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ON A LOOP EXPANSION FORMULA FOR ENUMERATION OF EULER SUBGRAPHS OF A PLANE GRAPH

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Abstract. The Vdovichenko formula is a formula for counting Euler subgraphs of a labelled graph which is embedded into a plane. We interpret it as a loop expansion formula. The nature of the loop expansion is explored in detail.

1. Introduction

An Euler graph is a graph of which each vertex has even degree. If an Euler graph P is a subgraph of a graph G we say that P is an Euler subgraph of G.

Let G be a graph embedded into a plane, and consider the generating function for Euler subgraphs of the graph with a given number of edges (see (1) and (2)).

Vdovichenko [1] gave a nice formula for the square lattice graph G on a plane which expresses the generating function in terms of a "transition matrix" using an analogy with a random walk problem on graphs. In a recent paper the author [2] gave a generalization which is applicable for any graph G embedded into a plane. However, it seems that no rigorous proof of the original Vdovichenko formula (and so, the generalized one by the author) has been appeared in the literatures.

This paper presents a preparatory consideration on the formula(s) (hopefully) toward a rigorous proof. We shall clarify the geometric meaning of the Vdovichenko formula (in the generalized form, presented in [2]); the formula is interpreted as a loop expansion where loops are drawn on the graph; see (10). The nature of this loop expansion will be explored in Section 5.

2. Terminology on graphs

To begin with we give precise definitions of Euler subgraphs, a generating function for them, and a plane graph.

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2.1. **A graph and its Euler subgraphs.** Abstractly, a *graph* (or *digraph*) is a triple $G = (V(G), E(G), \phi)$ consisting of a finite set $V(G)$ of *vertices* and a finite set $E(G)$ of *edges*, and a mapping $\phi : E(G) \to V(G) \times V(G)$, $\phi(e) = (I(e), T(e))$. The vertices $I(e)$ and $T(e)$ are called the *initial vertex* and the *terminal vertex* of e, respectively, and they are called the *endpoints* of e. We allow closed edges (an edge joining a vertex to itself) and parallel edges (more than one edge joining two vertices, or one vertex in case of a closed edge).

Let G be a graph. By an *Euler subgraph* of G we shall mean a subgraph P which satisfies $V(P) = V(G)$, $E(P) \subset E(G)$, and

$$\deg_P(\alpha) \text{ is even for every } \alpha \in V(P)$$

where the *degree* of a vertex $\alpha \in V(P)$ with respect to P is defined by

$$\deg_P(\alpha) = \#\{e \in E(P) \mid I(e) = \alpha\} + \#\{e \in E(P) \mid T(e) = \alpha\}.$$

Note that Euler subgraphs are called *polygons* in the Ising model literatures.

Let $\mathcal{P}_p(G)$ denote the set of all Euler subgraphs with p edges and let $\mathcal{P}(G) = \cup_{p=0}^{\infty}\mathcal{P}_p(G)$ (this set is finite). Let $\mathbf{x} = \{x_e \mid e \in E(G)\}$ be a set of commutative variables (indeterminates). Consider a polynomial in these variables defined by

$$S(\mathbf{x}) = \sum_{P \in \mathcal{P}(G)} \prod_{e \in E(P)} x_e. \tag{1}$$

We shall call $S(\mathbf{x})$ the generating function for Euler subgraphs, for if we substitute $x_e = x$ for all $e \in E(G)$ we obtain a generating function for Euler subgraphs with a given number of edges:

$$[S(\mathbf{x})]_{\text{all } x_e = x} = \sum_{p=0}^{\infty} A_p x^p \tag{2}$$

where A_p is the number of Euler subgraphs with p edges.

2.2. **A plane graph.** Usually a graph G is regarded as a topological space: first we suppose that each vertex is homeomorphic to a point and each edge is homeomorphic to the segment $[0, 1]$ of the real line, and that the initial and the terminal vertices correspond to 0 and 1 in the segment respectively; consider a direct sum space $(\sqcup_{\alpha \in V(G)}\{\alpha\}) \sqcup (\sqcup_{e \in E(G)}\{e\})$ and then identify the point $I(e)$ with α if $I(e) = \alpha$ and $T(e)$ with α if $T(e) = \alpha$, for each $e \in E(G)$ and each $\alpha \in V(G)$; in this way we obtain a topological space G. Note that an orientation is naturally defined on each edge e through the homeomorphism $e \mapsto [0, 1]$.

