Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
33 (2000), pp.21–30

PSEUDOCONSISTENT LOGIC AND TENSE LOGIC

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(Received: December 24, 1999)

Dedicated to Professor Maretsugu Yamasaki for his 60th birthday

1. Introduction

In our usual logic, we do not infer arbitrary proposition from a contradictory one. Also in executing programs, there is a state that a proposition A holds in some program and in another there is a state in which A does not hold. To explain these situations, recently, the logic called *paraconsistent* is proposed and investigated. ([1, 2, 3] etc.) Since the logic has two kinds of *negation* operators, there are cases such that both A and *not* A are theorems and hence it is difficult to obtain the concept of *truth*. To the contrary, De Glas has proposed in [4] a *pseudoconsistent logic (PCL)* in which $A \wedge \sim A \to \bot$ is not a theorem but so $\sim (A \wedge \sim A)$ is. He also gave the axiomatization of PCL and proved the completeness theorem by two kinds of models, *PC-models* and *I-models*. These models are based on *PC-algebras* and *partially ordered sets*, respectively.

But there is an important question which is not referred : Is the logic PCL decidable ?

In the present paper we prove the decidabily of PCL according to the following steps:

1. PCL is characterized by the the class of *pre-ordered* sets instead of that of partially ordered sets, that is $\vdash_{PCL} A \iff A$: *PO-valid*;
2. TL is characterized by the class of some kinds of Kripke-type models, that is, $\vdash_{TL} A \iff A$: *TL-valid*;
3. PCL can be embedded into a certain tense logic (TL), that is, for some map ξ, A : PO-valid $\iff \xi(A)$: *TL-valid*;
4. TL is decidable and hence so PCL is.

2. Pseudoconsistent logic and its semantics

First of all we define *(propositional) pseudoconsistent logic* according to De Glas [4]. In the following we simply write PCL. The logic has the language as follows:

1991 *Mathematics Subject Classification.* 03B50.

Key words and phrases. pseudoconsistent logic, tense logic.

- $p_0, p_1, p_2, \cdots$: denumerable propositional variables
- $\perp$: constant
- $\sim, \wedge, \vee, \rightarrow$: logical symbols

We put $\Pi_0 = \{p_0, p_1, p_2, \cdots\} \cup \{\perp\}$ and denote by Π the set of all formulas and by $A, B, C, \cdots$ formulas.

PCL is the following axiomatic system:

Axioms

1. $\perp \rightarrow A$
2. $A \rightarrow (B \rightarrow A)$
3. $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$
4. $A \wedge B \rightarrow A \qquad B \wedge A \rightarrow A$
5. $A \rightarrow (B \rightarrow (A \wedge B))$
6. $A \rightarrow A \vee B \qquad A \rightarrow B \vee A$
7. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
8. $(A \rightarrow B) \rightarrow (\sim A \vee B)$
9. $A \vee \sim A$
10. $\sim\sim A \rightarrow A$
11. $\sim (A \wedge \sim A)$

Rules of inference

1. B is inferred from A and $A \rightarrow B$
2. $\sim A \rightarrow B$ is inferred from $A \rightarrow B$

If A is a provable in PCL, then we write $\vdash_{PCL} A$. It is proved in [4] that

Proposition 1. 1. $\vdash_{PCL} A \wedge (A \rightarrow B) \rightarrow B$

2. $\vdash_{PCL} (\sim A \rightarrow B) \rightarrow (A \vee B)$
3. *if* $\vdash_{PCL} A \rightarrow B$ *and* $\vdash_{PCL} A \rightarrow (B \rightarrow C)$ *then* $\vdash_{PCL} A \rightarrow C$
4. *if* $\vdash_{PCL} A \rightarrow B$ *and* $\vdash_{PCL} B \rightarrow C$ *then* $\vdash_{PCL} A \rightarrow C$
5. *if* $\vdash_{PCL} \sim A \rightarrow B$ *then* $\vdash_{PCL} \sim B \rightarrow A$
6. *if* $\vdash_{PCL} A \rightarrow B$ *then* $\vdash_{PCL} \sim B \rightarrow \sim A$
7. *if* $\vdash_{PCL} A \rightarrow B$ *then* $\vdash_{PCL} \sim A \vee B$
8. *if* $\vdash_{PCL} A \rightarrow B$ *and* $\vdash_{PCL} A \rightarrow \sim B$ *then* $\vdash_{PCL} \sim A$
9. *if* $\vdash_{PCL} A$ *then* $\vdash_{PCL} \sim A \rightarrow B$ *for any* B
10. *if* $\vdash_{PCL} A$ *and* $\vdash_{PCL} \sim B$ *then* $\vdash_{PCL} B$ *for any* B
11. *if* $\vdash_{PCL} A$ *then* $\vdash_{PCL} \sim\sim A$.

