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GEOMETRY OF A COMPLEX PROJECTIVE SPACE FROM THE VIEWPOINT OF ITS CURVES AND REAL HYPERSURFACES

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Dedicated to Professor Maretsugu Yamasaki for his 60th birthday

Abstract. This expository paper consists of two parts. In the first half we study circles in a complex projective space and give a characterization of homogeneous real hypersurfaces in a complex projective space by using geometric properties of circles. In the latter half we study geodesics of geodesic spheres, which are the simplest real hypersurfaces in a complex projective space, and investigate their length spectrum in detail. Finally we characterize real and complex space forms from this point of view.

0. Introduction.

The study of circles is one of the interesting objects in differential geometry. We here recall the definition of circles.

A smooth curve $\gamma : \mathbb{R} \to M$ in a complete Riemannian manifold M is called a *circle* of curvature $\kappa(\geqq 0)$ if it is parametrized by its arclength s and satisfies the following equation:

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(s) = -\kappa^2\dot\gamma(s),$$

where κ is a constant, which is called the curvature of the circle γ, and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. A circle of null curvature is nothing but a geodesic. The notion of circles is an extension of the notion of geodesics. As a matter of course, a circle of curvature $\kappa(>0)$ in a Euclidean m-space $\mathbb{R}^m$ is a circle of radius $\frac{1}{\kappa}$ in the sense of Euclidean geometry and it is closed. However, in general a circle of positive curvature in a Riemannian manifold is not closed.

In section 1, we survey several results on geometric properties of circles in an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c (Theorems 1, 2). In this section we first show that for each positive

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constant κ, there exist infinitely many closed circles of curvature κ and infinitely many open circles of curvature κ in $\mathbb{C}P^n(c)$ up to an action of the isometry group of $\mathbb{C}P^n(c)$. The main purpose of section 1 is to give an answer to the following question: If two closed circles γ_1 and γ_2 in $\mathbb{C}P^n(c)$ have the same length, are they congruent with respect to an isometry of $\mathbb{C}P^n(c)$?

In section 2, we study real hypersurfaces M^{2n-1} in $\mathbb{C}P^n$. Typical examples of real hypersurfaces are *homogeneous real hypersurfaces*, namely they are given as orbits under subgroups of the projective unitary group $PU(n+1)$. Takagi ([17]) determined all homogeneous real hypersurfaces in $\mathbb{C}P^n$. Due to his work we find that a homogeneous real hypersurface in $\mathbb{C}P^n$ is locally congruent to one of the six model spaces of type $\mathrm{A}_1, \mathrm{A}_2$, B, C, D and E. They are realized as tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or rank 2 (see Theorem A).

In the study of real hypersurfaces in $\mathbb{C}P^n$, there can be the following two problems:

(i) Give a characterization of homogeneous real hypersurfaces.
(ii) Construct non-homogeneous *nice* real hypersurfaces and characterize such examples.

In this paper we are devoted to the problem (i). It is well-known that a hypersurface M^n isometrically immersed into $\mathbb{R}^{n+1}$ is locally congruent to a standard sphere if and only if every geodesic of M is a circle of positive curvature in $\mathbb{R}^{n+1}$. But we remark that there does not exist a real hypersurface M^{2n-1} all of whose geodesics are circles in $\mathbb{C}P^n$. The main purpose of section 2 is to give a characterization of all homogeneous real hypersurfaces M^{2n-1} in $\mathbb{C}P^n$ by observing the extrinsic shape of geodesics of M in $\mathbb{C}P^n$ (see Theorem 5).

In section 3, we restrict ourselves on geodesics of geodesic spheres in $\mathbb{C}P^n$. Geodesic spheres in $\mathbb{C}P^n$, which are nothing but homogeneous real hypersurfaces of type A_1, are nice objects in intrinsic geometry as well as extrinsic geometry (that is, submanifold theory).

From the view point of intrinsic geometry, we know that geodesic spheres of sufficiently large radius in $\mathbb{C}P^n$ are examples of *Berger spheres.* Namely, these spheres are homogeneous Riemannian manifolds which are diffeomorphic to a sphere, whose sectional curvatures lie in the interval $[\delta K, K]$ for some $\delta \in (0, \frac{1}{9})$, and which have closed geodesics of length less than $\frac{2\pi}{\sqrt{K}}$. These show that odd-dimensional version of Klingenberg's lemma does not hold (for details, see [18]).

From the view point of submanifold theory, they are the simplest real hypersurfaces. In $\mathbb{C}P^n$ ($n \geqq 3$), geodesic spheres are the only examples of real hypersurfaces with at most two distinct principal curvatures at its each point (see [6]). In this context it is natural to study geodesics of these real hypersurfaces. The main purpose of section 3 is to investigate the length spectrum of geodesic spheres in $\mathbb{C}P^n$ (see Theorems 7, 8, 9, 10). In this section we suppose that a complex projective space $\mathbb{C}P^n$ is furnished with the standard metric of constant holomorphic sectional curvatute 4.

In section 4, motivated by the discussion in section 3, we characterize real and complex space forms by observing the extrinsic shape of geodesics on their geodesic spheres.

Real space forms are Riemannian manifolds of constant curvature, which are locally isometric to either one of standard spheres, Euclidean spaces or real hyperbolic spaces. In a real space form M, a geodesic sphere $G_m(r) = \{p \in M|\ d(p,m) = r\}$ with center m and radius r is a totally umbilic but not totally geodesic hypersurface (with parallel second fundamental form). Here, d denotes the distance function induced by Riemannian metric $\langle\ ,\ \rangle$ on M. This fact tells us that every geodesic on $G_m(r)$ is a circle of positive curvature in the ambient manifold M.

Next we consider geodesic spheres in complex space forms. Complex space forms are Kähler manifolds of constant holomorphic sectional curvature. It is well-known that these are locally complex analytically isometric to either one of complex projective spaces, complex Euclidean spaces or complex hyperbolic spaces. For a geodesic sphere $G_m(r)$ of sufficiently small radius r in these complex space forms M, we know that both of all geodesics orthogonal to ξ on $G_m(r)$ and all integral curves of ξ are circles of positive curvature in M, where ξ is the characteristic vector field of $G_m(r)$ in M (for details, see [14]).

