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On Homogeneous Systems I

Michihiko KIKKAWA
Department of Mathematics, Shimane University, Matsue, Japan
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In this paper, homogeneous systems which have been introduced in [4] will be considered on differentiable manifolds. It is intended to show that the various results in [2], [3] for a homogeneous Lie loop G are essentially those results for the homogeneous system of G. Let (G, η) be a differentiable homogeneous system on a connected differentiable manifold G. The canonical connection and the tangent Lie triple algebra of (G, η) are defined in §§1, 2 in the same way as in the case of homogeneous Lie loops [2]. At any point e, G can be expressed as a reductive homogeneous space A/K with the canonical connection and with the decomposition $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$ of the Lie algebra of A, where $\mathfrak{G}$ is the tangent L. t. a. of (G, η) at e. In §3 we shall treat of the regular homogeneous system, a geodesic homogeneous system G in which the linear representation of K on $\mathfrak{G}$ coincides with the holonomy group at e. The following fact will be shown in §4; if (G, η) is a regular homogeneous system, then there exists a 1–1 correspondence between the set of invariant subsystem of G and the set of invariant subalgebras of its tangent L. t. a. (Theorem 5).

§1. Canonical Connection of Homogeneous Systems

Let η be a differentiable *homogeneous system* on a connected (C^∞-class) differentiable manifold G of dimension n; that is, a differentiable map η: $G\times G\times G\to G$ satisfying (1) $\eta(x, x, y)=\eta(x, y, x)=y$, (2) $\eta(x, y, \eta(y, x, z))=z$ and (3) $\eta(x, y, \eta(u, v, w))=\eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w))$ for any $x, y, z, u, v, w\in G$ (cf. [4]). Each *displacement* $\eta(x, y)$: $G\to G$ of η defined by $z\mapsto\eta(x, y, z)$, $z\in G$, is a diffeomorphism of G. Throughout this paper the manifold G is always assumed to be connected and second countable.

Now we introduce a linear connection on G associated with η as follows: Let π: $P(G)\to G$ be the bundle of linear frames on G. For each frame $\boldsymbol{u}\in P(G)$ at $a=\pi(\boldsymbol{u})\in G$, let $\sum_{\boldsymbol{u}}$ be a global cross section of $P(G)$ through $\boldsymbol{u}$ which is given by $\sum_{\boldsymbol{u}}(x)=\eta_*(a, x)\boldsymbol{u}$ for $x\in G$, where $\eta_*(a, x)$ denotes the map of linear frames induced from the diffeomorphism $\eta(a, x)$. For any $g\in GL(n, \boldsymbol{R})$ acting on $P(G)$ on the right we have $(\sum_{\boldsymbol{u}}(x))g=\sum_{\boldsymbol{u}g}(x)$. Hence the n-dimensional distribution Q which assigns to each $\boldsymbol{u}\in P(G)$ the tangent space $Q_{\boldsymbol{u}}$ to the section $\sum_{\boldsymbol{u}}$ at $\boldsymbol{u}$ is right invariant and so the infinitesimal connection of $P(G)$ with Q as its horizontal subspaces is defined and it induces a linear connection ∇ on G, which will be called the *canonical connection* of η.

EXAMPLE. Let G be a connected homogeneous Lie loop under the binary operation $xy=\mu(x, y)$ (cf. [2]), that is, μ is a differentiable loop on the manifold G whose left translations L_x: $y\mapsto xy$ satisfy the conditions; (1) for each $x\in G$ there exists an inverse x^{-1} of x such that $(L_x)^{-1}=L_{x^{-1}}$ and (2) every *left inner mapping* $L_{x,y}:=L_{xy}^{-1}L_xL_y$ is an automorphism of μ. If we set $\eta(x, y, z):=\mu(x, \mu(x^{-1}y, x^{-1}z))$ for $x, y, z \in G$, we see that η is a homogeneous system on G. In this case the canonical connection of μ introduced in [2] is nothing but the canonical connection of η defined above. In particular, if μ is a Lie group then the connection is the $(-)$-connection of E. Cartan.

