Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
34 (2001), pp. 53–60

KLEENE ALGEBRA AND MODAL LOGIC

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(Received: January 19, 2001)

Abstract. In this paper we define a Kleene algebra L and a certain modal logic Ω and consider the relation between them. We shall show that if L has countable generators then some quotient algebra $L/_{\sim}$ is isomorphic to a subalgebra of the Lindenbaum-Tarski algebra $\Omega/_{\equiv}$ for Ω.

1. Introduction

There are well-known properties between algebras and logics. For example, the Lindenbaum-Tarski algebra $L/_{\sim}$ for the classical (or intuitionistic) propositional logic L is a Boolean (Heyting, respectively) algebra. And the intuitionistic propositional logic can be embedded into the Lewis' modal logic $S4$. Hence we can conclude that the Heyting algebra with countable generators can be embedded into the modal logic $S4$. This implies us an expectation that in a sense many familiar algebras are embeddable into modal logics. Since the set of formulas in logics is essentially countable, every algebra can not be embedded into modal logics. Hence any embeddable algebra has a certain countability property, e.g. has a countable set of generators.

On the other hand, there is so-called a fuzzy algebra which is defined as a Kleene algebra in this paper. The algebra has as a model the set of all fuzzy subsets of some non-empty set X, more precisely, the set of all functions from X to the closed interval $I = [0, 1]$. Many results about the fuzzy set theory are applied to many branches of technologies and makes a great success. But, from the view-points of mathematics, we have a little one. In [2] we prove the embedding theorem of s-fuzzy algebras into frame algebras in terms of Kripke frames. It seems that this approach may shed new light on the relation between s-fuzzy algebras and modal logics.

Now we have the following question:

> What is the modal logic to which the Kleene algebra with countable generators can be embedded?

In this paper we define a Kleene algebra L and a certain modal logic Ω and consider the relation between them. We shall show that the quotient algebra $L/_{\sim}$ for the Kleene algebra L with countable generators is embeddable in the Lindenbaum-Tarski

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1991 *Mathematics Subject Classification.* 03B45, 06D72.
Key words and phrases. Kleene algebra, Modal logic.

algebra $\Omega/_{\equiv}$ for Ω. More precisely, the algebra $L/_{\sim}$ is isomorphic to some subalgebra $\xi(L)/_{\equiv}$ of $\Omega/_{\equiv}$, where ξ is a certain homomorphism defined below. The modal logic Ω is the answer of the question above.

2. Kleene algebra L and modal logic Ω

Referring to [2] and [3], we define a Kleene algebra L. It has as a model the set of all fuzzy subsets of some non-empty set with Zadeh's complementation. By a Kleene algebra, we mean the algebra $L = (L; \wedge, \vee, N, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that

1. $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice
2. $N : L \to L$ is a map satisfying the following conditions
   (a) $N0 = 1, N1 = 0$
   (b) $x \leq y$ implies $Ny \leq Nx$
   (c) $N^2x = x$, where $N^2x = N(Nx)$
   (d) $x \wedge Nx \leq y \vee Ny$ (Kleene's law).

As examples of the Kleene algebras, we introduce here two models as follows. First, the simple Kleene algebra $\mathbf{3} = \{0, 1/2, 1\}$ defined by for every x and y in $\mathbf{3}$,

$$x \wedge y = min\{x, y\}$$
$$x \vee y = max\{x, y\}$$
$$Nx = 1 - x.$$

Second, I^X the set of all functions from some non-empty set X to the closed interval $I = [0, 1]$, where for any $f, g \in I^X$ we define

$$(f \wedge g)(x) = min\{f(x), g(x)\}$$
$$(f \vee g)(x) = max\{f(x), g(x)\}$$
$$(Nf)(x) = 1 - f(x).$$

In the rest of this paper, we assume that L is the Kleene algebra generated by $\{x_1, x_2, x_3, \cdots\}$. That is, every element of L is constructed from finite combinations of x_i by use of the operations $\wedge, \vee$, and N. We regard the element 1 as an empty conjunction of x_j and hence do the element 0 as $N1$.