Along these contexts we shall give some characterizations of real and complex space forms (see Theorems 11, 12 and 13).

In this paper we study Riemannian manifolds without boundary.

1. Geometric properties of circles in $\mathbb{C}P^n$.

First of all we shall recall the Frenet formula for a smooth Frenet curve in a Riemannian manifold M with Riemannian metric $\langle\ ,\ \rangle$. A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s is called a *Frenet curve of proper order d* if there exist orthonormal frame fields $\{V_1 = \dot\gamma, \cdots, V_d\}$ along γ and positive functions $\kappa_1(s), \cdots, \kappa_{d-1}(s)$ satisfying the following system of ordinary equations

$$(1.1)\qquad \nabla_{\dot\gamma} V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j = 1, \cdots, d,$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ. We call Equation (1.1) the Frenet formula for the Frenet curve γ. The functions $\kappa_j(s) (j = 1, \cdots, d-1)$ and the orthonormal frame $\{V_1, \cdots, V_d\}$ are called the curvatures and the Frenet frame of γ, respectively.

A Frenet curve is called a *Frenet curve of order d* if it is a Frenet curve of proper order $r(\leqq d)$. For a Frenet curve of order d which is of proper order $r(\leqq d)$, we use the convention in (1.1) that $\kappa_j \equiv 0$ $(r \leqq j \leqq d-1)$ and $V_j \equiv 0$ $(r+1 \leqq j \leqq d)$. We call a smooth Frenet curve γ closed if there exists a nonzero constant s_0 with $\gamma(s+s_0) = \gamma(s)$ for every s. The minimum positive constant with this property is called the length of γ and denoted by length(γ). For an open Frenet curve, that is a Frenet curve which is not closed, we put its length as $\mathrm{length}(\gamma) = \infty$.

We call a smooth Fenet curve a *helix* when all its curvatures are constant. A helix of order 1 is nothing but a geodesic and a helix of order 2, that is a curve which satisfies $\nabla_{\dot\gamma} V_1(s) = \kappa V_2(s), \nabla_{\dot\gamma} V_2(s) = -\kappa V_1(s), V_1(s) = \dot\gamma(s)$, is called a circle of curvature κ.

Needless to say, a curve generated by a Killing vector field is a helix. But in general, the converse does not hold. From this point of view we are interested in the following well-known result.

Proposition 1. *In a complete and simply connected real space form, which is a Euclidean space $\mathbb{R}^n$, a standard sphere S^n or a hyperbolic space H^n, a smooth Frenet curve is a helix if and only if it is generated by a Killing vector field in this space.*

In order to obtain the complex version of Proposition 1, we review the definition of complex torsions of Frenet curves in Kähler manifolds. Let M be an n-dimensional Kähler manifold with complex structure J and Riemannian metric $\langle\ ,\ \rangle$. For a Frenet curve $\gamma = \gamma(s)$ in M of order $d(\leqq 2n)$ with the associated Frenet frame $\{V_1, \cdots, V_d\}$, we set $\tau_{ij}(s) = \langle V_i(s), JV_j(s)\rangle$ for $1 \leqq i < j \leqq d$ and call them its *complex torsions*. In the study of Frenet curves in a Kähler manifold their complex torsions play an important role. The following is the complex version of Proposition 1 (see [15]).

Proposition 2. *In a complete and simply connected complex space form, which is a complex Euclidean space $\mathbb{C}^n$, a complex projective space $\mathbb{C}P^n$ or a complex hyperbolic space $\mathbb{C}H^n$, a smooth Frenet curve is generated by a holomorphic Killing vector field in this space if and only if all its curvatures and all its complex torsions are constant functions.*

In a Kähler manifold, we call a smooth curve γ a *holomorphic helix* when both of all curvatures and all complex torsions of γ are constant.

Let $\gamma = \gamma(s)$ be a circle in a Kähler manifold M (with complex structure J) satisfying the equations $\nabla_{\dot\gamma}\dot\gamma = \kappa V_2(s), \nabla_{\dot\gamma} V_2(s) = -\kappa\dot\gamma$.
Then,

$$\begin{aligned}\nabla_{\dot\gamma}\langle\dot\gamma, JV_2(s)\rangle &= \langle\nabla_{\dot\gamma}\dot\gamma, JV_2(s)\rangle + \langle\dot\gamma, J\nabla_{\dot\gamma}V_2(s)\rangle \\ &= \kappa\cdot\langle V_2(s), JV_2(s)\rangle - \kappa\cdot\langle\dot\gamma, J\dot\gamma\rangle = 0.\end{aligned}$$

Hence the only complex torsion τ_{12}, say τ $(-1 \leqq \tau \leqq 1)$, of each circle in a Kähler manifold M is constant. Circles of complex torsion ± 1 are called *holomorphic circles* and circles of null complex torsion are called *totally real circles*. As an immediate consequence of Proposition 2 we can see that in a complete and simply connected complex space form M, every circle of curvature $\kappa(> 0)$ is generated by a Killing vector field of M. In order to get rid of the influence of the action of the full isometry group, we say that two smooth curves γ_1 and γ_2 are congruent each other if there exist an isometry φ and a constant s_0 with $\gamma_2(s) = \varphi \circ \gamma_1(s + s_0)$ for every s.

The congruence theorem for circles in a non-flat complex space form M is stated as follows (see Theorem 5.1 in [15]):

Proposition 3. *Two circles in a complete and simply connected non-flat complex space form $M(c)$, that is $M(c) = \mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$, are congruent if and only if they have the same curvatures and the same absolute values of complex torsion.*

Therefore the moduli space of all congruency classes of circles on a non-flat complex space form $M(c)$ is bijective to the set $[0, \infty) \times [0, 1]/\sim$, where (κ, τ) and (κ', τ') are equivalent if and only if $(\kappa, \tau) = (\kappa', \tau')$ in case of $\kappa\kappa' \neq 0$ or $\kappa = \kappa' = 0$ in case of $\kappa\kappa' = 0$.