An *embedding* of a graph G into the (Euclidean) plane is a one-to-one continuous mapping $f : G \to \boldsymbol{R}^2$. Not all graphs are embeddable into the plane; a graph which is embeddable into the plane is called *planar* (or *of genus* 0). An embedding f is said to be *smooth* if a curve $f(e)$ is smooth (except at endpoints) in the plane for each edge e. By a *plane graph* we shall mean a topological space $f(G)$ where $f : G \to \boldsymbol{R}^2$ is a smooth embedding of a graph G.

In what follows, since we shall fix a graph G and a smooth embedding f, we shall identify $f(G)$ with G; thus a plane graph will be sometimes called a graph for short.

3. The Vdovichenko formula

The Vdovichenko formula for $S(\mathbf{x})$, in the generalized sense in [2], is described as follows.

Let G be a plane graph.

Let $\mathcal{I} = \{e \mid e \in E(G)\} \sqcup \{-e \mid e \in E(G)\}$ (disjoint union) be an index set and define operations on $\mathcal{I}$ by

$$-m = -e,\ I(m) = I(e),\ T(m) = T(e),\ \epsilon(m) = e \text{ if } m = e \text{ for some } e \in E(G);$$

$$-m = e,\ I(m) = T(e),\ T(m) = I(e),\ \epsilon(m) = e \text{ if } m = -e \text{ for some } e \in E(G)$$

$(m \in \mathcal{I})$. When $e \in E(G)$ we shall identify $e \in \mathcal{I}$ with a "move" on the edge e in the direction of the orientation assigned to it, and we shall identify $-e \in \mathcal{I}$ with a "move" on the edge in the opposite direction. Also, sometimes $m \in \mathcal{I}$ will be identified with a plane curve travelling from $I(m)$ to $T(m)$.

For a pair (m, m') of members in $\mathcal{I}$ satisfying

$$m \neq -m' \quad \text{and} \quad T(m') = I(m) \tag{3}$$

define

$$W_{mm'} = x_{\epsilon(m')} \cdot \exp\left(\frac{i\theta_{mm'}}{2}\right), \qquad \theta_{mm'} = \varphi_{mm'} + \int_{m'} \kappa ds \tag{4}$$

where $\varphi_{mm'}$ is a counterclockwise angle $(-\pi \leq \varphi_{mm'} \leq \pi)$ a vector tangent to edges turns at the vertex $T(m')$ when moving from m' to m (When $\varphi_{mm'} \equiv \pi \bmod 2\pi$, then $\varphi_{mm'} = \pi$ if m lies in the left side of m', and $\varphi_{mm'} = -\pi$ otherwise), κ the geodesic curvature of the directed edge m' whose arc-length parameter is s.

Let $W_{mm'} = 0$ for pairs (m, m') which do not satisfy (3) and write

$$W = \left(W_{mm'} \,\middle|\, m \in \mathcal{I}, m' \in \mathcal{I} \right).$$

The generating function $S(\mathbf{x})$ can be expressed in terms of this $2\#E(G) \times 2\#E(G)$ matrix.

The Vdovichenko formula.

$$S(\mathbf{x}) = \exp\left(-\sum_{p=1}^{\infty} \frac{1}{2p} \operatorname{trace}\left(W^p\right)\right) \tag{5}$$

in the sense of formal power series in $\mathbf{x}$.

As was mentioned before there exits no rigorous proof; but it has been tested for many examples (see section 3.2 in [2] for example) and it is believed that the formula holds in general.

Since any square matrix can be transformed into an upper-triangle matrix by some similarity transformation, we have

Corollary.

$$S(\mathbf{x}) = \Big[\det \Big(I - W \Big) \Big]^{1/2}. \tag{6}$$

4. A loop expansion formula

Let us rewrite the formula (5) and consider its meaning more closely. To do this we introduce concepts of closed walks and loops on a graph.