We define a modal logic Ω. The language of Ω is

propositional variables; $p_1, p_2, p_3, \cdots$
primitive symbols; $\wedge, \neg, \Box$

The formulas are defined as usual. We use the meta-variables $A, B, C, \cdots$ for formulas of Ω. The other operators $\vee, \to$, and $\Box$ are defined by

$$A \vee B = \neg(\neg A \wedge \neg B),$$
$$A \to B = \neg A \vee B,$$
$$\Diamond A = \neg\Box\neg A.$$

The logic Ω has the following axiom schemata and rules of inference.

Axiom:

(a1-1) $A \to (B \to A)$
(a1-2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
(a1-3) $(\neg A \to \neg B) \to (B \to A)$
(a2) $\Box(A \to B) \to (\Box A \to \Box B)$
(a3) $A \to \Box\Diamond A$
(a4) $(\Box\Diamond A \to \Diamond A) \vee (\Diamond\Box B \to \Box B)$

Rules of inference:

(MP) infer B from A and $A \to B$;
(G) infer $\Box A$ from A.

We denote $\vdash_\Omega A$ (or simply $\vdash A$ when no confusion arises) and say that A is provable in Ω if there is a sequence $A_1, A_2, \cdots, A_n$ of finite number of formulas of Ω such that $A = A_n$ and for every j

1. A_j is an axiom of Ω,
2. A_j is obtained from A_i and A_k by (MP) for some $i, k < j$, or
3. A_j is obtained from A_i by (G) for some $i < j$.

We note that the logical system Ω is consistent, that is, there is a formula A which is not provable in Ω, or equivalently $\vdash_\Omega B \wedge \neg B$ for no formula B. For if $\vdash_\Omega B \wedge \neg B$ for some B, then there is a sequence of formulas $B_1, \cdots, B_n (= B \wedge \neg B)$ of a proof $B \wedge \neg B$ in Ω. If we delete the symbol $\Box$ (and also $\Diamond$) from Ω, then we obtain the classical propositional logic (CPL) and the sequence of the proof $B \wedge \neg B$ turns to be the sequence of the proof of $B' \wedge \neg B'$ in CPL, where B' indicates the formula obtained from B by deleting the symbol $\Box$ (and $\Diamond$). This yields that CPL is inconsistent. But this is not true. Thus Ω is the consistent logical system.

We call a structure $< W, R >$ an Ω-frame (or simply frame) if W is a non-empty set and R is a relation on W satisfying the next conditions:

(R1): $\forall x \forall y (xRy \Rightarrow yRx)$ i.e., R is symmetric
(R2): $\forall x (\exists y (xRy \ \& \ \forall z (yRz \Rightarrow xRz))$ or $\forall u \forall v (xRv \ \& \ xRv \Rightarrow uRv))$.

By an Ω-model (or simply model) on the Ω-frame, we mean the structure $< W, R, V >$, where $< W, R >$ is the Ω-frame and V is the valuation function from the set of all propositional variables to the power set of W, that is, $V(p_j) \subseteq W$ for every propositional variable p_j. The domain of the valuation function V can be uniquely extended to the set of all formulas recursively as follows: For every formula A and B,

$$V(A \wedge B) = V(A) \cap V(B)$$
$$V(\neg A) = 1 - V(A)$$
$$V(\Box A) = \{x \in W | \forall y (xRy \Rightarrow y \in V(A))\}.$$

A formula A is said to be true at x on the model $M =< W, R, V >$ denoted by $M \models_x A$ when $x \in V(A)$. If no confusion arises, we denote simply $\models_x A$. It follows from definition that $\models_x \Box A$ if and only if (iff) $\models_x A$ for every y such that xRy

and hence dually that $\models_x \Diamond A$ iff $\models_y A$ for some y with xRy. A formula A is called true on the Ω-model $M =< W, R, V >$ when $M \models_x A$ for every $x \in W$, that is, $V(A) = W$. We say that a formula A is Ω-valid if it is true on every Ω-model.

3. Completeness theorem of Ω

In this section we show that our modal logic Ω is complete with respect to the set of Ω-models, that is, if a formula A is provable in Ω iff it is Ω-valid. First we shall show that if a formula A is provable in Ω then it is Ω-valid.