For a fixed $e \in G$, we have seen in [4] that the homogeneous system η defines a binary operation $\mu^{(e)}(x, y):=\eta(e, x, y)$ whose left translations have the same properties as those of homogeneous loops; that is, e is the identity of $\mu^{(e)}$ and the conditions (1) and (2) in the example above are satisfied for $x^{-1}=\eta(x, e, e)$. Moreover, the equality $\eta(x, y, z)=\mu^{(e)}(x, \mu^{(e)}(x^{-1}y, x^{-1}z))$ holds and, since η is differentiable in the present case, $\mu^{(e)}$ is a homogeneous Lie loop if and only if all of the right translations of $\mu^{(e)}$ are diffeomorphisms of G (cf. Theorem 2 of [4]). Thus we see that the concept of differentiable homogeneous systems on manifolds is a slight generalization of that of homogeneous Lie loops, under the base point (identity element) free version. In fact we have;

PROPOSITION 1. *For each point $e \in G$, the multiplication $\mu^{(e)}$ forms a local homogeneous Lie loop around e.*

PROOF. We are only to show the fact that every right translation of $\mu^{(e)}$ restricted to a neighbourhood of e is a local diffeomorphism. This follows directly from the differentiability of $\mu^{(e)}$ and the equality $\mu^{(e)}(x, e)=x$. *q.e.d.*

Let $Aut(\eta)$ denote the group of all differentiable automorphisms of η. It contains the group $D(\eta)$ of displacements of η. For a point $e \in G$ fixed, denote by Λ_e the *left inner mapping group* of $\mu^{(e)}$, which is the same as the isotropy subgroup of the group $D(\eta)$ at e (see [2, § 1] and [4, § 3]).

The sections $\{\sum_u; u \in P(G)\}$ of the frame bundle $P(G)$ considered above are invariant under any automorphism of η. Hence we have

PROPOSITION 2. *The automorphism group $Aut(\eta)$ of a homogeneous system η on G is a closed subgroup of the affine transformation group of the canonical connection ∇ of η.*

COROLLARY. *The group of displacements of η and the left inner mapping group Λ_e of η at e are subgroups of the affine transformation group of ∇.*

In the analogous way to the case of homogeneous Lie loops [2, 3], we can show that the homogeneous system (G, η) with a base point e is identified in a natural manner to a reductive homogeneous space with the canonical connection of the second kind. An outline of the construction is as follows: Let K_e be the closure of the left inner mapping group A_e in the isotropy subgroup of the Lie transformation group $Aut(\eta)$ at e. Then K_e is a Lie group and the product manifold $A=G\times K_e$ is again a Lie group under the group multiplication $(x, \alpha)(y, \beta): =(\mu^{(e)}(x, \alpha(y)), L_{x,\alpha(y)}\alpha\beta)$, for (x, α), $(y, \beta)\in A$. The closed subgroup $K: =\{e\}\times K_e$ of A is isomorphic to K_e under the natural projection. As the tangent space at the identity $e\times 1_G$, the Lie algebra $\mathfrak{A}$ of A is decomposed into the direct sum $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$ of the tangent space $\mathfrak{G}=T_e(G)$ to G at e and the Lie algebra $\mathfrak{K}$ of the Lie group K_e. Since the submanifold $G\times 1_G$ of A is invariant under the inner automorphisms by elements of K, we have $Ad(K)\mathfrak{G}\subset\mathfrak{G}$ and we see that A/K is a reductive homogeneous space. In this case A acts effectively on A/K. The map $f: A/K\to G$ defined by $f(x\times K_e)=x$, $x\in G$, is a diffeomorphism which induces an affine isomorphism of the canonical connection of A/K to the canonical connection of η on G. We summarize these facts as follows; (cf. [5])