A *walk* on G of length p ($p \geq 1$) is an ordered p-tuples $(m_0, \dots, m_{p-1})$ of members of $\mathcal{I}$ which satisfies $T(m_i) = I(m_{i+1})$ for $i = 0, \dots, p-2$. A walk $(m_0, \dots, m_{p-1})$ is *closed* if $T(m_{p-1}) = I(m_0)$. For a closed walk $w = (m_0, \dots, m_{p-1})$ let $\mu(w)$ denote its *winding number* (in the sense of differential geometry; here we regard the closed walk as a piecewise smooth closed curve in the plane. See eq.(9)). We remark that the number '$\mu(w) - 1 \bmod 2$' can be interpreted as the *number of intersection points of* w *modulo 2*. Thus define the sign of w, $\mathrm{sgn}\,(w)$, to be $(-1)^{\mu(w)-1}$.

Let $\mathbf{D}_p$ be the dihedral group of order $2p$ defined by the relations $S^p = R^2 = 1$ and $RSR^{-1} = S^{-1}$:

$$\mathbf{D}_p = \Big\langle\ S, R \ \Big|\ S^p = R^2 = 1, RSR^{-1} = S^{-1}\ \Big\rangle.$$

$\mathbf{D}_p$ consists of $2p$ elements of the form $S^j R^k$, $0 \leq j \leq p-1$, $k = 0, 1$. Define an action of $\mathbf{D}_p$ on closed walks of length p by

$$\begin{aligned} S((m_0, \dots, m_{p-1})) &= (m_1, \dots, m_{p-1}, m_0), \\ R((m_0, \dots, m_{p-1})) &= (-m_{p-1}, \dots, -m_0). \end{aligned}$$

For a closed walk w of length p let

$$q(w) = \min\{ j \mid 1 \leq j \leq p, S^j(w) = w \}, \quad d(w) = p/q(w);$$

the numbers $q(w)$ and $d(w)$ are said to be the *period* and the *degeneracy* of w respectively. In other words, if $\mathbf{C}_p = \langle S \mid S^p = 1 \rangle$ denotes the cyclic group of order p which is a subgroup of $\mathbf{D}_p$ and if $\mathbf{C}_{p,w} = \{ X \in \mathbf{C}_p \mid X(w) = w \}$ denotes the stabilizer of w under the action of $\mathbf{C}_p$, then $d(w) = \#\mathbf{C}_{p,w}$ and $q(w) = \#\mathbf{C}_p / \#\mathbf{C}_{p,w}$. Write

$$\mathbf{x}^w = \prod_{i=1}^{p} x_{\epsilon(m_i)}$$

(the set of variables $\mathbf{x}$ was introduced in (1)). We note that for any closed walk w and any $X \in \mathbf{D}_p$ hold

$$\begin{aligned} \mu(X(w)) &\equiv \mu(w) \mod 2, \\ \operatorname{sgn}(X(w)) &= \operatorname{sgn}(w), \\ d(X(w)) &= d(w), \\ \mathbf{x}^{X(w)} &= \mathbf{x}^w. \end{aligned} \tag{7}$$

A closed walk $(m_0, \ldots, m_{p-1})$ is said to be *allowed* if $m_{i+1} \neq -m_i$, $i = 0, \ldots, p-2$, and $m_0 \neq -m_{p-1}$ (In other words, an allowed closed walk is a closed walk which contains no about-face turns). Let $\mathcal{W}_p(G)$ denote the set of all allowed closed walks of length p. The group $\mathbf{D}_p$ acts on $\mathcal{W}_p(G)$. We note that for $w \in \mathcal{W}_p(G)$ the stabilizer $\mathbf{D}_{p,w} = \{X \in \mathbf{D}_p \mid X(w) = w\}$ coincides with $\mathbf{C}_{p,w}$ [*Proof.* Let $w = (m_0, \ldots, m_{p-1}) \in \mathcal{W}_p(G)$ and suppose that $S^j R(w) = w$ for some j, $0 \leq j \leq p-1$; then we have $-m_{p-j-1-k} = m_k$, $0 \leq k \leq p-j-1$. If $p-j$ is odd we have $-m_i = m_i$ for some i, a contradiction. Otherwise we have $-m_{i+1} = m_i$ for some i; again impossible since w is assumed to be allowed.]; hence $d(w) = \#\mathbf{D}_{p,w}$ and $2q(w) = \#\mathbf{D}_p / \#\mathbf{D}_{p,w}$.

