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L-Compatible Orthodox Semigroups

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In the previous papers [6] and [7], the structure of L-compatible orthodox semigroups has been studied. In particular, it has been shown that an orthodox semigroup S is L-compatible if and only if S is an orthodox right regular band of left groups. In this paper, a construction theorem for L-compatible orthodox semigroups is established. Further, the construction of L- and H-compatible orthodox semigroups is also discussed.

§1. Introduction

A band B is said to be *left regular, right regular, regular, left semiregular* or *right semiregular* if B satisfies the corresponding identity, $xyx=xy$, $xyx=yx$, $xyxzx=xyzx$, $xyzxyxz=xyxz$ or $zxyxzyx=zxyx$. Let I be a band. A semigroup S is called *a band I of semigroups* S_i if it satisfies the following conditions:

(1.1) (1) $S=\sum\{S_i: i\in I\}$ (disjoint sum), and

(2) $S_jS_k\subset S_{jk}$ for all $j, k\in I$.

In this case, we shall denote it by $S\equiv\sum\{S_i: i\in I\}$. If S and each S_i are orthodox semigroups, then S is especially called *an orthodox band I of orthodox semigroups* S_i. In Schein [2], the construction of bands of monoids has been studied and in particular a nice construction was given for [left, right] regular bands of monoids. A semigroup S is said to be L [R, H]-*compatible* if the Green's L [R, H]-relation on S is a congruence. In the previous papers [6], [7] of the author, the structure of L [R, H]-compatible orthodox semigroups has been studied and the following results were established:

I. An orthodox semigroup S is L-compatible if and only if S is an orthodox right regular band of left groups.[1)]

II. An orthodox semigroup S is L- and H-compatible if and only if S is both an orthodox right regular band of left groups and a band of groups.

In the case II, the set $E(S)$ of idempotents of S is a right regular band of left zero semigroups (see [4]). Therefore, $E(S)$ is a right semiregular band. Hence, the result

1) A semigroup S is called a *left group* if S is isomorphic to the direct product of a left zero semigroup and a group.

II above can be rewritten as follows:

III. An orthodox semigroup S is L- and H-compatible if and only if S is an orthodox right semiregular band of groups.

The construction of L-compatible orthodox (or more generally, regular) semigroups has been investigated by Warne [3].[2] Let I denote a lower associative semilattice Y of left groups, and J an associative semilattice Y of right zero semigroups. Warne [3] proved that a semigroup S is a band of maximal left groups if and only if S is a Schreier product of I and J for some I and J.

In this short note, we shall give a construction for L-compatible orthodox semigroups and that for L- and H-compatible orthodox semigroups in the direction of the method given by Schein [2].

§2. L-compatible orthodox semigroups

Let S be an L-compatible orthodox semigroup, and $E(S)$ the right semiregular band of idempotents of S. As was shown in [6], S is of course a right regular band Λ of left groups S_λ: $S \equiv \sum\{S_\lambda : \lambda \in \Lambda\}$. Let G_λ be one of the maximal subgroups of S_λ for each $\lambda \in \Lambda$. Then, every $x \in S_\lambda$ can be written in the form $x = eg$, where $e \in E(S_\lambda)$ (the set of idempotents of S_λ) and $g \in G_\lambda$. For any $x, y \in S_\lambda$, it follows that $x = eg$, $y = ft$, $e, f \in E(S_\lambda)$ and $g, t \in G_\lambda$ imply $xy = efgt = egt$. Hence, the mapping $\varphi: S_\lambda \to E(S_\lambda) \times G_\lambda$ (the direct product of $E(S_\lambda)$, G_λ) defined by $x\varphi = (e, g)$, if $e \in E(S_\lambda)$, $g \in G_\lambda$ and $x = eg$, gives an isomorphism of S_λ onto $E(S_\lambda) \times G_\lambda$.

Hereafter, we shall simply denote $E(S_\lambda)$ by E_λ. Now, define a relation ρ on S as follows:

(2.1) $x\rho y$ if and only if $x, y \in S_\lambda$ for some $\lambda \in \Lambda$, and $x = eg$ and $y = fg$ for some $e, f \in E_\lambda$ and $g \in G_\lambda$.