Theorem 1. *If $\vdash_\Omega A$, then A is Ω-valid.*

Proof. We show the theorem by induction on the length of a proof of A. For brevity, we shall only prove that the axiom (a4) is Ω-valid. Suppose that $(\Box\Diamond A \to \Diamond A) \vee (\Diamond\Box B \to \Box B)$ is not Ω-valid. There exists an Ω-model $< W, R, V >$ such that neither $\models_x \Box\Diamond A \to \Diamond A$ nor $\models_x \Diamond\Box B \to \Box B$ for some $x \in W$. By condition (R2) of the Ω-model, we have two cases: $\exists y(xRy \,\&\, \forall z(yRz \Rightarrow xRz))$ or $\forall u\forall v(xRu \,\&\, xRv \Rightarrow uRv)$ for that x. Suppose that $\exists y(xRy \,\&\, \forall z(yRz \Rightarrow xRz))$. Since $\models_x \Box\Diamond A$ but not $\models_x \Diamond A$, we have $\models_y \Diamond A$ for that y. This means that $\models_z A$ for some z such that yRz. Since xRy and yRz yields xRz, It follows that $\models_x \Diamond A$. But this is a contradiction.

On the other hand we assume that $\forall u\forall v(xRu \,\&\, xRv \Rightarrow uRv)$. Since $\models_x \Diamond\Box B$ but not $\models_x \Box B$, there exist u and v such that xRu, $\models_u \Box B$, xRv, but not $\models_v B$. By supposition, we get that uRv and hence that $\models_v B$ by $\models_u \Box B$. This is a contradiction.

Therefore (a4) is Ω-valid. This completes the proof of the theorem. $\Box$

Conversely we shall prove that if a formula A is Ω-valid then it is provable in Ω. In order to establish the fact we need some definitions and lemmas.

Let Γ be a set of formulas in Ω. We say that Γ is inconsistent if there are finite number of formulas $A_1, \cdots, A_n$ of Γ such that $\vdash_\Omega \neg(A_1 \wedge \cdots \wedge A_n)$. Otherwise Γ is called a consistent set. It is easy to show the next lemmas, so we omit their proofs.

Lemma 1. *If Γ is a consistent set, then there exists a maximal cosistent set Δ such that Δ includes Γ.*

Lemma 2. *Let Δ be any maximal consistent set. Then for every formula A,B in Ω, we have that*

(1) if $\vdash_\Omega A$ then $A \in \Delta$;
(2) $A \wedge B \in \Delta$ iff $A \in \Delta$ and $B \in \Delta$;
(3) $A \vee B \in \Delta$ iff $A \in \Delta$ or $B \in \Delta$;
(4) $\neg A \in \Delta$ iff $A \notin \Delta$.

Corollary 1. *If $A \in \Delta$ and $A \to B \in \Delta$, then $B \in \Delta$.*

In order to prove the completeness theorem for Ω, we assume that A is not provable in Ω. Then it is sufficient to construct an Ω-model in which A is not Ω-valid.

We shall construct such an Ω-model of A. Let W_Ω be the collection of all maximal consistent sets in Ω. Since A is not provable in Ω, the set $\{\neg A\}$ is consistent. There exists a maximal consistent set Δ including $\{\neg A\}$ by lemma 2. This implies that W_Ω is not empty. For every $x, y \in W_\Omega$, we define R_Ω: $xR_\Omega y$ iff $(\forall A)(\Box A \in x \to A \in y)$. We also define the valution V_Ω as $x \in V_\Omega(p)$ iff $p \in x$, where p is any propositional variable. We call the structure $M_\Omega =< W_\Omega, R_\Omega, V_\Omega >$ a canonical Ω-model. By easy calculation, we can show that the canonical Ω-model is indeed an Ω-model.

Lemma 3. *The canonical Ω-model $M_\Omega =< W_\Omega, R_\Omega, V_\Omega >$ is the Ω-model.*

Proof. We only show that R_Ω satisfies the condition (R1) and (R2). For (R1), suppose that $xR_\Omega y$. Unless $yR_\Omega x$, there is a formula A such that $\Box A \in y$ but $A \notin x$. Since x is a maximal consistent set, we have that $\neg A \in x$ by lemma 3. It follows that $\vdash_\Omega \neg A \to \Box\Diamond\neg A$ and hence that $\Box\Diamond\neg A \in x$ by lemma 3. Since $xR_\Omega y$, we also obtain that $\Diamond\neg A \in y$ and thus $\neg\Box A \in y$ by definition. But this is a contradiction.