An orbit of the action of $\mathbf{D}_p$ on $\mathcal{W}_p(G)$ is called a *loop* of length p. Let $\mathcal{L}_p(G)$ denote the set of all loops of length p. $\mathcal{L}_p(G) = \mathbf{D}_p \backslash \mathcal{W}_p(G)$. By virtue of (7) the following definitions make sense for $L \in \mathcal{L}_p(G)$:

$$\operatorname{sgn}(L) = \operatorname{sgn}(w), \quad d(L) = d(w), \quad \mathbf{x}^L = \mathbf{x}^w$$

where $w \in L$. Now let us define the *weight* of L to be

$$\omega(L) = \operatorname{sgn}(L) \cdot \frac{1}{d(L)}$$

The following lemma clarifies the meaning of $\operatorname{trace} W^p$ in the Vdovichenko formula.

Lemma 1.

$$\sum_{L \in \mathcal{L}_p(G)} \omega(L)\, \mathbf{x}^L = -\frac{1}{2p} \operatorname{trace}\left(W^p\right). \tag{8}$$

Write $W(w) = W_{m_{p-1},m_{p-2}} \cdot \cdots \cdot W_{m_1,m_0} W_{m_0,m_{p-1}}$ for $w = (m_0, \ldots, m_{p-1}) \in \mathcal{W}_p(G)$. Then

$$\operatorname{trace}\left(W^p\right) = \sum_{w \in \mathcal{W}_p(G)} W(w).$$

For an allowed closed walk $w = (m_0, \ldots, m_{p-1})$ we have an explicit expression for the winding number

$$\theta_{m_{p-1},m_{p-2}} + \cdots + \theta_{m_1,m_0} + \theta_{m_0,m_{p-1}} = 2\pi\mu(w) \tag{9}$$

and hence

$$W(w) = \exp(i\pi\mu(w)) \cdot \mathbf{x}^w = -\operatorname{sgn}(w) \cdot \mathbf{x}^w.$$

Thus $W(X(w)) = W(w)$ for $w \in \mathcal{W}_p(G)$ and $X \in \mathbf{D}_p$ (because of (7)), and therefore $W([w]) = W(w)$ is well-defined (where $[w] \in \mathcal{L}_p(G)$). Since the number of distinct representatives (allowed closed walks) of a loop $L = [w] \in \mathcal{L}_p(G)$ ($w \in \mathcal{W}_p(G)$) is

$$\#L = \frac{\#\mathbf{D}_p}{\#\mathbf{D}_{p,w}} = \frac{2p}{d(w)} = \frac{2p}{d(L)}$$

we have

$$\begin{aligned}\sum_{w\in\mathcal{W}_p(G)} W(w) &= \sum_{L\in\mathcal{L}_p(G)} \frac{2p}{d(L)} \cdot W(L) \\ &= -\sum_{L\in\mathcal{L}_p(G)} \frac{2p}{d(L)} \cdot \operatorname{sgn}(L) \cdot \mathbf{x}^L = -2p \sum_{L\in\mathcal{L}_p(G)} \omega(L)\, \mathbf{x}^L.\end{aligned}$$

Thus (8) is proved. □

Thus we have the following exponential relationship for $S(\mathbf{x})$ and a summation over all loops.

Theorem 1. *Let G be a plane graph and let $S(\mathbf{x})$ be the generating function (1) in $\#E(G)$ variables $\mathbf{x} = \{x_e \mid e \in E(G)\}$. The following is equivalent to the Vdovichenko formula (5):*

$$S(\mathbf{x}) = \exp\left(\sum_{p=1}^{\infty} \sum_{L\in\mathcal{L}_p(G)} \omega(L)\, \mathbf{x}^L\right). \tag{10}$$

Proof. An immediate consequence of Lemma 1. □

5. Some lemmas partly justifying the formula

Although the author has not completed a proof of the formula (10) or (5), there are many pieces of "evidence" each of which would form a part of a complete proof. We describe a few of them.

5.1. **Products of loops.** Let $\mathcal{L}(G) = \cup_{p=1}^{\infty} \mathcal{L}_p(G)$ and refer to it as the set of all loops. For loops $L_1, L_2 \in \mathcal{L}(G)$ we consider a formal commutative product $L_1 L_2$ and define

$$\operatorname{sgn}(L_1 L_2) = \operatorname{sgn}(L_1)\operatorname{sgn}(L_2), \qquad \mathbf{x}^{L_1 L_2} = \mathbf{x}^{L_1}\mathbf{x}^{L_2}.$$

As usual we will write $L^2 = LL$, etc.