THEOREM 1. *Let G be a connected differentiable manifold admitting a homogeneous system η, A_e be its left inner mapping group at a base point e. Then, w.r.t. the point e, G can be expressed as a reductive homogeneous space A/K such that the canonical connection of η is expressed as the canonical connection of A/K, where $A=G\times K_e$, $K_e=\bar{A}_e$ and the subgroup K of A is isomorphic to K_e.*

§2. Tangent Lie Triple Algebras

Let $G=A/K$ be the reductive homogeneous space at $e\in G$ of the homogeneous system η on G, with the canonical decomposition of the Lie algebra $\mathfrak{A}$ of A; $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$, $Ad(K)\mathfrak{G}\subset\mathfrak{G}$. Since A acts effectively on G, K_e can be identified with its linear representation on $\mathfrak{G}$, the latter is identified with the adjoint representation of K restricted on the subspace $\mathfrak{G}$ of the Lie algebra $\mathfrak{A}$. (cf. [6], [7], [8]).

Let S and R denote the torsion and curvature tensors of the canonical connection ∇ of η respectively, opposite in their signs to the usually defined ones. Since ∇ is identified with the canonical connection of A/K, we see that $\nabla S=0$ and $\nabla R=0$, and they are evaluated at e as follows; $S_e(X, Y)=[X, Y]_{\mathfrak{G}}$, $R_e(X, Y)Z=[[X, Y]_{\mathfrak{K}}, Z]$ for $X, Y, Z\in\mathfrak{G}$, where $[\ ]_{\mathfrak{G}}$ (resp. $[\ ]_{\mathfrak{K}}$) is the $\mathfrak{G}$-component (resp. $\mathfrak{K}$-component) of the bracket in the Lie algebra $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$. Furthermore, applying the results on reductive homogeneous space ([8]) to our case, we get the following propositions.

PROPOSITION 3. *The restricted holonomy group Φ_e^0 of the canonical connection (with reference point e) is a normal subgroup of the linear representation of $K_e=\bar{A}_e$ on $\mathfrak{G}=T_e(G)$.*

PROPOSITION 4. *For each $X\in\mathfrak{G}$, the 1-parameter subgroup $\exp tX=(x(t), \alpha(t))$ of $A=G\times K_e$ generated by X acts as parallel displacement along the geodesic $x(t)$ from $e=x(0)$.*

A homogeneous system η on G will be said to be *geodesic* if, for any two points x, y on any geodesic arc c (*w.r.t.* ∇), the displacement $\eta(x, y)$ induces the parallel displacement of tangent vectors from x to y along c. By using Proposition 4 we can show that the following proposition whose proof will be omitted (cf. [5]).

PROPOSITION 5. *A homogeneous system* (G, η) *is geodesic if and only if the Lie group* $A=G\times K_e$ *has the following property: For each* $X\in\mathfrak{G}$ *the* 1*-parameter subgroup* $\exp tX$ *is contained in the submanifold* $G\times 1_G$ *of* A.

REMARK. In [2] we have introduced the concept of geodesic homogeneous Lie loops. The proposition above shows that a homogeneous Lie loop is geodesic if and only if its homogeneous system given in the example in §1 is geodesic.

In the same way as in the case of geodesic homogeneous Lie loops, we introduce the concept of tangent Lie triple algebras of geodesic homogeneous systems. On the tangent space $\mathfrak{G}_e=T_e(G)$ to a homogeneous system (G, η) at e, we define a bilinear operation XY and a trilinear operation $[X, Y, Z]$, $X, Y, Z\in\mathfrak{G}_e$ as $XY:=S_e(X, Y)$ and $[X, Y, Z]:=R_e(X, Y)Z$. The tangent space $\mathfrak{G}_e$ equipped with these operations will be called the *tangent Lie triple algebra* of (G, η) at e.