It is easily seen that this condition (2.1) is equivalent to the following:

(2.2) $x\rho y$ if and only if $x, y \in S_\lambda$ for some $\lambda \in \Lambda$, and $ex = ey$ for some $e \in E_\lambda$.

LEMMA 1. *The relation ρ is a congruence on S, and S/ρ is an orthodox right regular band Λ of the groups S_λ/ρ: $S/\rho \equiv \sum\{S_\lambda/\rho : \lambda \in \Lambda\}$.*

PROOF. It is obvious that ρ is reflexive, symmetric and transitive. Now, let $c \in S_\tau$ and $x\rho y$. Then, there exists $\lambda \in \Lambda$ such that $x, y \in S_\lambda$. Since $x\rho y$, there exists also $e \in E_\lambda$ such that $ex = ey$. Hence, $xy^{-1} \in E_\lambda$ (where y^{-1} denotes an inverse of y in the maximal subgroup containing y). Now, $cxy^{-1}c^{-1} = f \in S_{\lambda\tau}$ (since $cxy^{-1}c^{-1} \in S_{\tau\lambda\tau} = S_{\lambda\tau}$), and f is an idempotent. Further, we have $cxy^{-1}c^{-1}cy = fcy$, $cc^{-1}cxy^{-1} \cdot cc^{-1}yy^{-1}y = fcy$, $cxy^{-1}y = fcy$, $cxx^{-1}xy^{-1}y = fcy$, $cx = fcy$, and consequently fcx

2) The general theory of orthodox unions of groups has been studied by many papers (for the outlines of their papers, see Clifford [1]).

$=fcy$. It is easy to see that $cx, cy \in S_{\tau\lambda}$ and both $S_{\tau\lambda}$ and $S_{\lambda\tau}$ are contained in a rectangular group component (kernel) of the greatest semilattice decomposition of S. Hence, $fhe_1cx = fhe_2cy$, where e_1, e_2 are the identities of the maximal groups containing cx and cy respectively, and h is an element of $E_{\tau\lambda}$. Since $fh \in S_{\lambda\tau\lambda}$ and Λ is a right regular band, $fh \in S_{\tau\lambda}$. Hence, $fhcx = fhcy$ and $fh \in S_{\tau\lambda}$. This implies that $cx\rho cy$. Next, we shall show that $xc\rho yc$. First we have $ex = ey$, $exc = eyc$, $ee_{\lambda\tau}xc = ee'_{\lambda\tau}yc$ (where $e_{\lambda\tau}$ and $e'_{\lambda\tau}$ are the identities of maximal subgroups containing xc and yc respectively), $e_{\lambda\tau}ee_{\lambda\tau}xc = e_{\lambda\tau}ee'_{\lambda\tau}yc$, and consequently $e_{\lambda\tau}xc = e_{\lambda\tau}yc$. Thus, $xc\rho yc$. It is easily proved that each S_λ/ρ is a group and S/ρ is an orthodox right regular band Λ of the groups S_λ/ρ.

In $S/\rho \equiv \sum\{S_\lambda/\rho\colon \lambda \in \Lambda\}$, define $\varphi_\lambda\colon S_\lambda/\rho \to G_\lambda$ by $\overline{eg}\varphi_\lambda = g$ (where $\overline{eg}$ is the ρ-class containing eg, and $e \in E_\lambda$, $g \in G_\lambda$). Then, φ_λ is an isomorphism. Therefore, if $\overline{eg}$ is identified with g and if S_λ/ρ is identified with G_λ then S/ρ can be considered as an orthodox right regular band Λ of the groups G_λ: $S/\rho \equiv \sum\{G_\lambda\colon \lambda \in \Lambda\}$. We shall denote the multiplication in S/ρ by $*$.

Now, for $eg \in S_\lambda$ ($e \in E_\lambda$, $g \in G_\lambda$) and $ft \in S_\tau$ ($f \in E_\tau$, $t \in G_\tau$),

$$egft = egfg^{-1}gft \quad (g^{-1} \text{ denotes an inverse of } g \text{ in } G_\lambda).$$

Define $\tilde{g}\colon E \to E$ by $f^{\tilde{g}}\ (=f\tilde{g}) = gfg^{-1}$ (where $E = E(S)$, that is, the band of all idempotents of S).