Let us introduce the "empty loop" $\emptyset$ which plays a role of unit element: i.e., $\emptyset L = L\emptyset = L$, $L \in \mathcal{L}(G)$. Define $\operatorname{sgn}(\emptyset) = 1$ and $\mathbf{x}^{\emptyset} = 1$.

Let $\mathcal{Z}(G)$ denote the set which consists of all products of loops and the empty loop:

$$\mathcal{Z}(G) = \{\, L_1 \cdots L_k \mid k \geq 0, L_i \in \mathcal{L}(G)(i = 1, \ldots, k) \,\}.$$

Each element $Z \in \mathcal{Z}(G)$ is factorized as

$$Z = L_1^{j_1} \cdots L_k^{j_k}$$

where the factors $L_1, \ldots, L_k$ are distinct loops and $j_1 \geq 1, \ldots, j_k \geq 1$. The factorization is unique apart from the order of the factors.

When

$$Z = L_1^{j_1} \cdots L_k^{j_k}, \quad Z' = L_1^{j'_1} \cdots L_k^{j'_k};$$

$0 \leq j'_1 \leq j_1, \ldots, 0 \leq j'_k \leq j_k$, we shall say that Z' is a divisor of Z. If Z and Z' have no common divisor but the empty loop $\emptyset$ they are said to be relatively prime.

Lemma 2. *Define $U(Z)$, $Z \in \mathcal{Z}(G)$, by*

$$\exp\left(\sum_{L \in \mathcal{L}(G)} \omega(L)\, \mathbf{x}^L\right) = \sum_{Z \in \mathcal{Z}(G)} U(Z)\, \mathbf{x}^Z. \tag{11}$$

The coefficients $U(Z)$ have the following properties:

(a) *$U(\emptyset) = 1$ and $U(L) = \omega(L) = \mathrm{sgn}\,(L)/d(L)$ for $L \in \mathcal{L}(G)$, and in general*

$$\begin{aligned} U(L_1^{j_1} \cdots L_k^{j_k}) &= \frac{1}{j_1! \cdots j_k!}\, \omega(L_1)^{j_1} \cdots \omega(L_k)^{j_k} \\ &= \frac{1}{j_1! \cdots j_k!} \frac{\mathrm{sgn}\,(L_1)^{j_1} \cdots \mathrm{sgn}\,(L_k)^{j_k}}{d(L_1)^{j_1} \cdots d(L_k)^{j_k}} \end{aligned}$$

where $L_1, \ldots, L_k$ are distinct loops and $j_1, \ldots, j_k$ nonnegative integers.

(b) *If Z and Z' in $\mathcal{Z}(G)$ are relatively prime then*

$$U(ZZ') = U(Z)U(Z').$$

(c) *Let $L_1, \ldots, L_k$ be distinct loops and let $j_1 \geq 1, \ldots, j_k \geq 1$. Then*

$$|U(L_1^{j_1} \cdots L_k^{j_k})| = 1$$

if and only if $j_1 = \cdots = j_k = 1$ and $d(L_1) = \cdots = d(L_k) = 1$.

Proof. The definition of $U(Z)$ implies (a). (b) and (c) are corollaries of (a). □

5.2. **Sum of the coefficients corresponding to an Euler subgraph is 1.** Let $\mathbf{x} = \{x_e \mid e \in E(G)\}$. Define a mapping $\mathbf{x} : \mathcal{Z}(G) \to \{\text{monomials}\}$ by $\mathbf{x}(Z) = \mathbf{x}^Z$. It is known that any Euler graph can be partitioned into loops. Thus, for each Euler subgraph $P \in \mathcal{P}(G)$ there exists a product of loops $Z \in \mathcal{Z}(G)$ such that $\prod_{e \in E(P)} x_e = \mathbf{x}^Z$. Let $\mathcal{Z}_P$ denote the set of all such Z; namely, for each $P \in \mathcal{P}(G)$ define

$$\mathcal{Z}_P = \left\{ Z \in \mathcal{Z}(G) \;\middle|\; \prod_{e \in E(P)} x_e = \mathbf{x}^Z \right\}.$$

This is an easy lemma:

Lemma 3. *Let $P \in \mathcal{P}(G)$ be an Euler subgraph. Then,* $\sum_{Z\in\mathcal{Z}_P} U(Z) = 1$.