PROPOSITION 6. *The tangent L.t.a. at any point* e *of* (G, η) *is a general Lie triple system.*

The tangent L.t.a. $\mathfrak{G}_e$ at e of the geodesic homogeneous system (G, η) is related to the multiplication $\mu^{(e)}$ in the following manner: Consider the product $(x, 1_G)(y, 1_G)=(\mu^{(e)}(x, y), L_{x,y})$ of any elements of $G\times 1_G$ in the Lie group $A=G\times K_e$. For any $X, Y\in\mathfrak{G}_e$, if $x=\exp tX$ and $y=\exp sY$ then the G-component $\mu^{(e)}(x, y)$ and the K_e-component $L_{x,y}$ induce the infinitesimal operators $\mu_*{}^{(e)}$ and λ_* such that $\mu_*{}^{(e)}(X, Y)=[X^*, Y^*]_e$ for the vector fields $X^*(z)=\eta_*(e, z)X$, $Y^*(z)=\eta_*(e, z)Y$, $z\in G$, and that $\lambda_*: \mathfrak{G}\times\mathfrak{G}\to\mathfrak{K}$ is a bilinear map induced from $\lambda(x, y)=L_{x,y}$ by taking account of the properties $L_{x,e}=L_{e,x}=1_G$. Then we have; (cf. [5])

PROPOSITION 7. *The bilinear and trilinear operations* XY *and* $[X, Y, Z]$ *of the tangent L.t.a. of a geodesic homogeneous system* (G, η) *at* e *is expressed, respectively, as* $XY=\mu_*{}^{(e)}(X, Y)$ *and* $[X, Y, Z]=ad(\lambda_*(X, Y)-\lambda_*(Y, X))Z$ *for* $X, Y, Z\in\mathfrak{G}_e$.

Since every displacement of η is an automorphism of η, it induces an isomorphism of the multiplications $\mu^{(e)}$ and $\mu^{(e')}$ at any e and e'. Therefore, the tangent L.t.a.'s of a geodesic homogeneous system are isomorphic with each other.

§3. Regular Homogeneous Systems

Let (G, η) be a geodesic homogeneous system and $\mathfrak{G}=\mathfrak{G}_e$ its tangent Lie triple algebra at e. An *inner derivation* $D(X, Y)$ of $\mathfrak{G}$ for $X, Y \in \mathfrak{G}$ is an endomorphism of $\mathfrak{G}$ given by $D(X, Y)Z = [X, Y, Z]$ for $Z \in \mathfrak{G}$. We denote by $D(\mathfrak{G})$ the *inner derivation algebra* of $\mathfrak{G}$ which is by definition a Lie subalgebra of $\mathrm{End}(\mathfrak{G})$ generated by all inner derivations of $\mathfrak{G}$. $D(\mathfrak{G})$ is an ideal of the Lie algebra $ad\,\mathfrak{K}|_{\mathfrak{G}} = \mathfrak{K}$. With respect to the canonical connection ∇, $D(X, Y) = R_e(X, Y)$. Since $\nabla R = 0$, $D(\mathfrak{G})$ is equal to the Lie algebra of the holonomy group of ∇ at e.

Suppose that G is a Lie group and (G, η) is the homogeneous system on G induced by the group multiplication of G. Then the left inner mapping group A_e at the identity e is the trivial group consisting of 1_G. Hence $A = G \times 1_G$ and $\mathfrak{K} = D(\mathfrak{G}) = \{0\}$. In this case the tangent L.t.a. $\mathfrak{G}$ is reduced to the Lie algebra of G with the bracket $[X, Y] = XY$ and $[X, Y, Z] = 0$ for any $X, Y, Z \in \mathfrak{G}$. ∇ is an invariant connection on G such that $\nabla S = 0$ and $R = 0$, that is the $(-)$-connection.

A geodesic homogeneous system (G, η) will be said to be *regular* if, at some point e, the linear representation dK_e of the Lie group K_e on the tangent L.t.a. $\mathfrak{G}$ at e coincides with the holonomy group Φ_e of ∇ at e. It is easy to see that the definition for (G, η) to be regular does not depend on the choice of the base point.