For $x \in S_\lambda$ and $y \in S_\tau$, where $x = eg$, $y = ft$, $e \in E_\lambda$, $g \in G_\lambda$, $f \in E_\tau$ and $t \in G_\tau$,

(2.3) $xy = egft = ef^{\tilde{g}}gft = ef^{\tilde{g}}u_{\lambda\tau}g*t$ (where $u_{\lambda\tau}$ is some element of $E_{\lambda\tau}$) $= ef^{\tilde{g}}u_{\lambda\tau}1_{\lambda\tau}g*t$ (where 1_λ is a representative of E_λ for each $\lambda \in \Lambda$) $= ef^{\tilde{g}}1_{\lambda\tau}g*t$ (since $f^{\tilde{g}} \in E_{\tau\lambda}$ and the elements $f^{\tilde{g}}$, $1_{\lambda\tau}$ and $u_{\lambda\tau}$ are contained in the same rectangular band component (kernel) of the greatest semilattice decomposition of E).

Therefore, for $z \in S_\delta$ (where $z = hv$, $h \in E_\delta$ and $v \in G_\delta$)

$$(xy)z = x(yz) \text{ implies that } e(fh^{\tilde{t}}1_{\tau\delta})^{\tilde{g}}1_{\lambda\tau\delta}x*y*z$$
$$= ef^{\tilde{g}}1_{\lambda\tau}h^{\widetilde{g*t}}1_{\lambda\tau\delta}x*y*z\,.$$

Hence, we have the following:

(2.4) (1) For any $g \in G_\lambda$ ($\lambda \in \Lambda$), $\tilde{g}$ maps E_τ into $E_{\tau\lambda}$ ($\tilde{g}$ is necessarily a homomorphism on E_τ), and the restriction of $\tilde{g}$ to E_λ (that is, $\tilde{g}|E_\lambda$) maps E_λ to a single element of E_λ,

(2) $e(fh^{\tilde{t}}1_{\tau\delta})^{\tilde{g}}1_{\lambda\tau\delta} = ef^{\tilde{g}}1_{\lambda\tau}h^{\widetilde{g*t}}1_{\lambda\tau\delta}$ for $e \in E_\lambda$, $f \in E_\tau$, $h \in E_\delta$, $g \in G_\lambda$ and $t \in G_\tau$.

If we denote x by (e, g) if $x = eg$, $e \in E_\lambda$ and $g \in G_\lambda$, then $S = \{(e, g)\colon e \in E_\lambda, g \in G_\lambda,$

$\lambda \in \Lambda\}$ and the multiplication in S is given as follows:

(2.5) For $x=(e, g)$, $y=(f, t)$ (where $x \in S_\lambda$, $y \in S_\tau$),

$$(e, g)(f, t)=(ef^{\tilde{g}}1_{\lambda\tau}, g*t).$$

Conversely, let Λ be a right regular band. Suppose that $G \equiv \sum\{G_\lambda : \lambda \in \Lambda\}$ is an orthodox right regular band Λ of groups G_λ and $E \equiv \sum\{E_\lambda : \lambda \in \Lambda\}$ is a right regular band Λ of left zero semigroups E_λ. Let 1_λ be a representative of E_λ for each $\lambda \in \Lambda$. For each $g \in G$, let $\tilde{g}$ be a mapping of E into E such that the system $\{\tilde{g} : g \in G\}$ of all $\tilde{g}$ satisfies the condition (2.4) above. Then, $S=\sum\{E_\lambda \times G_\lambda : \lambda \in \Lambda\}$ becomes an orthodox right regular band Λ of the left groups $E_\lambda \times G_\lambda$ under the multiplication defined as follows:

For $(e, g) \in E_\lambda \times G_\lambda$ and $(f, t) \in E_\tau \times G_\tau$, $(e, g)(f, t)=(ef^{\tilde{g}}1_{\lambda\tau}, g*t)$,

where $*$ is the multiplication in G.