In order to develop the algebraic semantics for the logic PCL we define a PC-algebra.

We call a structure $(L, \wedge, \vee, \rightarrow, \neg, 0, 1)$ a PC-algebra when it satisfies the conditions:

1. $(L, \leq, \wedge, \vee)$ is a complete distributive lattice
2. $x \rightarrow y = \bigvee\{z : x \wedge z \leq y\}$
3. $\neg x = \bigwedge\{y : x \vee y = 1\}$

It is clear that any Boolean algebra is a PC-algebra. We note that if we define $\neg x$ as the element $\bigvee\{z : x \wedge z = 0\}$ then the algebra is called a *Heyting algebra.*

So the difference of PC-algebra and Heyting one is only the definition of $\neg x$. We have the following results of PC-algebra in [4].

Proposition 2. 1. $x \leq y \iff x \to y$
2. $x \wedge \neg x \geq 0$
3. $\neg(x \wedge y) = \neg x \vee \neg y$
4. $\neg(x \vee y) = \neg x \wedge \neg y$
5. $\neg\neg x \leq x$
6. $\neg x \leq y \iff x \vee y = 1$
7. $x = 1 \iff \neg\neg x = 1$
8. *if* $\neg x = 0$ *then* $x = 1$
9. *if* $\neg x = 1$ *then* $x \geq 0$
10. $\neg y \to x = 1 \implies \neg x \to y = 1$
11. $\neg y \to \neg x = 1 \implies x \to y = 1$
12. $x \to 1 = 1$
13. $0 \to x = 1$
14. $x \to (y \to x) = 1$
15. $x \to (y \to x \wedge y) = 1$
16. $x \to x \vee y = 1$

We have a well-known example of the PC-algebra. Let X be a topological space and Ω be the set of closed subspaces. For any element $A, B \in \Omega$ if we define

1. $A \wedge B = A \cap B$
2. $A \vee B = A \cup B$
3. $A \to B = \bigcup\{C \in \Omega : A \cap C \subseteq B\}$
4. $\neg A =$ closure of $(X - A)$

then $(\Omega, \wedge, \vee, \to, \neg, \phi, X)$ is the PC-algebra. Moreover it is proved in [4] that any PC-algebra $\mathcal{A}$ can be embedded into a topological PC-algebra Ω.

We define a global semantics. A mapping $v : \Pi_0 \to \mathcal{A}$ is called a *valuation* on a PC-algebra $\mathcal{A}$. The valuation v can be extended to the set Π of formulas recursively:

- $v(\perp) = 0$
- $v(\sim A) = \neg v(A)$
- $v(A \wedge B) = v(A) \wedge v(B)$
- $v(A \vee B) = v(A) \vee v(B)$
- $v(A \to B) = v(A) \to v(B)$

A formula A is called an $\mathcal{A}$-valid if $v(A) = 1$ for every valuation $v : \Pi \to \mathcal{A}$. We say that A is *PC-valid* when it is $\mathcal{A}$-valid for any PC-algebra $\mathcal{A}$. It is proved in [4] the completeness theorem of PCL.

Theorem 1. $\vdash_{PCL} A$ *if and only if* A *is* PC*-valid.*

Another semantics called local semantics is given in [4]. This is a Kripke-like style and is based on the partially ordered sets (poset). Let $(I, \leq)$ be a poset. A subset $S \subseteq I$ is said to be *anti-hereditary* if whenever $i \in S$ and $j \leq i$ then $j \in S$.