Assume that (R2) does not hold. We suppose $\forall y(xR_\Omega y \Rightarrow \exists z(yR_\Omega z \;\&\; \textit{not}\; xR_\Omega z))$ and $\exists u \exists v(xR_\Omega u \;\&\; xR_\Omega v \;\&\; \textit{not}\; uR_\Omega v)$ for some $x \in W_\Omega$. That is, we assume that for some $x \in W_\Omega$

(1) If $xR_\Omega y$, then there exists $z \in W_\Omega$ such that $yR_\Omega z$ but not $xR_\Omega z$;

(2) There are $u, v \in W_\Omega$ such that $xR_\Omega u$ and $xR_\Omega v$ but not $uR_\Omega v$.

By (2), since not $uR_\Omega v$, there is a formula B such that $\Box B \in u$ but $B \notin v$. Because of $xR_\Omega u$ and $xR_\Omega v$, we get that $\Diamond\Box B \in x$ and $\Box B \notin x$. Thus we have $\Diamond\Box B \to \Box B \notin x$.

Take any y such that $xR_\Omega y$ (we note that such an element y exists by condition (2)). Then we have $yR_\Omega z$ but not $xR_\Omega z$ for some $z \in W_\Omega$ by (1). This implies the existence of a formula A such that $\Box A \in x$ but $A \notin z$. Since $yR_\Omega z$, we have $\Box A \notin y$ for any y such that $xR_\Omega y$. On the other hand, since $\vdash (\Box\Diamond\neg A \to \Diamond\neg A) \vee (\Diamond\Box B \to \Box B)$ for those formulas, we have $(\Box\Diamond\neg A \to \Diamond\neg A) \vee (\Diamond\Box B \to \Box B) \in x$. Hence it follows that $\Box\Diamond\neg A \to \Diamond\neg A \in x$ by $\Diamond\Box B \to \Box B \notin x$. We obtain that $\neg(\Box\Diamond\neg A \to \Diamond\neg A) \notin x$, $\Box\Diamond\neg A \wedge \Box A \notin x$ and hence that $\Box\Diamond\neg A \notin x$ by $\Box A \in x$. This yields to $\Diamond\Box A \in x$. In that case, we can conclude that $\Box A \in s$ for some s such that $xR_\Omega s$. But this is a contradiction. Therefore, R_Ω satisfies the condition (R2).

Thus we can prove this lemma completely. □

The next fundamental lemma is important to establish the completeness theorem.

Lemma 4. *For every formula A and $x \in W_\Omega$, we have $M_\Omega \models_x A$ iff $A \in x$.*

Proof. We show this lemma by induction on the construction of the formula A.

If A is the propositional variable p, then it holds evidently by definition of V_Ω. We only show the case of $\Box B \in x$. If $\Box B \in x$, then we obtain that $B \in y$ for every y satisfying $xR_\Omega y$. By induction hypothesis (IH), we get that $\models_y B$ and hence that $\models_x \Box B$.

Conversely, suppose that $\Box B \notin x$. There is a maximal consistent set $z \in W_\Omega$ such that $xR_\Omega z$ and $\neg B \in z$, because the set $\{C | \Box C \in x\} \cup \{\neg B\}$ is consistent. By IH, this means that it is not $\models_x \Box B$.

The lemma can be proved completely. □

Now we shall show the completeness theorem of our modal logic Ω. Suppose that A is not probable in Ω. Since $\{\neg A\}$ is consistent, by lemma 2, there exists a maximal consistent set $\Delta \in W_\Omega$ such that $\neg A \in \Delta$. For the canonical Ω-model $M_\Omega =< W_\Omega, R_\Omega, V_\Omega >$, we have that A is not true at Δ in M_Ω by lemma 6. This means the next result.

Theorem 2. *If a formula A is Ω-valid, then it is provable in Ω.*

4. Relation between L and Ω

In this section we consider the relation between the Kleene algebra L with countable generators and the modal logic Ω.