Summarizing the results above, we obtain the following theorem:

THEOREM 2. *Let Λ be a right regular band. Let $G \equiv \sum\{G_\lambda : \lambda \in \Lambda\}$ be an orthodox right regular band Λ of groups G_λ, and $E \equiv \sum\{E_\lambda : \lambda \in \Lambda\}$ a right regular band Λ of left zero semigroups E_λ. Let 1_λ be a representative of E_λ for each $\lambda \in \Lambda$. For each $g \in G$, let $\tilde{g}$ be a mapping of E into E such that the system $\{\tilde{g} : g \in G\}$ satisfies (2.4). Then, $S=\sum\{E_\lambda \times G_\lambda : \lambda \in \Lambda\}$ becomes an orthodox right regular band Λ of the left groups $E_\lambda \times G_\lambda$ under the multiplication defined as follows:*

(2.6) *For $(e, g) \in E_\alpha \times G_\alpha$ and $(f, t) \in E_\beta \times G_\beta$, $(e, g)(f, t)=(ef^{\tilde{g}}1_{\alpha\beta}, g*t)$,*

where $*$ denotes the multiplication in G. Accordingly, S is an L-compatible orthodox semigroup. Conversely, every L-compatible orthodox semigroup can be constructed in this way.

REMARK 1. A construction for orthodox right regular bands of groups can be obtained as a special case of Theorem 3 of Schein [2].
2. It has been proved by [7] that an orthodox semigroup S is both L- and R-compatible if and only if S is an orthodox regular band of groups. On the other hand, a construction for such semigroups can be obtained as a special case of Theorem 4 of Schein [2].

§3. L- and H-compatible orthodox semigroups

It has been shown by [6] that an orthodox semigroup S is both L- and H-compatible if and only if S is a band of groups and an orthodox right regular band of left groups. Accordingly, S is L- and H-compatible if and only if S is an orthodox right semiregular band of groups.

In this section, we shall consider the construction of L- and H-compatible orthodox semigroups.

Let S be an L- and H-compatible orthodox semigroup. Then, there exist a right semiregular band Γ which is a right regular band Λ of left zero semigroups Γ_λ (that is, $\Gamma \equiv \sum\{\Gamma_\lambda : \lambda \in \Lambda\}$) and a left group S_λ for each $\lambda \in \Lambda$ such that

(1) S is an orthodox right regular band Λ of the left groups S_λ: $S \equiv \sum\{S_\lambda : \lambda \in \Lambda\}$,

(2) S_λ is a left zero semigroup Γ_λ of groups G_{λ_i}: $S_\lambda \equiv \sum\{G_{\lambda_i} : \lambda_i \in \Gamma_\lambda\}$ (hereafter, elements of Γ_λ are denoted by λ_i, λ_j, etc.), and

(3) for any $\lambda_i \in \Gamma_\lambda$ and $\tau_j \in \Gamma_\tau$, $G_{\lambda_i} G_{\tau_j} \subset G_{\lambda_i \tau_j}$ holds.

Denote the identity of G_{λ_i} by e_{λ_i}.

Now, we introduce a quasiorder $\leq$ in Λ as follows (see [2]): For $\alpha, \beta \in \Lambda$, $\beta \leq \alpha$ if and only if $\beta\alpha\beta = \beta$. Since Λ is a right regular band, in this case $\alpha\beta = \beta$ holds.

For any $\alpha_i \in \Gamma_\alpha$, $\beta_j \in \Gamma_\beta$ such that $\alpha \geq \beta$, define a mapping f_{α_i,β_j}: $G_{\alpha_i} \to G_{\beta_j}$ by

$$x f_{\alpha_i,\beta_j} = e_{\beta_j} x e_{\beta_j}.$$

For $x, y \in G_{\alpha_i}$, we have $e_{\beta_j} x e_{\beta_j} y e_{\beta_j} = e_{\beta_j} x e_{\beta_j} e_{\alpha_i} y e_{\beta_j} = e_{\beta_j} x y e_{\beta_j}$ (since $e_{\beta_j} e_{\alpha_i}$ is the identity of $G_{\beta_j \alpha_i}$ ($\ni e_{\beta_j} x$)). Hence, f_{α_i,β_j} is a homomorphism. Now, let us consider the system $\{f_{\alpha_i,\beta_j} : \alpha, \beta \in \Lambda, \alpha \geq \beta, \alpha_i \in \Gamma_\alpha, \beta_j \in \Gamma_\beta\}$. For $\alpha_i \in \Gamma_\alpha$, $\beta_j \in \Gamma_\beta$ and $\gamma_k \in \Gamma_\gamma$ such that $\alpha \geq \beta \geq \gamma$, we can prove by simple calculation that $x f_{\alpha_i,\beta_j} f_{\beta_j,\gamma_k} = x f_{\alpha_i,\gamma_k}$ for all $x \in G_{\alpha_i}$. Further, $x f_{\alpha_i,\alpha_j} = e_{\alpha_j} x e_{\alpha_j} = e_{\alpha_j} x$ for $\alpha_i, \alpha_j \in \Gamma_\alpha$ and $x \in G_{\alpha_i}$. Therefore, f_{α_i,α_j} is the left multiplication by e_{α_j}.