We are now in a position to study circles in $\mathbb{C}P^n(c)$ (for details, see [2,3]). We call a smooth curve *simple* if it does not have self-intersection points. More precisely, an open curve σ is called simple if $\sigma(s_1) \neq \sigma(s_2)$ for every s_1, s_2 $(s_1 \neq s_2)$, and a closed curve σ is called simple if $\sigma(s_1) \neq \sigma(s_2)$ for every s_1, s_2 $(0 \leqq s_1 < s_2 < \mathrm{length}(\sigma))$.

Theorem 1. *Let γ be a circle of curvature κ and complex torsion τ in a complex projective space $\mathbb{C}P^n(c)$ of holomorphic sectional curvature c. Then the following hold*:

(1) *When $\tau = 0$, γ is a simple closed curve whose length is $\frac{4\pi}{\sqrt{4k^2+c}}$.*
(2) *When $\tau = \pm 1$, γ is a simple closed curve whose length is $\frac{2\pi}{\sqrt{k^2+c}}$.*
(3) *When $\tau \neq 0, \pm 1$, we denote by a, b and $d (a < b < d)$ the nonzero solutions for*
$$c\lambda^3 - (4k^2 + c)\lambda + 2\sqrt{c}\kappa\tau = 0.$$
Then we find the following:
(i) *If one of the three ratios $a/b, b/d$ and d/a is rational, γ is a simple closed curve. Its length is the least common multiple of $\frac{4\pi}{\sqrt{c}(b-a)}$ and $\frac{4\pi}{\sqrt{c}(d-a)}$.*
(ii) *If each of the three ratios $a/b, b/d$ and d/a is irrational, γ is a simple open curve.*

Theorem 1 tells us that there exist infinitely many congruency classes of open circles in $\mathbb{C}P^n(c)$. In the following we pay attention to closed circles in $\mathbb{C}P^n(c)$.

Theorem 2.

(1) *Let γ_1 and γ_2 be closed circles with common length l in $\mathbb{C}P^n(c)$. Suppose that l satisfies the inequalities $\frac{2}{\sqrt{c}}\pi < l \leqq \frac{4}{3}\sqrt{\frac{5}{c}}\pi$. Then γ_1 and γ_2 are congruent by an isometry of $\mathbb{C}P^n(c)$.*
(2) *For each positive $l \notin \left(\frac{2}{\sqrt{c}}\pi, \frac{4}{3}\sqrt{\frac{5}{c}}\pi\right]$, there exist at least two congruency classes of closed circles in $\mathbb{C}P^n(c)$ whose length is l.*

We consider lengths of closed circles of the fixed curvature κ.

Proposition 4.

(1) *For each* $\kappa(>0)$*, the set of* $\mathrm{length}(\gamma)$ *of all closed circles of curvature* κ *in* $\mathbb{C}P^n(c)$ *is an unbounded discrete subset of the real positive line* $(0,\infty)$.

(2) *The bottom of this unbounded subset is* $\frac{2\pi}{\sqrt{\kappa^2+c}}$*. The holomorphic circle of curvature* κ *is the only example of a closed circle of curvature* κ *whose length is* $\frac{2\pi}{\sqrt{\kappa^2+c}}$.

(3) *The second lowest element of this unbounded subset is* $\frac{4\pi}{\sqrt{4\kappa^2+c}}$*. The totally real circle of curvature* κ *is the only example of a closed circle of curvature* κ *whose length is* $\frac{4\pi}{\sqrt{4\kappa^2+c}}$.

Proposition 4 shows that holomorphic circles and totally real circles are nice examples in the class of closed circles in $\mathbb{C}P^n(c)$. These two closed circles are plane curves in $\mathbb{C}P^n(c)$. Namely, they are lying on some real 2-dimensional totally geodesic submanifolds of $\mathbb{C}P^n(c)$. In fact, every holomorphic circle in $\mathbb{C}P^n(c)$ lies on $\mathbb{C}P^1(c)$ which is a holomorphic totally geodesic submanifold of $\mathbb{C}P^n(c)$, and every totally real circle in $\mathbb{C}P^n(c)$ lies on $\mathbb{R}P^2(\frac{c}{4})$ of curvature $\frac{c}{4}$ which is a totally real totally geodesic submanifold of $\mathbb{C}P^n(c)$. We remark that every circles in $\mathbb{C}P^n(c)$ lies on a holomorphic totally geodesic $\mathbb{C}P^2(c)$.

We shall provide characterizations of holomorphic circles and totally real circles in $\mathbb{C}P^n(c)$ from the viewpoint of submanifold theory. Let f_0 be an isometric minimal imbedding of $\mathbb{C}P^n(c)$ into a sphere $S^{n(n+2)-1}(\frac{n+1}{2n}c)$ of curvature $\frac{n+1}{2n}c$ and ι be a totally umbilic imbedding of $S^{n(n+2)-1}(\frac{n+1}{2n}c)$ into Euclidean space $\mathbb{R}^{n(n+2)}$. Put $f=\iota\circ f_0$. Then f is an isometric parallel imbedding of $\mathbb{C}P^n(c)$ into $\mathbb{R}^{n(n+2)}$. It is well-known that the imbedding f is the only parallel full imbedding of $\mathbb{C}P^n(c)$ into a Euclidean space $\mathbb{R}^N$. This imbedding f has many nice properties. For example, the imbedding f has a property that for each geodesic γ of $\mathbb{C}P^n(c)$ the curve $f\circ\gamma$ is a circle in $\mathbb{R}^{n(n+2)}$.

By using this imbedding f, we shall characterize holomorphic circles and totally real circles in $\mathbb{C}P^n(c)$ (cf. [7]).

Theorem 3. *Let* γ *be a curve in* $\mathbb{C}P^n(c)$*. Then* γ *is a geodesic or a holomorphic circle if and only if the curve* $f\circ\gamma$ *is a circle in* $\mathbb{R}^{n(n+2)}$*, where* f *is a parallel imbedding of* $\mathbb{C}P^n(c)$ *into* $\mathbb{R}^{n(n+2)}$.