We denote by I^* the set of all anti-hereditary subsets of I. A *PCL-model* on I is a structure $\mathcal{I} = (I, \leq, \varphi)$, where $\varphi : \Pi_0 \to I^*$ is a mapping which is called an I-valuation in [4]. Of course the map φ can be extended to one with the domain Π the set of all formulas. By $\mathcal{I} \models_i A$ or simply $\models_i A$, we mean that a formula A is true in the model $\mathcal{I}$ at a state $i \in I$, which is defined as follows:

- $\models_i p$ iff $i \in \varphi(p)$, where $p \in \Pi_0$;
- $\not\models_i \perp$, that is, it is not the case $\models_i \perp$;
- $\models_i \sim A$ iff there exists j such that $i \leq j$ and not $\models_j A$;
- $\models_i A \wedge B$ iff $\models_i A$ and $\models_i B$;
- $\models_i A \vee B$ iff $\models_i A$ or $\models_i B$;
- $\models_i A \to B$ iff for any $j \leq i$ if $\models_j A$ then $\models_j B$.

A formula A is called true in $\mathcal{I}$, denoted $\mathcal{I} \models A$, if $\mathcal{I} \models_i A$ for every $i \in I$. Also A is said to be $\mathcal{I}$-valid, denoted $\mathcal{I} \models A$, when A is true in the model $\mathcal{I} = (I, \leq)$. For the local semantics it is proved in [4]

Proposition 3. *$\vdash_{PCL} A$ iff A is $\mathcal{I}$-valid for any poset $\mathcal{I}$.*

We note that in the proof of the above we do not use the condition of asymmetricity of the relation. This implies that it can be proved for the class of pre-ordered sets. That is,

Theorem 2. *$\vdash_{PCL} A$ iff A is $\mathcal{I}$-valid for any* ***pre-ordered*** *set $\mathcal{I}$.*

3. Tense logic TL and its semantics

In this section we define a certain tense logic (TL) and show the completeness theorem by the class of Kripke-type models. The logic has the following language :

- $p_0, p_1, p_2, \cdots$: denumerable propositional variables
- H, G : unary tense operators
- $\neg, \wedge, \vee, \to$: logical symbols

TL-formulas are defined as usual, especially, we denote $\neg G \neg A$ and $\neg H \neg A$ by FA and PA, respectively. We put Φ_0 the set of all propositional variables, that is, $\Phi_0 = \{p_0, p_1, p_2, \cdots\}$, Φ the set of all TL-formulas, and $A, B, C, \cdots$ the TL-formulas. TL is the axiomatic system which is defined as the smallest tense logic K_t with the axioms expressing reflexivity and transitivity, that is,

Axioms

1. *every tautology*
2. $G(A \to B) \to (GA \to GB)$
3. $H(A \to B) \to (HA \to HB)$
4. $A \to HFA, \;\; A \to GPA$
5. $GA \to A, \;\; HA \to A$
6. $GA \to GGA, \;\; HA \to HHA$

As is well-known in the theory of modal (or tense) logic, axiom 4 expresses the reflexivity and axiom 5 transitivity.

Rules of inference

1. B inferred from A and $A \to B$
2. GA is inferred from A
3. HA is inferred from A

Now we define a semantics called a *Kripke-model*. Let $< W, R >$ be a structure where (i) W is a non-empty set (ii) R is a reflexive and transitive relation on W, that is, W is the pre-ordered set with the relation R. We write xRy instead of $(x, y) \in R$. Hence the condition (ii) is expressed by

- xRx for every $x \in W$
- $xRy, yRz \Longrightarrow xRz$ for any $x, y, z \in W$

A function $V : \Phi_0 \to 2^W$ is also called a *valuation* on the strucutre $< W, R >$. The valuation V is extended to V^* which domain is the set Φ of all the TL-formulas as follows:

1. $V^*(p) = V(p)$ if $p \in \Phi_0$
2. $V^*(\neg A) = \{x | x \notin V^*(A)\}$
3. $V^*(A \wedge B) = V^*(A) \cap V^*(B)$
4. $V^*(A \vee B) = V^*(A) \cup V^*(B)$
5. $V^*(A \to B) = \{x | x \in V^*(A) \Longrightarrow V^*(B)\}$
6. $V^*(GA) = \{x | \forall y(xRy \Longrightarrow y \in V^*(A))\}$
7. $V^*(HA) = \{x | \forall y(yRx \Longrightarrow y \in V^*(A))\}$

For the sake of simplicity we use the same symbol V as the extended valuation. A formula $A \in \Phi$ is said to be true at x in a model $M =< W, R, V >$ when $x \in V(A)$ and denoted by $M \models_x A$. We say that a formula A is TL-valid if A is true at each element $x \in W$ in every model $M =< W, R, V >$, that is, $M \models_x A$ for every model $M =< W, R, V >$ and $x \in W$. For the semantics we can show the soundness theorem of the logic TL.

Theorem 3. $\vdash_{TL} A \implies$ *A is TL-valid.*

Proof. By induction on the length of the proof. □

In order to prove the converse (Completeness Theorem), we define a special model called the *canonical model*. For any set of formulas $\Gamma \subseteq \Phi$, we say that Γ is inconsistent if there exist some formulas $A_i \in \Gamma$ such that $\vdash_{TL} \neg(A_1 \wedge \cdots \wedge A_n)$ and that *consistent* otherwise. We can show that every consistent set of formulas has a maximal one, that is,

Proposition 4. *If Γ is a consistent set, then there exists a maximal consistent set Γ^* containing Γ.*

Proof. We define a sequence $\{ \Gamma_n \}$ of subsets of formulas as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{A_n\} & \text{if } \Gamma_n \cup \{A_n\} \text{ is consistent} \\ \Gamma_n \cup \{\neg A_n\} & \text{otherwise} \end{cases}$$

It easy to show that each Γ_n is consistent and so $\Gamma^* = \bigcup_n \Gamma_n$ is. It is also clear that Γ^* is the maximal consistent set contaning Γ. □

Concerning to any maximal consistent set we have the results.

Proposition 5. *For every maximal consistent set* Δ*, we have*

1. $\vdash_{TL} A \implies A \in \Delta$
2. $A \notin \Delta \iff \neg A \in \Delta$
3. $A \wedge B \in \Delta \iff A \in \Delta$ *and* $B \in \Delta$
4. $A \vee B \in \Delta \iff A \in \Delta$ *or* $B \in \Delta$
5. $A \to B \in \Delta \iff$ *if* $A \in \Delta$ *then* $B \in \Delta$.

Proof. We only show the case of (1). Suppose that $\vdash_{TL} A$ and $A \notin \Delta$. Since $\Delta \cup \{A\}$ is inconsistent by maximality of Δ, there exist some formulas $B_i \in \Delta$ such that $\vdash_{TL} \neg(B_1 \wedge \cdots B_n \wedge A)$. This yields to $\vdash_{TL} A \to \neg(B_1 \wedge \cdots \wedge B_n)$. It follows from $\vdash_{TL} A$ that $\vdash_{TL} \neg(B_1 \wedge \cdots \wedge B_n)$ and hence that Δ is inconsistent. But this is a contradiction. Hence if $\vdash_{TL} A$ then $A \in \Delta$. □

For any maximal consistent sets x, y, we have

Lemma 1. *For every formula* $A \in \Psi$*, the two conditions*

1. *If* $GA \in x$ *then* $A \in y$.
2. *If* $HA \in y$ *then* $A \in x$.

are equivalent to each other.

Proof. Suppose that (1). If $HA \in y$ but $A \notin x$, since x is the maximal consistent set, then $\neg A \in x$. From $\vdash_{TL} \neg A \to GP\neg A$, we have $GP\neg A \in x$ and hence $P\neg A \in y$ by (1). This means that $\neg HA \in y$. This contradicts to the fact y being consistent.