Let ξ be a map from the generator set $\{x_1, x_2, x_3, \cdots\}$ of L to the set of formulas of Ω defined by $\xi(x_i) = \Box p_i$, where p_i is the propositional variable. Then ξ can be uniquely extended to all elements of L as follows:

$\xi(x \wedge y) = \xi(x) \wedge \xi(y)$

$\xi(Nx) = \Box\neg\xi(x)$.

Concerning to $\xi(x \vee y)$, we do not have $\xi(x \vee y) = \xi(x) \vee \xi(y)$ but only have $\xi(x \vee y) = \xi(N(Nx \wedge Ny)) = \Box\neg(\Box\neg\xi(x) \wedge \Box\neg\xi(y))$.

If we introduce the order relation $\sqsubseteq$ in the modal logic Ω as $A \sqsubseteq B$ iff $\vdash_\Omega A \to B$, then $< \Omega, \sqsubseteq$ becomes a partially ordered set and moreover does a lattice, where $sup\{A, B\} = A \vee B$ and $inf\{A, B\} = A \wedge B$ for any formulas A and B in Ω. In the following we denote $A \equiv B$ when $A \sqsubseteq B$ and $B \sqsubseteq A$.

Proposition 1. *If $x \leq y$ in L, then we have $\xi(x) \sqsubseteq \xi(y)$, that is, ξ is a monotone map.*

Proof. Suppose that $x \leq y$ in L. Since $x \wedge y = y$, we have $\xi(x) = \xi(x \wedge y) = \xi(x) \wedge \xi(y)$ and hence $\xi(x) \sqsubseteq \xi(y)$. □

Proposition 2. *For every $x \in L$, there exists a formula A such that $\xi(x) \equiv \Box A$.*

Proof. We show the proposition by induction on the construction of element of L step by step.

(1) If x is identical with the generator x_i of L, then it is evident that $\xi(x_i) \equiv \Box p_i$

(2) If x is the form of $y \wedge z$, then we have $\xi(y) \equiv \Box B$ and $\xi(z) \equiv \Box C$ for some formulas B and C by IH. Then we have $\xi(y \wedge z) = \xi(y) \wedge \xi(z) \equiv \Box B \wedge \Box C \equiv \Box(B \wedge C)$ because $\vdash \Box(B \wedge C) \leftrightarrow \Box B \wedge \Box C$, where the symbol $\vdash A \leftrightarrow B$ means that $\vdash (A \to B) \wedge (B \to A)$.

(3) If x is the form of Ny, then clearly $\xi(x)$ is the form of $\Box\neg\xi(y)$. □

Proposition 3. *We have $\xi(x) \vee \xi(y) \sqsubseteq \Box\neg(\Box\neg\xi(x) \wedge \Box\neg\xi(y))$ for any $x, y \in L$.*

Proof. Suppose that $\xi(x) = \Box A$ and $\xi(y) = \Box B$. By (a3), we have that $\vdash (\Box A \vee \Box B) \to \Box\Diamond(\Box A \vee \Box B)$ and hence that $\vdash (\Box A \vee \Box B) \to \Box\neg(\Box\neg\Box A \wedge \Box\neg\Box B)$. This yields the desired result. □

The fact that $\xi(x \vee y) \not\equiv \xi(x) \vee \xi(y)$ in general means that $\xi(L)$ may not be a lattice which is a homomorphic image of L, that is, the map ξ does not need a homomorphism. To overcome this difficulty, we newly define operations $\triangle$, $\bigtriangledown$, $*$ in Ω as follows: For any formulas A and B in Ω,

$$A \triangle B = A \wedge B,$$
$$A \bigtriangledown B = (A^* \wedge B^*)^*$$
$$A^* = \Box\neg A.$$

Clearly, $< \Omega, \triangle, \bigtriangledown >$ is a lattice under these operations. In this case, we have in $\xi(L)$

$$\xi(x \wedge y) = \xi(x) \triangle \xi(y)$$
$$\xi(x \vee y) = \xi(x) \bigtriangledown \xi(y)$$
$$\xi(Nx) = (\xi(x))^*$$

Thus ξ is the onto homomorphism from L to $\xi(L)$. Since the map ξ is homomorphic, $\xi(L) = \{\xi(x)|x \in L\}$ becomes a distributive lattice. The lattice $\xi(L)$ has a minimum element $\Box\neg\Box t$ and a maximum element $\Box t(= t)$, where t means any CPL-tautologous formula such as $A \to A$. Thus $\xi(L) =< \xi(L), \triangle, \bigtriangledown, \Box\neg\Box t, \Box t >$ is a bounded distributive lattice. Moreover, the lattice $< \xi(L), \triangle, \bigtriangledown, *, \Box\neg\Box t, \Box t >$ is the Kleene algebra as shown in the following lemma.