Theorem 3 is a generalization of the fact that a curve γ on $S^2(c)$ is a circle, that is γ is a great circle or a small circle of $S^2(c)$, if and only if γ is a circle in $\mathbb{R}^3$. By virtue of Theorem 3 we obtain a characterization of geodesics of $\mathbb{C}P^n(c)$ from the viewpoint of submanifold theory:

Theorem 3'. *Let* γ *be a curve in* $\mathbb{C}P^n(c)$*. Then* γ *is a geodesic if and only if the curve* $f\circ\gamma$ *is a circle of curvature* $\sqrt{c}$ *in* $\mathbb{R}^{n(n+2)}$.

We establish a characterization of totally real circles of $\mathbb{C}P^n(c)$.

Theorem 4. *Let γ be a curve in $\mathbb{C}P^n(c)$. Then γ is a totally real circle if and only if γ lies on totally real totally geodesic $\mathbb{R}P^n(\frac{c}{4})$ of $\mathbb{C}P^n(c)$ and the curve $f\circ\gamma$ is a helix of proper order 4 in $\mathbb{R}^{n(n+2)}$.*

If we omit the condition that γ lies on $\mathbb{R}P^n(\frac{c}{4})$, Theorem 4 is not true (for details, see [7]). The statement of a characterization of other circles γ with complex torsion τ $(-1<\tau\neq 0<1)$ in $\mathbb{C}P^n(c)$ is more complicated (see Theorem 6 in page 182 in [7]).

Finally we remark that in $\mathbb{C}P^n(c)$ even if closed circles have the same curvature κ and the same length l, they are not necessarily congruent. For example, let γ_1 be a circle of curvature $\frac{\sqrt{2c}}{4}$ and complex torsion $\tau=\frac{5698}{559\sqrt{559}}$ and γ_2 be a circle of curvature $\frac{\sqrt{2c}}{4}$ and complex torsion $\tau=\frac{12502}{559\sqrt{559}}$. These two circles are closed circles of the same curvature $\frac{\sqrt{2c}}{4}$ and the same length $\frac{4\sqrt{1118}}{3\sqrt{c}}\pi$. But they are not congruent each other (see [2]).

2. Characterizations of homogeneous real hypersurfaces in $\mathbb{C}P^n$.

Let M be a real hypersurface of $\mathbb{C}P^n$ (of constant holomorphic sectional curvature 4) and let $\mathcal{N}$ be a unit normal vector field on M. The Riemannian connections $\widetilde{\nabla}$ on $\mathbb{C}P^n$ and ∇ on M are related by

$$\widetilde{\nabla}_X Y=\nabla_X Y+\langle AX,Y\rangle\mathcal{N}$$

and

$$\widetilde{\nabla}_X\mathcal{N}=-AX,$$

where $\langle\ ,\ \rangle$ denotes the Riemmanian metric of M induced from the Fubini-Study metric of $\mathbb{C}P^n$ and A is the shape operator of M in $\mathbb{C}P^n$. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively. It is known that M admits an almost contact metric structure $(\phi,\xi,\eta,\langle\ ,\ \rangle)$ induced from the Kähler structure of $\mathbb{C}P^n$, which satisfies

$$\phi^2=-I+\eta\otimes\xi,\quad \eta(\xi)=1$$

and

$$\langle\phi X,\phi Y\rangle=\langle X,Y\rangle-\eta(X)\eta(Y).$$

We recall the following ([17]).

Theorem A. *Let M be a homogeneous real hypersurface of $\mathbb{C}P^n$. Then M is a tube of radius r over the following Kähler submanifolds*:

(A_1) *hyperplane $\mathbb{C}P^{n-1}$, where $0<r<\frac{\pi}{2}$,*
(A_2) *totally geodesic $\mathbb{C}P^k(1\leqq k\leqq n-2)$, where $0<r<\frac{\pi}{2}$*
(B) *complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0<r<\frac{\pi}{4}$,*
(C) *$\mathbb{C}P^1\times\mathbb{C}P^{\frac{n-1}{2}}$, where $0<r<\frac{\pi}{4}$ and $n(\geqq 5)$ is odd,*
(D) *complex Grassmann $\mathbb{C}G_{2,5}$, where $0<r<\frac{\pi}{4}$ and $n=9$,*
(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0<r<\frac{\pi}{4}$, and $n=15$.*

The numbers of distinct principal curvatures of homogeneous real hypersurfaces are 2, 3, 3, 5, 5, 5, respectively.

In introduction we remark that there exist no real hypersurfaces all of whose geodesics are circles in $\mathbb{C}P^n$. However, for each homogeneous real hypersurface M by taking orthonormal vectors $v_1, \cdots, v_{2n-2}$ orthogonal to ξ at each point p of M in such a way that $v_1, \cdots, v_{2n-2}$ are principal curvature vectors, all geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ are circles in $\mathbb{C}P^n$ with positive curvature. Considering the converse of this fact, we establish the following characterization of all homogeneous real hypersurfaces in $\mathbb{C}P^n$ (cf. [1]).

Theorem 5. *Let M be a connected real hypersurface of $\mathbb{C}P^n$. Then M is locally congruent to a homogeneous real hypersurface if and only if there exist orthonormal vectors $v_1, \cdots, v_{2n-2}$ orthogonal to ξ at each point p of M such that all geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ are circles in $\mathbb{C}P^n$ with positive curvature.*

In the hypothesis of Theorem 5 we do not need to suppose that we take the vectors $\{v_1, \cdots, v_{2n-2}\}$ as *a local field of orthonormal frames* in M. But, for every homogeneous real hypersurface M in $\mathbb{C}P^n$, we can take a local field of orthonormal frames in M satisfying the hypothesis of Theorem 5. Every circle in Theorem 5 is a totally real circle in $\mathbb{C}P^n$. We note that for any homogeneous real hypersurface M, at each point p of M the geodesic $\gamma = \gamma(s)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$ is a holomorphic circle of $\mathbb{C}P^n$.