The converse can be proved similarly. □

Now we are ready to define a *canonical* model. Let W_{TL} be the set of all maximal consistent sets of TL. For every $x, y \in W_{TL}$, we define a relation R_{TL} on W_{TL} as

$$xR_{TL}y \iff \forall A \in \Phi\ (GA \in x \Longrightarrow A \in y).$$

Lastly, a valuation V_{TL} is defined by $V_{TL}(p) = \{x \in W_{TL} | p \in x\}$. In this case we call $< W_{TL}, R_{TL}, V_{TL} >$ the canonical model of TL. The model plays an important role in the theory of modal or tense logics. We note from the above

$$xR_{TL}y \text{ if and only if } \forall A \in \Phi\ (HA \in y \Longrightarrow A \in x).$$

Lemma 2. $< W_{TL}, R_{TL}, V_{TL} >$ *is indeed our model, that is,* R_{TL} *is reflective and transitive.*

Proof. First of all we shall prove that R_{TL} is reflexive. Suppose that $GA \in x$ for any $x \in W_{TL}$. Since $\vdash_{TL} GA \to A$, we have $GA \to A \in x$ and thus $A \in x$. This implies that $xR_{TL}x$. Secondly, suppose that $xR_{TL}y$, $yR_{TL}z$, and $GA \in x$. Since $\vdash_{TL} GA \to GGA$, we have $GGA \in x$. It follows from $xR_{TL}y$ that $GA \in y$. We also obtain $A \in z$ from $yR_{TL}z$. This means that R_{TL} is transitive. Thus $< W_{TL}, R_{TL}, V_{TL} >$ is certainly our considered model. □

As to the canonical model we have the fundamental theorem to be able to show the completeness.

Theorem 4. *For every formula $A \in \Phi$ and maximal consistent set $x \in W_{TL}$,*

$$x \in V_{TL}(A) \iff M_{TL} \models_x A.$$

Proof. We shall prove the theorem by induction on the construction of a formula.

If A is a propositional variable $p \in \Phi_0$, then it is evident from definition that $x \in V_{TL}(p)$ iff $p \in x$.

If A is a formula of the form $\neg B$, then we have $x \in V_{TL}(\neg B)$ iff $x \notin V_{TL}(B)$ iff $M_{TL} \not\models_x B$ by induction hypothesis (IH) iff $M_{TL} \models \neg B$.

It is similar to the cases of $B \wedge C$, $B \vee C$, and of $B \to C$.

Let A be a formula of the form GB. Suppose that $x \in V_{TL}(GB)$ and $xR_{TL}y$ for every $y \in \Phi$. By definition of V_{TL} we have $y \in V_{TL}$. It follows from IH that $M_{TL} \models_y B$ for every y such that $xR_{TL}y$. Thus $M_{TL} \models_x GB$ by definition.

Conversely, we assume that $M_{TL} \models_x GB$. If $x \notin V_{TL}(GB)$, then there exists $y \in \Phi$ such that $xR_{TL}y$ and $y \notin V_{TL}(B)$ by definition of V_{TL}. It follows from IH that $M_{TL} \not\models_y B$ for some y such that $xR_{TL}y$. This implies $M_{TL} \not\models_x GB$ and hence a contradiction. Thus we have $x \in V_{TL}(GB)$ iff $M_{TL} \models_x GB$.

We can also prove the theorem in the case of HB similarly. □

From the above we can show the completeness theorem of TL.

Theorem 5. *If A is TL-valid then $\vdash_{TL} A$. Hence*

$$\vdash_{TL} A \iff A : TL\text{-valid}.$$

Proof. Suppose that $\not\vdash_{TL} A$. Since $\{\neg A\}$ is a consistent set, there exists a maximal consistent set $x \in W_{TL}$ such that $\{\neg A\} \subseteq x$. It follows from the above that $M_{TL} \not\models_x A$. This means that A is not TL-valid. □

4. Decidability of TL

In this section we shall prove the decidability of the logic TL using the *filtration* method. The method is familiar to the theory of modal logics. Let $\Psi \subseteq \Phi$ be a subset of TL-formulas which is closed under subformulas, that is, if $A \in \Psi$ and B is a subformula of A then $B \in \Psi$. For every TL-model $M =< W, R, V >$, we define a relation $\equiv$ on W as follows: For any $x, y \in W$,

$$x \equiv y \iff \forall A \in \Psi\ (M \models_x A \text{ iff } M \models_y A).$$

It is easy to show that the relation is the equivalence one, so we omit the proof.