From the results above, this system satisfies the following:

(3.1) (1) For any $\alpha \in \Lambda$ and for any $\alpha_i, \alpha_j \in \Gamma_\alpha$, f_{α_i,α_j} = the left multiplication by e_{α_j},
(2) for $\alpha_i \in \Gamma_\alpha$, $\beta_j \in \Gamma_\beta$ and $\gamma_k \in \Gamma_\gamma$ such that $\alpha \geq \beta \geq \gamma$,

$$f_{\alpha_i,\beta_j} f_{\beta_j,\gamma_k} = f_{\alpha_i,\gamma_k}.$$

Now, it is easy to see that the multiplication in S is given as follows by using this system:

(3.2) For $x \in G_{\alpha_i}$ and $y \in G_{\beta_j}$,

$$\begin{aligned} xy &= x e_{\alpha_i} e_{\beta_j} y = x e_{\alpha_i \beta_j} y = (e_{\alpha_i \beta_j} x e_{\alpha_i \beta_j})(e_{\alpha_i \beta_j} y e_{\alpha_i \beta_j}) \\ &= (x f_{\alpha_i,\alpha_i\beta_j})(y f_{\beta_j,\alpha_i\beta_j}). \end{aligned}$$

Conversely, we have the following:

LEMMA 3. *Let Γ be a right semiregular band which is a right regular band*

Λ of left zero semigroups Γ_λ: $\Gamma \equiv \sum\{\Gamma_\lambda: \lambda \in \Lambda\}$. For each $\lambda \in \Lambda$, let S_λ be a left zero semigroup Γ_λ of groups G_{λ_i} (hence, S_λ is a left group): $S_\lambda \equiv \sum\{G_{\lambda_i}: \lambda_i \in \Gamma_\lambda\}$. Let e_{λ_i} be the identity of G_{λ_i} for $\lambda_i \in \Gamma_\lambda$, $\lambda \in \Lambda$. Now, let $F=\{f_{\alpha_i,\beta_j}: \alpha, \beta \in \Lambda, \alpha \geq \beta, \alpha_i \in \Gamma_\alpha, \beta_j \in \Gamma_\beta\}$ be a system of homomorphisms $f_{\alpha_i,\beta_j}: G_{\alpha_i} \to G_{\beta_j}$ such that it satisfies (3.1) (such a system is called a direct system on $\{G_{\lambda_i}: \Gamma_\lambda, \Gamma(\Lambda)\}$). Then, $S=\sum\{S_\lambda: \lambda \in \Lambda\}$ becomes an orthodox right regular band Λ of the left groups S_λ and is a right semiregular band Γ of the groups G_{λ_i} under the multiplication $*$ defined as follows:

(3.3) *For $x \in G_{\alpha_i}$ and $y \in G_{\beta_j}$,*

$$x*y=(xf_{\alpha_i,\alpha_i\beta_j})(yf_{\beta_j,\alpha_i\beta_j}).$$

That is, $S(*)$ is an L- and H-compatible orthodox semigroup.

PROOF. First we shall show that $S(*)$ is a semigroup. For $x \in G_{\alpha_i}$, $y \in G_{\beta_j}$ and $z \in G_{\gamma_k}$,

$$\begin{aligned}(x*y)*z &= ((xf_{\alpha_i,\beta_j\gamma_k}yf_{\beta_j,\alpha_i\beta_j})f_{\alpha_i\beta_j,\alpha_i\beta_j\gamma_k})(zf_{\gamma_k,\alpha_i\beta_j\gamma_k}) \\ &= (xf_{\alpha_i,\alpha_i\beta_j\gamma_k})(yf_{\beta_j,\alpha_i\beta_j\gamma_k})(zf_{\gamma_k,\alpha_i\beta_j\gamma_k}).\end{aligned}$$