In the following, we pay particular attention to *geodesic spheres* in $\mathbb{C}P^n(4)$. It is known that each geodesic sphere of radius r $(0 < r < \frac{\pi}{2})$ in $\mathbb{C}P^n(4)$ is congruent to a tube of radius $\frac{\pi}{2} - r$ around a totally geodesic complex hyperplane $\mathbb{C}P^{n-1}(4)$ in $\mathbb{C}P^n(4)$, which is the simplest homogeneous real hypersurface of $\mathbb{C}P^n$. The following is a direct consequence of Theorem 5 (see [14]).

Theorem 6. *Let M be a connected real hypersurface of $\mathbb{C}P^n$. Then M is locally congruent to a geodesic sphere of radius r $(0 < r < \frac{\pi}{2})$ if and only if there exist orthonormal vectors $v_1, \cdots, v_{2n-2}$ orthogonal to ξ at each point p of M such that all geodesics of M through p in the direction $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-2)$ are circles in $\mathbb{C}P^n$ with positive curvature.*

In the next section we shall investigate the extrinsic shape of *every geodesic* of geodesic spheres in $\mathbb{C}P^n(4)$ (for details, see [4]).

3. Extrinsic shape of geodesics of geodesic spheres.

Unless otherwise stated we here adopt the same terminology as that of the preceding section. Let M be a geodesic sphere of radius r in $\mathbb{C}P^n(4)$. Then the shape operator A of M in $\mathbb{C}P^n(4)$ is expressed as:

$$A\xi = (2\cot 2r)\xi \quad \text{and} \quad Au = (\cot r)u$$

for every tangent vector $u \in TM$ orthogonal to ξ. Moreover, this real hypersurface M satisfies the following (cf. [16]).

1) The structure tensor ϕ and the shape operator A of M in $\mathbb{C}P^n(4)$ are commutative: $\phi A = A\phi$.
2) The covariant derivative of the shape operator A satisfies

$$(\nabla_X A)Y = -\{\langle \phi X, Y\rangle \xi + \eta(Y)\phi X\}.$$

Let ι be an isometric imbedding of a geodesic sphere M into $\mathbb{C}P^n(4)$. We shall show that for every geodesic γ on M the curve $\iota \circ \gamma$ in $\mathbb{C}P^n(4)$ is a helix of order 4. Note that $\langle \dot\gamma(s), \xi\rangle$ is constant along γ. Indeed,

$$\nabla_{\dot\gamma}\langle \dot\gamma(s), \xi\rangle = \langle \dot\gamma(s), \phi A\dot\gamma\rangle = \langle \dot\gamma, A\phi\dot\gamma\rangle = -\langle \phi A\dot\gamma, \dot\gamma\rangle = 0.$$

We shall call this constant the *structure torsion* of γ and denote by $\sin\theta$ with $0 \leqq |\theta| \leqq \frac{\pi}{2}$. By direct computation we obtain the following:

Proposition 5. *Let M be a geodesic sphere of radius r $(0 < r < \frac{\pi}{2})$ in $\mathbb{C}P^n(4)$. We denote by ι an isometric imbedding of M into $\mathbb{C}P^n(4)$. Then the extrinsic shape $\iota \circ \gamma$ of a geodesic γ on M is as follows*:

(1) *Suppose the radius r satisfies $\frac{\pi}{4} \leqq r < \frac{\pi}{2}$. If the structure torsion of γ is $\pm\cot r$, the curve $\iota \circ \gamma$ is a geodesic.*
(2) *When $r \neq \frac{\pi}{4}$, if the structure torsion of γ is ± 1(i.e. $\dot\gamma = \pm\xi$), the curve $\iota \circ \gamma$ is a circle of curvature $2|\cot 2r|$ and of complex torsion ∓ 1 in $\mathbb{C}P^n(4)$. This circle lies on a totally geodesic $\mathbb{C}P^1(4)$.*
(3) *If γ has null structure torsion (i.e. $\dot\gamma$ is orthogonal to ξ), the curve $\iota \circ \gamma$ is a circle of curvature $\cot r$ and null complex torsion in $\mathbb{C}P^n(4)$. This circle lies on a totally geodesic $\mathbb{R}P^2(1)$.*
(4) *Generally, if the structure torsion of γ is of the form $\sin\theta$ $(0 < |\theta| < \frac{\pi}{2},\ \sin\theta \neq \pm\cot r)$, then the curve $\iota \circ \gamma$ is a holomorphic helix of proper order 4 whose curvatures are described as*

$$\kappa_1 = |\cot r - \tan r \cdot \sin^2\theta|, \kappa_2 = \tan r \cdot |\sin\theta| \cos\theta, \kappa_3 = \cot r.$$

Its complex torsions are described as

$$\tau_{12} = \begin{cases} -\sin\theta, & if\ \ \cot r - \tan r \cdot \sin^2\theta > 0, \\ \ \ \sin\theta, & if\ \ \cot r - \tan r \cdot \sin^2\theta < 0, \end{cases}$$
$$\tau_{14} = \begin{cases} -\mathrm{sgn}(\sin\theta)\cos\theta, & if\ \ \cot r - \tan r \cdot \sin^2\theta > 0, \\ \mathrm{sgn}(\sin\theta)\cos\theta, & if\ \ \cot r - \tan r \cdot \sin^2\theta < 0, \end{cases}$$
$$\tau_{23} = \mathrm{sgn}(\sin\theta)\cos\theta,\ \tau_{34} = \sin\theta,\ \tau_{13} = \tau_{24} = 0,$$

where $\mathrm{sgn}(a)$ denotes the signature of a real number a. This helix $\iota \circ \gamma$ lies on a totally geodesic $\mathbb{C}P^2(4)$.