Proposition 6. $\equiv$ *is the equivalence relation.*

We define a filtration model $M^* =< W^*, R^*, V^* >$ of $M =< W, R, V >$ through Ψ. We put $W^* = \{[x] | x \in W\}$ and $[x] = \{y \in W \mid x \equiv y\}$. For each $[x], [y] \in W^*$, a binary relation R^* is define as

$$[x]R^*[y] \iff \forall GA \in \Psi(M \models_x GA \Longrightarrow M \models_y GA \wedge A) \text{ and } \forall HB \in \Psi(M \models_y HB \Longrightarrow M \models_x HB \wedge B).$$

The valuation V^* is also defined by

$$V^*(p) = \begin{cases} \{[x] \mid M \models_x p\} & \text{if } p \in \Psi_0 \\ W^* & \text{otherwise} \end{cases}$$

The following is proved from the definition of R^*.

Lemma 3. *If xRy in M, then $[x]R^*[y]$ in M^*.*

Proof. Suppose xRy. For every formula of the form $GA \in \Psi$, if $M \models_x GA$, then $M \models_y A$ by xRy. On the other hand, since $M \models_x GA \to GGA$, we have $M \models_x GGA$ and hence $M \models_y GA$. This implies that $M \models_y GA \wedge A$. In case of $HB \in \Psi$, we assume $M \models_y HB$. It follows from xRy that $M \models_x B$. Since $M \models_y HB \to HHB$, we have $M \models_y HHB$, $M \models_x HB$ and thus $M \models_x HB \wedge B$. Hence these mean that $[x]R^*[y]$. □

According to the definition, we can show that the filtration model is indeed a TL-model.

Lemma 4. *$M^* =< W^*, R^*, V^* >$ is the TL-model.*

Proof. We only show that R^* is a reflexive and transitive relation on W^*. Firstly suppose that $[x] \in W^*$. For every formula $GA \in \Psi$, if $M \models_x GA$, since $M \models_x GA \to A$, we have $M \models_x A$ and hence $M \models_x GA \wedge A$. It is similar for any formula of the form $HB \in \Psi$. This implies that $[x]R^*[x]$ and that R^* is reflexive.

Secondly, suppose that $[x]R^*[y]$ and $[y]R^*[z]$. For any formula $GA \in \Psi$, if $M \models_x GA$, since $M \models_y GA \wedge A$ by supposition, then we have $M \models_y GA$. It follows that $M \models_z GA \wedge A$. This means that R^* is transitive. □

For the filtration model M^* of M through Ψ, we establish the fundamental theorem.

Theorem 6. *For every $A \in \Psi$ and $x \in W$,*

$M^* \models_{[x]} A \iff M \models_x A$

Proof. By induction on A.

- If $p \in \Psi_0$ then $M^* \models_{[x]} p$ iff $p \in V^*([x])$ iff $M \models_x p$.
- For the formula of the form $B \wedge C \in \Psi$, since $B, C \in \Psi$, we have $M^* \models_{[x]} B \wedge C$ iff $M^* \models_{[x]} B$ and $M^* \models_{[x]} C$ iff $M \models_x B$ and $M \models_x C$ iff $M \models_x B \wedge C$.
- For the case of $GB \in \Psi$, suppose that $M \not\models_x GB$. There exists an element $y \in W$ such that xRy and $M \not\models_y B$. Since $[x]R^*[y]$ and $M^* \not\models_{[y]} B$ by induction hypothesis (IH), we have $M^* \not\models_{[x]} GB$. Conversely, if $M^* \not\models_{[x]} GB$ then there exists $[y] \in W^*$ such that $[x]R^*[y]$ and $M^* \not\models_{[y]} B$. It follows from IH that $M \not\models_y B$ and hence $M \not\models_y GB \wedge B$. Since $[x]R^*[y]$, this means that $M \not\models_x GB$.
- The other cases are proved similarly. □