Similarly, we have

$$x*(y*z)=(xf_{\alpha_i,\alpha_i\beta_j\gamma_k})(yf_{\beta_j,\alpha_i\beta_j\gamma_k})(zf_{\gamma_k,\alpha_i\beta_j\gamma_k}).$$

Hence, $S(*)$ is a semigroup. Next, for $x \in G_{\alpha_i}$ and $y \in G_{\alpha_j}$ it follows that

$$\begin{aligned}x*y &= (xf_{\alpha_i,\alpha_i\alpha_j})(yf_{\alpha_j,\alpha_i\alpha_j}) = (xf_{\alpha_i,\alpha_i})(yf_{\alpha_j,\alpha_i}) = e_{\alpha_i}xe_{\alpha_i}y \\ &= xy \text{ (in } S_\alpha).\end{aligned}$$

Therefore, S_α is embedded in $S(*)$. Further, for any $x \in G_{\alpha_i}$ and $y \in G_{\beta_j}$

$$x*y=(xf_{\alpha_i,\alpha_i\beta_j})(yf_{\beta_j,\alpha_i\beta_j}) \subset G_{\alpha_i\beta_j} \subset S_{\alpha\beta}.$$

Thus, $S(*)$ is a right regular band Λ of the left groups S_λ and is a right semiregular band Γ of the groups G_{α_i}. Especially, if we put $e_{\alpha_i}=x$ and $e_{\beta_j}=y$ in the equality above then we have $e_{\alpha_i}*e_{\beta_j}=e_{\alpha_i\beta_j}e_{\alpha_i\beta_j}=e_{\alpha_i\beta_j}$. Hence, the set $E(S(*))$ of idempotents of $S(*)$ is a band which is isomorphic to Γ. Therefore, $S(*)$ is an orthodox semigroup.

From the results above, we have the following theorem:

THEOREM 4. *Let Γ be a right semiregular band which is a right regular band Λ of left zero semigroups Γ_λ: $\Gamma \equiv \sum\{\Gamma_\lambda: \lambda \in \Lambda\}$. For each $\lambda \in \Lambda$, let S_λ be a left zero semigroup Γ_λ of groups G_{λ_i}: $S_\lambda \equiv \sum\{G_{\lambda_i}: \lambda_i \in \Gamma_\lambda\}$ (hence, S_λ is a left group). Let e_{λ_i} be the identity of G_{λ_i} for $\lambda_i \in \Gamma_\lambda$, $\lambda \in \Lambda$. Let $\{f_{\alpha_i,\beta_j}: \alpha, \beta \in \Lambda, \alpha \geq \beta, \alpha_i \in \Gamma_\alpha, \beta_j \in \Gamma_\beta\}$ be a direct system on $\{G_{\lambda_i}: \Gamma_\lambda, \Gamma(\Lambda)\}$. Then, $S=\sum\{S_\lambda: \lambda \in \Lambda\}$ is an orthodox right* *regular band Λ of the left groups S_λ and is a right semiregular band Γ of the groups G_{λ_i} under the multiplication $*$ defined by* (3.3). *That is, $S(*)$ is an L- and H-compatible orthodox semigroup. Conversely, every L- and H-compatible orthodox semigroup can be constructed in this way.*

REMARK. The following result was established by [5]: Let Γ be a right semiregular band, and $\Gamma \sim \sum\{\Gamma_\delta : \delta \in \Delta\}$ the structure decomposition[3] of Γ. Let G be a semilattice Δ of groups G_δ: $G \equiv \sum\{G_\delta : \delta \in \Delta\}$. Then, the spined product $\Gamma \bowtie G(\Delta)$ of Γ and G is an orthodox right semiregular band of groups, and accordingly $\Gamma \bowtie G(\Delta)$ is an L- and H-compatible orthodox semigroup. Further, every L- and H-compatible orthodox semigroup can be constructed in this way.

Theorem 4 above gives another construction for L- and H-compatible orthodox semigroups in the direction of the method given by Schein [2].

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3) Any band B can be uniquely expressed as a semilattice Δ of rectangular bands B_δ: $B \equiv \sum\{B_\delta : \delta \in \Delta\}$. In this case, this expression is called *the structure decomposition of* B, and denoted by $B \sim \sum\{B_\delta : \delta \in \Delta\}$.