It follows from Propositions 2 and 5 that every geodesic on a geodesic sphere in $\mathbb{C}P^n$ is generated by a Killing vector field on $\mathbb{C}P^n$ as a curve in $\mathbb{C}P^n$. Thus we have

Corollary 1. *Every geodesic on a geodesic sphere in a complex projective space is a simple curve.*

The fact that the isometric imbedding ι of a geodesic sphere M into $\mathbb{C}P^n$ is an equivariant map, together with Propositions 2 and 5, shows the following congruence theorem about geodesics on M:

Proposition 6. *On a geodesic sphere M in a complex projective space, two geodesics are congruent with respect the isometry group of M if and only if the absolute values of their structure torsion coincide.*

We are now in a position to study length of closed geodesics on a geodesic sphere in a complex projective space. Let $\Pi : S^{2n+1}(1) \to \mathbb{C}P^n(4)$ denote the Hopf fibration of a unit sphere. For a smooth curve σ on $\mathbb{C}P^n$ a smooth curve $\tilde{\sigma}$ is called a horizontal lift of σ if $\dot{\tilde{\sigma}}(s)$ is a horizontal vector and $d\Pi(\dot{\tilde{\sigma}}(s)) = \dot{\sigma}(s)$ for all s. Our idea lies on considering a horizontal lift of a holomorphic helix $\iota \circ \gamma$ for every geodesic γ on a geodesic sphere. The following elementary lemma is useful in our argument.

Lemma 1. *Let σ be a smooth simple curve on $\mathbb{C}P^n(4)$. Suppose a horizontal lift $\tilde{\sigma}$ of σ on $S^{2n+1}(1)$ is represented as*

$$\tilde{\sigma}(s) = Ae^{\sqrt{-1}as} + Be^{\sqrt{-1}bs} + Ce^{\sqrt{-1}cs} + De^{\sqrt{-1}ds},$$

which is a curve on $\mathbb{C}^{n+1}$ with non-zero vectors $A, B, C, D \in \mathbb{C}^{n+1}$ and mutually distinct real numbers a, b, c, d which satisfy $a + b + c + d = 0$ and $a \neq 0$. Then σ is closed if and only if all the ratios $b/a, c/a, d/a$ are rational. In this case, its length is

$$\text{length}(\sigma) = 2\pi \times \text{L.C.M.}\left(\frac{1}{|b-a|}, \frac{1}{|c-a|}, \frac{1}{|d-a|}\right).$$

Here for positive numbers $\alpha_1, \alpha_2, \alpha_3$, we denote by L.C.M.$\{\alpha_1, \alpha_2, \alpha_3\}$ *the minimum value of the set $\{j\alpha_1 \mid j = 1, 2, \dots\} \cap \{j\alpha_2 \mid j = 1, 2, \dots\} \cap \{j\alpha_3 \mid j = 1, 2, \dots\}$.*

By virtue of this lemma we establish the following:

Theorem 7. *Let γ be a geodesic on a geodesic sphere M of radius r $(0 < r < \pi/2)$ in $\mathbb{C}P^n$ of holomorphic sectional curvature 4.*

1. *If the structure torsion of γ is ± 1, then γ is closed and its length is $\pi \sin 2r$.*
2. *If γ has null structure torsion, then γ is also closed and its length is $2\pi \sin r$.*

(3) *When the structure torsion of* γ *is of the form* $\sin\theta$ $(0 < |\theta| < \frac{\pi}{2})$, *then it is closed if and only if*

$$\sin\theta = \frac{\pm q}{\sin r\sqrt{p^2\tan^2 r + q^2}}$$

with some relatively prime positive integers p *and* q *with* $q < p\tan^2 r$. *In this case, its length is*

$$\mathrm{length}(\gamma) = \begin{cases} 2\pi\sqrt{p^2\sin^2 r + q^2\cos^2 r}, & \textit{if } pq \textit{ is even} \\ \pi\sqrt{p^2\sin^2 r + q^2\cos^2 r}, & \textit{if } pq \textit{ is odd.} \end{cases}$$

Remark. When the structure torsion is not equal to ± 1, every horizontal lift $\hat{\gamma}$ of $\iota\circ\gamma$ for a geodesic γ on M is closed if and only if γ is closed.

When we study length spectrum of geodesics on a Riemannian manifold N, in order to avoid the influence of the action of the isometry group of N, we consider the moduli space of geodesics under the action of isometries. The moduli space $\mathrm{Geod}(N)$ of geodesics on N is the quotient space of the set of all geodesics on N under the congruency relation. We define the *length spectrum* $\mathcal{L}_N : \mathrm{Geod}(N) \to \mathbb{R}\cup\{\infty\}$ of N by $\mathcal{L}_N([\gamma]) = \mathrm{length}(\gamma)$, where $[\gamma]$ denotes the congruency class containing a geodesic γ. We also call the image $\mathrm{Lspec}(N) = \mathcal{L}_N(\mathrm{Geod}(N))\cap\mathbb{R}$ the length spectrum of N. For example, the length spectrum of a standard unit sphere is $\mathrm{Lspec}(S^m(1)) = \{2\pi\}$.

As a direct consequence of Theorem 7, for a geodesic sphere M of radius r in $\mathbb{C}P^n(4)$, we can see that

$$\begin{aligned}\mathrm{Lspec}(M) = {} & \{\pi\sin 2r\}\bigcup\{2\pi\sin r\} \\ & \bigcup\left\{2\pi\sqrt{p^2\sin^2 r + q^2\cos^2 r}\;\middle|\;\begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is even and } q < p\tan^2 r\end{array}\right\} \\ & \bigcup\left\{\pi\sqrt{p^2\sin^2 r + q^2\cos^2 r}\;\middle|\;\begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is odd and } q < p\tan^2 r\end{array}\right\}.\end{aligned}$$

Therefore we obtain the following.

Theorem 8. *On a geodesic sphere* M *in* $\mathbb{C}P^n$, *there exist infinitely many congruency classes of closed geodesics. Moreover the length spectrum* $\mathrm{Lspec}(M)$ *of* M *is an unbounded discrete subset in the real line* $\mathbb{R}$.

For a length spectrum $\lambda\in\mathrm{Lspec}(N)$ we call the cardinality $m_N(\lambda)$ of the set $\mathcal{L}_N^{-1}(\lambda)$ the multiplicity of λ. When the multiplicity of a length spectrum is 1 we say it is *simple*. Clearly for a geodesic sphere M in a complex projective space, we see by the expression of $\mathrm{Lspec}(M)$ that $m_M(\lambda) < \infty$ at each λ. We here study the first, the second and the third length spectrum, that is the minimum, the second minimum and the third minimum of the length spectrum.