Now we show the decidability of TL. Let $\not\vdash_{TL} A$ and Ψ be the set of subformulas of A. It is clear that Ψ is finite and closed under subformulas. By completeness theorem of TL, there exists a TL-model $M =< W, R, V >$ and $x \in W$ such that $M \not\models_x A$. By the above we can construct the filtration model M^* of M through Ψ. Since Ψ is the finite set, M^* is also a finite TL-model. For that model we have $M^* \not\models_{[x]} A$. This means that if A is not a TL-theorem then it is not TL-valid in some finite model. Since the logic TL is finitely axiomatized, we can prove

Theorem 7. *TL is decidable.*

5. Embedding of PCL into TL

Let ξ be a map from the set Π of all PCL-formulas to the set Φ of all TL-formulas as follows:

1. $\xi(p) = Hp$, where $p \in \Pi_0 - \{\bot\} = \Phi_0$
2. $\xi(\bot) = p \wedge \neg p$ for some fixed $p \in \Phi_0$
3. $\xi(\sim A) = F\neg\xi(A)$
4. $\xi(A \wedge B) = \xi(A) \wedge \xi(B)$
5. $\xi(A \to B) = H(\xi(A) \to \xi(B))$

We can show that PCL can be embedded into TL in the sense that A is PCL-valid iff $\xi(A)$ is TL-valid for every formula $A \in \Pi$.

Theorem 8. *For every formula $A \in \Pi$,*
A is PCL-valid $\Longleftrightarrow \xi(A)$ is TL-valid.

Proof. If $\xi(A)$ is not TL-valid, then there exists a TL-model $M =< W, R, V >$ such that $M \not\models_x \xi(A)$ for some element $x \in W$. We note that the relation R is reflexive and transitive, that is, $< W, R >$ is a pre-ordered set. We construct a PCL-model $\mathcal{I} =< I, \leq, \varphi >$ as follows:

- $I = W$
- $\leq = R$, that is, $x \leq y$ if and only if xRy
- $\varphi(p) = \{x \in W | \forall y(yRx \Longrightarrow y \in V(p)\}$

We note that $\varphi(p)$ is anti-hereditary. For if $x \in \varphi(p)$ and $y \leq x$, then we have yRx for every z such that yRz. By transitivity, zRx. This yields to $z \in V(p)$. Hence $\varphi(p)$ is anti-hereditary and $\mathcal{I}$ is indeed the PCL-model. For that model we can show that $\mathcal{I} \models_x \alpha$ iff $M \models_x \xi(\alpha)$ for any formula $\alpha in \Pi$. Since $M \not\models_x \xi(A)$, we obtain that $\mathcal{I} \not\models_x A$ for the PCL-model $\mathcal{I}$. Thus A is not PCL-valid unless $\xi(A)$ is TL-valid.

Conversely, assume that A is not PCL-valid. There exists a PCL-model $\mathcal{I} =< I, \leq, \varphi >$ such that $\mathcal{I} \not\models_x A$ for some $x \in I$. From that model we can construct a TL-model $M =< W, R, V >$, where

1. $W = I$
2. $R = \leq$, that is, xRy is defined by $x \leq y$
3. $V(p) = \varphi(p)$.

It is obvious that M is the TL-model. It is also easy to show that $\mathcal{I} \models_u \alpha$ iff $M \models_u \xi(\alpha)$ for every formula $\alpha \in \Pi$ and $u \in W$. Since $\mathcal{I} \not\models_x A$, we have $M \not\models_x \xi(A)$. This means that $\xi(A)$ is not TL-valid.

Therefore we can prove the theorem completely. □

From the above we can obtain the main result of our paper.

Theorem 9. *PCL is decidable.*

Proof.

$$\begin{aligned}
\nvdash_{PCL} A &\iff A : not\ PCL - valid \\
&\iff \xi(A) : not\ TL - valid \\
&\iff \exists M : finte\ TL - model, \exists x; M \not\models_x \xi(A) \\
&\iff \exists \mathcal{I} : finite\ PCL - model, \exists x; \mathcal{I} \not\models_x A
\end{aligned}$$

Since PCL is the finitely axiomatizable logic, it follows from the above that it is decidable. □

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