Proof. Let $1, \dots, k$ be the vertices of P whose degree $\deg_P \geq 4$. Write $2n_j = \deg_P(j)$, $1 \leq j \leq k$. At each vertex j there are $2n_j$ incident edges; the number of ways of joining two in pairs is $(2n_j - 1)!!$, of which $((2n_j - 1)!! + 1)/2$ ways have even transversal intersections (in a neighborhood of the vertex) and remaining $((2n_j - 1)!! - 1)/2$ ways have odd transversal intersections. Thus the number of ways of partitioning the Euler subgraph P into loops is $M_P = (2n_1 - 1)!! \times \cdots \times (2n_k - 1)!!$ [that is, $\#\mathcal{Z}_P = \prod_{j=1}^{k}(2n_j - 1)!!$]; among those partitions $(M_P + 1)/2$ ways correspond to products of loops having even intersections (i.e., $\operatorname{sgn}(Z) = 1$), and remaining $(M_P - 1)/2$ ways to products of loops having odd intersections (i.e., $\operatorname{sgn}(Z) = -1$). Thus $\sum_{Z\in\mathcal{Z}_P} U(Z) = (M_P + 1)/2 - (M_P - 1)/2 = 1$. □

5.3. **Cancellation of terms which do not correspond to Euler subgraphs.** This is the most nontrivial part. The author has not completed justification for the title of this subsection. Only partial results are exhibited here.

For a loop $L \in \mathcal{L}_q(G)$ and a positive integer n, take an allowed closed walk $w = (m_0, \dots, m_{q-1}) \in \mathcal{W}_q(G)$ such that $[w] = L$, and write

$$w^{*n} = (\underbrace{m_0, \dots, m_{q-1}, \dots\dots\dots, m_0, \dots, m_{q-1}}_{\text{pattern } m_0, \dots, m_{q-1} \text{ repeats } n \text{ times}}) \quad \in \mathcal{W}_{nq}(G),$$

and define a loop $L^{*n} \in \mathcal{L}_{nq}(G)$ to be $L^{*n} = [w^{*n}]$. Clearly $L^{*1} = L$, and $\mathbf{x}^{L^{*n}} = (\mathbf{x}^L)^n$. Each loop L has a representation $L = L_1^{*d}$ for some $L_1 \in \mathcal{L}(G)$ where $d = d(L)$.

When $d(L) \geq 2$ the loop L is said to be *degenerate*.

Lemma 4. *Let L be a nondegenerate loop $(d(L) = 1)$. Let $k \geq 1$. Then*

$$\operatorname{sgn}(L^{*k}) = -(-\operatorname{sgn}(L))^k, \quad d(L^{*k}) = k$$

and hence

$$U(L^{*k}) = -(-\operatorname{sgn}(L))^k \, \frac{1}{k}.$$

Proof. Take an allowed closed walk w such that $[w] = L$, and let $\mu = \mu(w)$ be its winding number; thus "the number of intersection points mod 2" of w is $\nu = \mu - 1$. The degenerate walk w^{*k} has the winding number $\mu(w^{*k}) = k\mu$. Thus its "number of intersection points mod 2" is

$$\mu(w^{*k}) - 1 = k\mu - 1 = k\nu + k - 1 \equiv \begin{cases} k - 1 \mod 2 & (\text{if } \nu \text{ is even}), \\ 1 \mod 2 & (\text{if } \nu \text{ is odd}). \end{cases}$$

Since $\operatorname{sgn}(L) = (-1)^{\mu(w^{*k})-1}$ the equality for sign is proved.

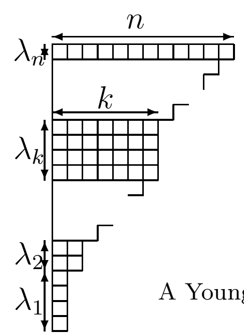
The statements on d and U are trivial. □

The following nice cancellation occurs.