Proposition 7. *Let M be a geodesic sphere of radius r $(0 < r < \pi/2)$ in $\mathbb{C}P^n$ of holomorphic sectional curvature* 4*.*

(1) *The first length spectrum of M is $\pi \sin 2r$, which is the length of geodesics with structure torsion ± 1. It is simple.*
(2) *The second length spectrum of M is also simple. When $0 < r \leqq \pi/4$, it is $2\pi \sin r$, which is the length of geodesics with null structure torsion. When $\pi/4 < r < \pi/2$, it is π, which is the length of geodesics with structure torsion $\pm \cot r$.*
(3) *The third length spectrum is also simple. When $\pi/4 < r < \pi/2$, it is $2\pi \sin r$, which is the length of geodesics with null structure torsion. When $\sqrt{2m-1} \leqq \cot r < \sqrt{2m+1}$ $(m = 1, 2, \dots)$, in particular, $0 < r \leqq \pi/4$, it is $\pi\sqrt{4m(m+1)\sin^2 r + 1}$, which is the length of geodesics with structure torsion $\pm 1/(\sin r\sqrt{(2m+1)^2 \tan^2 r + 1})$.*

Since the sectional curvature of a geodesic sphere M of radius r in $\mathbb{C}P^n(4)$ lies in the interval $[\cot^2 r, 4 + \cot^2 r]$, the first length spectrum of M is smaller than $2\pi/\sqrt{4 + \cot^2 r}$ if $\tan^2 r > 2$. Hence M is an example of a Bereger sphere, as was pointed out in [18]. But for other length spectrum we find that the following Klingenberg's lemma holds:

Corollary 2. *Let M be a geodesic sphere of radius r in $\mathbb{C}P^n$ of holomorphic sectional curvature* 4*. Except geodesics with structure torsion ± 1, every geodesic γ on M satisfies* $\mathrm{length}(\gamma) > 2\pi/\sqrt{4 + \cot^2 r}$*.*

Length spectrum is of course not necessarily simple. For example when M is a geodesic sphere of radius $\pi/4$ in $\mathbb{C}P^n$, we have

$$\begin{aligned}\mathrm{Lspec}(M) = \Big\{&\pi, \sqrt{2}\pi, \sqrt{10}\pi, 2\sqrt{5}\pi, \sqrt{26}\pi, \sqrt{34}\pi, \sqrt{50}\pi, 2\sqrt{13}\pi,\\ &\sqrt{58}\pi, 2\sqrt{17}\pi, \sqrt{74}\pi, \sqrt{82}\pi, 10\pi, \sqrt{106}\pi, 2\sqrt{29}\pi, \sqrt{130}\pi, \dots\Big\}\end{aligned}$$

and the multiplicity of $\sqrt{130}\pi$ is two; it is the length of geodesics of structure torsions $3/\sqrt{65}$ and $7/\sqrt{65}$. Every spectrum which is smaller than $\sqrt{130}\pi$ is simple. Our aim here is to establish the following:

Theorem 9. *Let M be a geodesic sphere of radius r $(0 < r < \pi/2)$ in $\mathbb{C}P^n$ of holomorphic sectional curvature* 4*.*

(1) *If $\tan^2 r$ is irrational, every length spectrum of M is simple.*
(2) *If $\tan^2 r$ is rational, the multiplicity of each length spectrum of M is finite. But it is not uniformly bounded*; $\limsup_{\lambda\to\infty} m_M(\lambda) = \infty$. *In this case, the growth order of m_M is not so rapid. It satisfies $\lim_{\lambda\to\infty} \lambda^{-\delta} m_M(\lambda) = 0$ for arbitrary positive δ.*

This theorem guarantees that on a geodesic sphere of radius r with irrational $\tan^2 r$ in a complex projective space, two closed geodesics are congruent if and only if they have the same length. On the other hand, if $\tan^2 r$ is rational, this theorem shows that we can not classify congruency classes of geodesics only by their length.

Now we make mention of the growth of the number of congruency classes of geodesics with respect to their length spectrum for a geodesic sphere in a complex projective space. For a Riemannian manifold N we denote by $n_N(\lambda)$ the cardinality of the set $\{[\gamma] \in \mathrm{Geod}(N) \mid \mathcal{L}_N([\gamma]) \leqq \lambda\}$.

Theorem 10. *For a geodesic sphere M of radius r $(0 < r < \pi/2)$ in $\mathbb{C}P^n$ of holomorphic sectional curvature 4 we have*

$$\lim_{\lambda\to\infty} \frac{n_M(\lambda)}{\lambda^2} = \frac{3r}{\pi^4 \sin 2r}.$$

4. Characterizations of space forms.

Our discussion is based on the expansion for the second fundamental form of geodesic spheres due to Chen and Vanheche ([8]). For the sake of reader's convenience, we here place their result.

Let M be a Riemannian manifold of dimension greater than 2 with Riemannian metric $\langle\ ,\ \rangle$. We denote by $G_m(r)$ a geodesic sphere with center m and radius r in M, and by $A_{m,r}$ the shape operator of $G_m(r)$ in M with respect to the outward unit normal vector field. The following is a key lemma in this section.

Lemma 2 (Theorem 3.1 in [8])**.** *For non-zero tangent vectors $v, w \in T_mM$ at a point $m \in M$, we choose a unit tangent vector $u \in T_mM$ orthogonal to both v and w. We respectively denote by $v_r, w_r \in T_{\exp_m(ru)}M$ the parallel displacements of v and w along the geodesic segment $\exp_m(su)$, $0 \leqq s \leqq r$. Then for sufficiently small r we have*

$$\langle A_{m,r}v_r, w_r\rangle = \frac{1}{r}\langle v, w\rangle + \frac{r}{3}\langle R(u,v)w, u\rangle + O(r^2),$$

where R is the curvature tensor of M.