PROPOSITION 8. *A geodesic homogeneous system (G, η) is regular if and only if $\mathfrak{K} = D(\mathfrak{G})$.*

PROOF. If (G, η) is regular then it is clear that $\mathfrak{K} = D(\mathfrak{G})$ since the Lie groups $dK_e = Ad(K)$ and Φ_e coincides with each other. Conversely, if $\mathfrak{K} = D(\mathfrak{G})$ then dK_e is equal to the restricted holonomy group Φ_e^0. Let c be any closed piecewise differentiable curve in G with the base point e. There exists a piecewise geodesic closed curve c' homotopic to c with the fixed end point e. Since (G, η) is supposed to be geodesic, the parallel displacement along c' is expressed as a composition of the differentials of displacements which must belong to the linear representation of A_e. From $\tau_{c'}^{-1}\tau_c \in \Phi_e^0 = dK_e$ we see that τ_c belongs to dK_e, where τ_c denotes the parallel displacement along the curve c. *q.e.d.*

As we have seen above the homogeneous system of any connected Lie group is regular with $D(\mathfrak{G}) = 0$. Conversely we have;

THEOREM 2. *If a geodesic homogeneous system (G, η) is regular with $D(\mathfrak{G}) = 0$, then the multiplication $\mu^{(e)}(x, y) = \eta(e, x, y)$ defines a Lie group on G whose homogeneous system is (G, η).*

PROOF. Since the maps $(x, y) \mapsto \mu^{(e)}(x, y)$ and $x \mapsto x^{-1} = \eta(x, e, e)$ are differentiable, it is sufficient to show that $\mu^{(e)}$ is an abstract group. From the assumption it follows that $dA_e \subset dK_e = \Phi_e^0 = \{1_{\mathfrak{G}}\}$. Since the linear representation of K_e is faithful, we get $1_G = L_{x,y}$ for $x, y \in G$ which shows that $\mu^{(e)}$ is a group (cf. [4]). *q.e.d.*

THEOREM 3. *A geodesic homogeneous system (G, η) is regular if the exponential* *map* Exp_e: $\mathfrak{G}\to G$ *of the canonical connection* ∇ *is surjective at some point* $e\in G$.

PROOF. If Exp_e is surjective then each point $y\in G$ is joined by a geodesic arc $c(t)=\mathrm{Exp}_e(tX)$ for some $X\in\mathfrak{G}$. By the displacements $\eta(e, x)$, $x\in G$, we see that any two points of G can be joined by a geodesic arc. Hence every displacement $\eta(x, y)$ induces the parallel displacement along a geodesic arc $c(x, y)$ joining x to y. Thus, for every left inner mapping $L_{x,y}=\eta(x^{-1}, e)\eta(y, x^{-1})\eta(e, y)$, its linear representation on $\mathfrak{G}$ is an element of the holonomy group corresponding to the geodesic triangle consisting of $c(e, y)$, $c(y, x^{-1})$ and $c(x^{-1}, e)$. It follows that $d\bar{A}_e=dK_e=\Phi_e^0$, which implies $\mathfrak{K}=D(\mathfrak{G})$.

THEOREM 4. *Let* $G=A/K$ *be the reductive homogeneous space at* $e\in G$ *of a geodesic homogeneous system* (G, η). *Suppose that* $\dim G\geqq 2$ *and the restricted holonomy group* Φ_e^0 *of the canonical connection is irreducible on the tangent L.t.a.* $\mathfrak{G}$. *If the Lie algebra* $\mathfrak{A}$ *of* A *is reductive, then* (G, η) *is regular.*