Lemma 5. *Let $Z_1 \in \mathcal{Z}(G)$ and let $L \in \mathcal{L}(G)$. Suppose that L is nondegenerate ($d(L) = 1$) and that L is not a divisor of Z_1. Let $n \geq 2$. Then*

$$\sum_{\lambda \vdash n} U(Z_1 L^{\lambda_1} (L^{*2})^{\lambda_2} \cdots (L^{*n})^{\lambda_n}) = 0.$$

Here "$\lambda \vdash n$" means that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an n-tuple of nonnegative integers such that $\sum_{k=1}^{n} k\lambda_k = n$ (see Figure). The summation is taken over all such λ.



A Young diagram $\lambda = (\lambda_1, \ldots, \lambda_n)$

Proof. Because of Lemma 2 (b) it is sufficient to prove the statement for $Z_1 = \emptyset$.

Let $\lambda \vdash n$ and put $Z_\lambda = \prod_{k=1}^{n} (L^{*k})^{\lambda_k}$. Using lemmas prepared above and $\sum_{k=1}^{n} k\lambda_k = n$ we have

$$\begin{aligned} U(Z_\lambda) &= U\Big(\prod_{k=1}^{n} (L^{*k})^{\lambda_k}\Big) = \prod_{k=1}^{n} U((L^{*k})^{\lambda_k}) \\ &= \prod_{k=1}^{n} \frac{1}{\lambda_k!} \operatorname{sgn}(L^{*k})^{\lambda_k} \frac{1}{d(L^{*k})^{\lambda_k}} \\ &= \prod_{k=1}^{n} \frac{1}{\lambda_k!} (-1)^{\lambda_k} (-\operatorname{sgn}(L))^{k\lambda_k} \frac{1}{k^{\lambda_k}} \\ &= (-1)^{\sum \lambda_k} (-\operatorname{sgn}(L))^{\sum k\lambda_k} \prod_{k=1}^{n} \frac{1}{\lambda_k! k^{\lambda_k}} \\ &= (-\operatorname{sgn}(L))^{n} (-1)^{\sum \lambda_k} \prod_{k=1}^{n} \frac{1}{\lambda_k! k^{\lambda_k}} \end{aligned}$$

and hence

$$\sum_{\lambda \vdash n} U(Z_\lambda) = (-\operatorname{sgn}(L))^{n} \sum_{\lambda \vdash n} (-1)^{\sum \lambda_k} \prod_{k=1}^{n} \frac{1}{\lambda_k! k^{\lambda_k}}.$$

Now recall a well-known fact that the number of elements of $\mathbf{S}_n$, the symmetric group of degree n, of type $\lambda = (\lambda_1, \ldots, \lambda_n)$, i.e., having λ_k cycles of length k, $k = 1, 2, \ldots, n$, is

$$h(\lambda) = \prod_{k=1}^{n} \frac{n!}{\lambda_k! k^{\lambda_k}}$$

(Cauchy's formula). The sign of a permutation of this type λ is

$$\prod_{k=1}^{n}\{(-1)^{k+1}\}^{\lambda_k} = (-1)^n(-1)^{\sum \lambda_k}.$$

Putting all this together we have

$$\sum_{\lambda\vdash n} U(Z_\lambda) \;=\; \frac{(\,\mathrm{sgn}\,(L))^n}{n!}\sum_{\lambda\vdash n}(-1)^n(-1)^{\sum \lambda_k}\, h(\lambda) \;=\; \frac{(\,\mathrm{sgn}\,(L))^n}{n!}\sum_{\sigma\in\mathbf{S}_n}\mathrm{sgn}\,\sigma.$$

This equals 0. □

Thus, what remains to be justified in order to complete a proof of the loop expansion formula (10) or the Vdovichenko formula (5) is the following claim. Here, for $L \in \mathcal{L}(G)$ and $n \geq 2$, the product $L^n = L \cdots L$ (n times) is said to be *duplicate loops*.

Claim. *Let $\mathcal{Z}'$ be the set of all elements Z of $\mathcal{Z}(G)$ such that*

Z has no duplicate loops nor degenerate loops in factors

and that

there exists at least one $e \in E(G)$ such that the degree of $\mathbf{x}^Z$ in the variable x_e is ≥ 2.

Then $\sum_{Z\in\mathcal{Z}'} U(Z)\,\mathbf{x}^Z = 0$.

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