In this section we characterize real space forms in terms of the extrinsic shape of geodesics on geodesic spheres. It is well-known that for a real space form, every geodesics on each geodesic sphere is a circle of positive curvature. We know the converse holds.

Theorem 11. *Let M be a Riemannian manifold with $\dim M \geqq 3$. Then the following conditions are equivalent.*

(1) *M is a real space form.*

(2) *At any point* $m \in M$*, every geodesic on a geodesic sphere* $G_m(r)$ *of* M *is a circle of positive curvature in* M *for each sufficiently small* r*.*
(3) *At any point* $m \in M$*, every geodesic sphere* $G_m(r)$ *of* M *is totally umbilic in* M *for each sufficiently small* r*.*

The following is an improvement of Theorem 11. The same discussion as in [14], together with Lemma 2 and Theorem 11, yields

Theorem 11'. *A Riemannian manifold* M *with* $\dim M = n \geqq 3$ *is a real space form if and only if at any point* $m \in M$ *for each sufficiently small geodesic sphere* $G_m(r)$ *of* M*, there exist orthonormal vectors* $v_1, v_2, \dots, v_{n-1}$ *at each point* p *of* $G_m(r)$ *such that all geodesics of* $G_m(r)$ *through* p *in the direction* $v_i + v_j$ $(1 \leqq i \leqq j \leqq n-1)$ *are circles of positive curvature in the ambient manifold* M*.*

Finally we study geodesic spheres in a Kähler manifold M with complex structure J. Let N be the outward unit normal vector field on $G_m(r)$. Since $G_m(r)$ is a real hypersurface in M, it admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced from the Kähler structure J of M.

Paying attention on the characteristic vector field ξ of sufficiently small geodesic spheres $G_m(r)$, we obtain the following characterization of complex space forms in the class of Kähler manifolds.

Theorem 12. *Let* M *be a complex* $n(\geqq 2)$*-dimensional Kähler manifold. Then the following conditions are equivalent each other.*

(1) M *is a complex space form.*
(2) *At any point* $m \in M$*, each sufficiently small geodesic sphere* $G_m(r)$ *of* M *is a Hopf hypersurface of* M*, that is, the characteristic vector* ξ *of* $G_m(r)$ *is a principal curvature vector in* M *at each point* $p \in G_m(r)$*.*
(3) *At any point* $m \in M$*, for each sufficiently small geodesic sphere* $G_m(r)$*, every integral curve of the vector field* ξ *is a geodesic on* $G_m(r)$*.*
(4) *At any point* $m \in M$*, for each sufficiently small geodesic sphere* $G_m(r)$*, the geodesic on* $G_m(r)$ *through* p *in the direction of the vector* ξ *is a circle of positive curvature in* M *at every point* $p \in G_m(r)$*.*

Next we give attention to the extrinsic shape of geodesics on $G_m(r)$ orthogonal to the characteristic vector ξ. The following is a complex version for Theorem 11 in some sense.

Theorem 13. *A complex* $n(\geqq 2)$*-dimensional Kähler manifold* M *is a complex space form if and only if at an arbitrary point* $m \in M$*, for any geodesic sphere* $G_m(r)$ *of sufficiently small radius* r*, every geodesic through any fixed point* p *of* $G_m(r)$*, which is orthogonal to the vector* ξ *at the point* p*, is a circle of positive curvature in the ambient manifold* M*.*

As an immediate consequence of our argument we find the following result which corresponds to Theorem 11'.

Theorem 13'. *A complex $n(\geqq 2)$-dimensional Kähler manifold M is a complex space form if and only if at an arbitrary point $m \in M$, for any geodesic sphere $G_m(r)$ of sufficiently small radius r, there exist orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ orthogonal to the characteristic vector ξ at each point p of $G_m(r)$ such that all geodesics on $G_m(r)$ through p in the direction $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-2)$ are circles of positive curvature in the ambient manifold M.*

References

[1] T. Adachi, M. Kimura and S. Maeda, *A characterization of all homogeneous real hypersurfaces in a complex projective space by observing the extrinsic shape of geodesics*, to appear in Archiv der Math..

[2] T. Adachi and S. Maeda, *Length spectrum of circles in a complex projective space*, Osaka J. Math. **35** (1998), 553–565.

[3] T. Adachi, S. Maeda and S. Udagawa, *Circles in a complex projective space*, Osaka J. Math. **32** (1995), 709–719.

[4] T. Adachi, S. Maeda and M. Yamagishi, *Length spectrum of geodesic spheres in a non-flat complex space form*, preprint.

[5] T. Adachi and S. Maeda, *Space forms from the viewpoint of their geodesic spheres*, preprint.

[6] T. Cecil and P. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[7] B.-y. Chen and S. Maeda, *Extrinsic characterizations of circles in a complex projective space imbedded in a Euclidean space*, Tokyo J. Math. **19** (1996), 169–185.

[8] B.-y. Chen and L. Vanhecke, *Differential geometry of geodesic spheres*, J. Reine Angew. Math. **325** (1981), 28-67.

[9] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–145.

[10] M. Kimura and S. Maeda, *On real hypersurfaces of a complex projective space*, Math. Z. **202** (1989), 299–311.

[11] S. Maeda, *Real hypersurfaces of complex projective spaces*, Math. Ann. **263** (1983), 473–478.

[12] S. Maeda, *Ricci tensors of real hypersurfaces in a complex projective space*, Proc. Amer. Math. Soc. **122** (1994), 1229–1235.

[13] S. Maeda and T. Adachi, *Holomorphic helices in a complex space form*, Proc. Amer. Math. Soc. **125** (1997), 1197–1202.

[14] S. Maeda and K. Ogiue, *Characterizations of geodesic hyperspheres in a complex projective space by observing the extrinsic shape of geodesics*, Math. Z. **225** (1997), 537–542.

[15] S. Maeda and Y. Ohnita, *Helical geodesic immersions into complex space forms*, Geom. Dedicata **30** (1989), 93–114.

[16] M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.

[17] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.

[18] A. Weinstein, *Distance spheres in complex projective spaces*, Proc. Amer. Math. Soc. **39** (1973), 649–650.

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