Lemma 5. *$\xi(L) =< \xi(L), \triangle, \bigtriangledown, *, \Box\neg\Box t, \Box t >$ is the Kleene algebra.*

Proof. We only show that the operation $*$ satisfies the condition (c3): $\xi(x) \triangle (\xi(x))^* \sqsubseteq \xi(y) \bigtriangledown (\xi(y))^*$. We can put $\xi(x) = \Box A$ and $\xi(y) = \Box B$ for some formulas A and B by proposition 2. For those formulas, since $\vdash (\Box\Diamond\neg A \to \Diamond\neg A) \vee (\Diamond\Box B \to \Box B)$, we get $\vdash (\neg\Box\Diamond\neg A \vee \Diamond\neg A) \vee (\neg\Diamond\Box B \vee \neg\Box B)$ and $\vdash \neg(\Box A \wedge \Box\neg\Box A) \vee (\Box B \to \Box\neg\Box B)$. This means $\vdash \neg(\xi(x) \wedge (\xi(x))^*) \vee (\xi(y) \vee (\xi(y))^*)$ and hence $\vdash \xi(x) \wedge (\xi(x))^* \to \xi(y) \vee (\xi(y))^*$. Thus we have $\xi(x) \wedge (\xi(x))^* \sqsubseteq \xi(y) \vee (\xi(y))^*$. By proposition 4, we have that $\xi(y) \vee (\xi(y))^* \sqsubseteq \xi(y) \bigtriangledown (\xi(y))^*$ and hence that $\xi(x) \triangle (\xi(x))^* \sqsubseteq \xi(y) \bigtriangledown (\xi(y))^*$.

$\Box$

We have now two Kleene algebras L and $\xi(L)$. It is meaningful to investigate the relation between them. Clearly $\xi(L)$ is a homomorphic image of L, but may not isomorphic.

We define the Lindenbaum-Tarski algebra $\Omega/_{\equiv}$. In order to define it, we need a congruence relation on Ω. For every A and B in Ω, we remark that $A \equiv B$ iff $\vdash_\Omega A \leftrightarrow B$. It is easy to show that the relation $\equiv$ is a congruence relation on Ω. Let $[A] = \{B \in \Omega| \vdash_\Omega A \leftrightarrow B\}$ and $\Omega/_{\equiv} = \{[A]|A \in \Omega\}$. If we define the operations $\sqcap$, $\sqcup$, $N^{\equiv}$ as below, then it is easy to show that the structure $\Omega/_{\equiv}$ becomes the bounded lattice.

$$[A] \sqcap [B] = [A \wedge B]$$
$$[A] \sqcup [B] = [A \vee B]$$
$$N^{\equiv}[A] = [A^*](= [\Box\neg A])$$

Since the operations above are closed in the subset $\xi(L)$ of Ω, we can define the subalgebra $\xi(L)/_{\sim}$ of $\Omega/_{\equiv}$. On the other hand we define a relation $\sim$ in L as $x \sim y$ iff $\xi(x) \equiv \xi(y)$. Clearly $\sim$ is the congruence relation and $L/_{\sim}$ is the Kleene algebra by usual argument, where $[x] \wedge [y] = [x \wedge y]$, $[x] \vee [y] = [N(Nx \wedge Ny)]$, and $N[x] = [Nx]$.

Now we state the main theorem.

Theorem 3. $L/_{\sim} \cong \xi(L)/_{\equiv}$.

Proof. The map $\psi : L/_{\sim} \rightarrow \xi(L)/_{\equiv}$ defined by $\psi([x]) = [\xi(x)]$ gives the desired result. □

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