PROOF. Let $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$ be the canonical decomposition of the Lie algebra $\mathfrak{A}$. By the assumption for Φ_e^0, the inner derivation algebra $D(\mathfrak{G})$ is not trivial. Put $\mathfrak{A}_1=\mathfrak{G}+D(\mathfrak{G})$ (direct sum). Then $\mathfrak{A}_1$ is an ideal of $\mathfrak{A}$. Since $\mathfrak{A}$ is supposed to be reductive there exists a complementary ideal $\mathfrak{A}_0$ in $\mathfrak{A}$ such that $\mathfrak{A}=\mathfrak{A}_0+\mathfrak{A}_1$ (direct sum). Let $\mathfrak{G}_0$ be the subspace of $\mathfrak{G}$ consisting of all $\mathfrak{G}$-component of the element of $\mathfrak{A}_0$. Then $[\mathfrak{K}, \mathfrak{G}_0]\subset\mathfrak{G}_0$ so $\mathfrak{G}_0$ is a dK-invariant subspace of $\mathfrak{G}$. Since the linear representation dK on $\mathfrak{G}$ contains Φ_e^0, it acts irreducibly on $\mathfrak{G}$. Thus $\mathfrak{G}_0=0$ or $\mathfrak{G}_0=\mathfrak{G}$. If $\mathfrak{G}_0=\mathfrak{G}$ then $[\mathfrak{G}, \mathfrak{A}_1]\subset[\mathfrak{A}_0, \mathfrak{A}_1]=0$ which implies $D(\mathfrak{G})=0$. This can not occur in our case. Hence $\mathfrak{G}_0=0$ and so $\mathfrak{A}_0$ is contained in $\mathfrak{K}$. Since A acts effectively on G, we get $\mathfrak{A}_0=0$ and $\mathfrak{A}=\mathfrak{A}_1$. *q.e.d.*

§4. Invariant Homogeneous Subsystems

Let (G, η) be a homogeneous system. A homogeneous *subsystem* (H, η_H) of (G, η) is a connected submanifold H of G in which η is closed and its restriction η_H: $H\times H\times H\to H$ is differentiable. We denote a homogeneous subsystem (H, η_H) by H.

A homogeneous subsystem H of (G, η) will be said to be *invariant* if the following condition is satisfied;

$$(*) \qquad \eta(x, y)\eta(H, x, H)=\eta(H, y, H) \qquad \text{for} \quad x, y\in G.$$

Suppose that H is an invariant subsystem of (G, η).

LEMMA 1. $\eta(H, x, H)=\eta(e, x)H$ *for any* $e\in H$.

PROOF. By the condition $(*)$ we get $\eta(H, x, H)=\eta(e, x)\eta(H, e, H)=\eta(e, x)H$, if $e\in H$. *q.e.d.*

We denote $\eta(H, x, H)$ by xH.

LEMMA 2. *$y \in xH$ if and only if $xH = yH$.*

PROOF. If $xH = yH$ then $y = \eta(e, y, e) \in yH = xH$. Conversely, if $y \in xH$ then $y = \eta(e, x, h)$ for some $e, h \in H$. By (*) and Lemma 1 we get $yH = \eta(x, y)xH = \eta(x, \eta(e, x, h))\eta(e, x)H = \eta(e, x)\eta(e, h)H = xH$. *q.e.d.*

LEMMA 3. *$\eta(x, y)(uH) = vH$ if $v = \eta(x, y, u)$ for $x, y, u \in G$.*

PROOF. For any $e \in H$, we get $\eta(x, y)(uH) = \eta(x, y)\eta(e, u)H = \eta(x, y)\eta(x, u)\eta(e, x)H = \eta(x, y)\eta(x, u)\eta(y, x)\eta(e, y)H = \eta(y, v)\eta(e, y)H = \eta(e, v)H = vH$. *q.e.d.*

COROLLARY. *$\eta(xH, y, xH) = \eta(H, y, H)$ for $x, y \in G$.*

PROOF. $\eta(xH, y, xH) = \eta(e, x)\eta(H, \eta(x, e, y), H) = \eta(e, x)(\eta(x, e, y)H) = yH = \eta(H, y, H)$, since $\eta(e, x, \eta(x, e, y)) = y$. *q.e.d.*

PROPOSITION 9. *If H is an invariant subsystem of (G, η), then for any $x \in G$, xH is again an invariant subsystem.*

