FQI^Z_{\lambda_1,l_1}(c_0, F)$. Hence by Lemma 4, we have $T\in FQI^Z_{\lambda_1,l_1}(E, F)$, which completes the proof.

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