PROOF. By Lemma 1 we see that xH is a submanifold of G diffeomorphic to H under the diffeomorphism $\eta(e, x)$, $e \in H$. Moreover, for $u, v, w \in xH$, there exist $a, b, c \in H$ such that $u = \eta(e, x, a)$, $v = \eta(e, x, b)$ and $w = \eta(e, x, c)$. Then $\eta(u, v, w) = \eta(e, x)\eta(a, b, c) \in xH$. Thus we see that xH is a homogeneous subsystem. Using the corollary to Lemma 3 we get $\eta(y, z)\eta(xH, y, xH) = \eta(y, z)\eta(H, y, H) = \eta(H, z, H) = \eta(xH, z, xH)$, $y, z \in G$. This shows that xH satisfies the condition (*). *q.e.d.*

Lemma 2 and Proposition 9 show that if H is an invariant subsystem of (G, η) then G is decomposed into the disjoint union of invariant subsystems isomorphic to H.

PROPOSITION 10. *A homogeneous subsystem H of (G, η) is invariant if and only if H is invariant under the left inner mapping group A_e of G at some $e \in H$.*

PROOF. Since A_e is the isotropy subgroup of the group $D(\eta)$ of displacements of η at e, a subsystem H of G is invariant under A_e if and only if $\eta(y, e)\eta(x, y)\eta(e, x)H = H$ hold for all $x, y \in G$. Therefore, if H is invariant under A_e we get $\eta(x, y)\eta(e, x, H) = \eta(e, y, H)$ for $x, y \in G$ and $\eta(e, y, H) = \eta(h, y, H)$ for $h \in H$, $y \in G$. Hence H satisfies (*). Conversely, if H is an invariant subsystem then the condition (*) implies that H is invariant under A_e. *q.e.d.*

Now we shall consider the tangent L.t.a.'s of invariant subsystems of (G, η). Let $\mathfrak{G}$ be a L.t.a. (general Lie triple system) and $\mathfrak{H}$ a (triple) subalgebra of $\mathfrak{G}$, i.e., $\mathfrak{H}$ satisfies $\mathfrak{H}\mathfrak{H} \subset \mathfrak{H}$ and $[\mathfrak{H}, \mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H}$ in $\mathfrak{G}$. $\mathfrak{H}$ will be called an *invariant subalgebra* of $\mathfrak{G}$ if it is invariant under the inner derivation algebra $D(\mathfrak{G})$ of $\mathfrak{G}$, i.e.,

$D(\mathfrak{G})\mathfrak{H}\subset\mathfrak{H}$.

Assume that the homogeneous system (G, η) is regular. If $\mathfrak{H}$ is an invariant subalgebra of the tangent L. t. a. $\mathfrak{G}$ of G at $e\in G$, then $\mathfrak{H}$ is invariant under the group Λ_e. Let $\mathfrak{S}$: $x\mapsto\mathfrak{S}_x=\eta(e, x)_*\mathfrak{H}$ be the distribution on G. Since $\mathfrak{H}$ is invariant under Λ_e, we have $\eta(x, y)_*\mathfrak{S}_x=\mathfrak{S}_y$. On the other hand, since the differential $\eta(e, x)_*$ of the displacement $\eta(e, x)$ is equal to the parallel displacement along some piecewise geodesic curve from e to x, and since $\mathfrak{H}$ is invariant under the holonomy group at e, the distribution $\mathfrak{S}$ is a parallel distribution on G.

PROPOSITION 11. *The distribution $\mathfrak{S}$ is completely integrable and each maximal integral manifold of $\mathfrak{S}$ is an invariant subsystem of (G, η). Moreover they are geodesic homogeneous systems.*

PROOF. Since $\mathfrak{S}$ is a parallel distribution, the torsion tensor S is parallel and $S_e(X, Y)\in\mathfrak{H}$ for $X, Y\in\mathfrak{H}$, we see that the vector field $S(X^*, Y^*)$ belongs to $\mathfrak{S}$ for any vector fields $X^*, Y^*\in\mathfrak{S}$. Hence S is inducible to $\mathfrak{S}$ in the sense of [1] and so $\mathfrak{S}$ is completely integrable by Proposition 1 in [1]. Let H be the maximal integral manifold of $\mathfrak{S}$ containing e. Since (G, η) is geodesic we see that any geodesic $x(t)$ in G tangent to $\mathfrak{H}$ at $e=x(0)$ is contained in H, and that the displacement $\eta(e, x(t))$ sends each geodesic through e to a geodesic through the point $x(t)$. The canonical connection of η is complete and so every point $y\in H$ can be joined by geodesic arcs contained in H. By using the facts seen above we can show that the submanifold H is an auto-parallel submanifold of G, and that if e' belongs to H, the displacement $\eta(e, e')$ sends H onto itself. If e is replaced by e' and $\mathfrak{H}$ by $\mathfrak{H}'=\eta(e, e')_*\mathfrak{H}$ in the discussion above, we get the same distribution on G as $\mathfrak{S}$. Therefore $\eta(H, H, H)=H$ holds. The differentiability of η on $H\times H\times H$ follows from the 2nd countability of G. Thus H is a homogeneous subsystem of G. Let x be any point of G. By the affine isomorphism $\eta(e, x)$ we have an auto-parallel submanifold $H'=\eta(e, x)H$. Since $\eta(e, x)$ preserves the parallel displacements along geodesic arcs, we can show that the tangent space to H' at each point of H' is equal to the value of $\mathfrak{S}$ at the point, that is, H' is an integral manifold of $\mathfrak{S}$. In the same way, we see that $\eta(x, y)\eta(e, x)H=\eta(e, y)H$ for $x, y\in G$, which shows that H is invariant under Λ_e. Thus the proof of the first half of the proposition is com pleted by Propositions 9 and 10. The remaining half is proved easily by the fact that the auto-parallel submanifold H of G has the connection induced from G in a natural manner and that it coincides with the canonical connection of (H, η_H). *q. e. d.*

THEOREM 5. *Let (G, η) be a regular geodesic homogeneous system, $\mathfrak{G}$ its tangent L. t. a. at $e\in G$. There exists a 1–1 correspondence between the set of (connected) invariant subsystems of G containing e and the set of invariant subalgebras of $\mathfrak{G}$. The correspondence is given in such a way as an invariant subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ is the tangent L. t. a. of an invariant subsystem H of G.*

PROOF. Let $\mathfrak{H}$ be an invariant subalgebra of $\mathfrak{G}$. Then from Proposition 11 there exists a unique connected invariant subsystem H of (G, η) such that $\mathfrak{H}$ is tangent to H at e. Since the canonical connection of H is equal to the connection induced from the canonical connection of G, we see that $\mathfrak{H}$ is the tangent L.t.a. of H. Conversely, if H is an invariant subsystem of (G, η), then its tangent L.t.a. $\mathfrak{H}$ at $e \in H$ is a subspace of $\mathfrak{G}$ such that $D(\mathfrak{G})\mathfrak{H} \subset \mathfrak{H}$, since it is invariant under the holonomy group of G at e which is equal to dK_e. The submanifold H must be the maximal integral manifold of the parallel distribution $\mathfrak{S}$ in Proposition 11 for the subspace $\mathfrak{H}$. Since (G, η) is geodesic so is the subsystem H and it is an auto-parallel submanifold of G w.r.t. the canonical connection. Thus the torsion and curvature of the canonical connection of (H, η_H) is the restriction of those of (G, η) on H. Hence the tangent L.t.a. $\mathfrak{H}$ of (H, η_H) is an invariant subalgebra of $\mathfrak{G}$ at e. *q.e.d.*

REMARK. In the preceding theorem, if (G, η) is a homogeneous system of a geodesic homogeneous Lie loop considered in §1, then we get the main theorem of [3] for the regular case. Especially, if (G, η) is a homogeneous system of a connected Lie group G, the correspondence of the invariant subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ and the invariant subsystem H of (G, η) is that of Lie subalgebra $\mathfrak{H}$ and Lie subgroup H.

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