Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
32 (1999), pp. 69–74

ON GEODESIC HOMOGENEOUS LEFT LIE LOOPS AS REDUCTIVE HOMOGENEOUS SPACES

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(Received: December 25, 1998)

Abstract. Two kinds of canonical connections have been introduced by the author for homogeneous (left) Lie loops, one of which is the canonical connection of the reductive homogeneous spaces induced from the homogeneous (left) Lie loops and the other is the canonical connection of the homogeneous systems associated with left Lie loops. In this paper, it is shown that they are coincident with each other. Moreover, the concept of *geodesic* reductive homogeneous spaces are introduced. It is shown that the reductive homogeneous space induced from a homogeneous left Lie loop is geodesic if and only if the left Lie loop is geodesic.

1. Preliminaries

For any linearly connected manifold with any fixed point e, the local binary operation is introduced by

$$\mu(x, y) = \mathrm{Exp}_x \circ \tau_{e,x} \circ \mathrm{Exp}_e^{-1}(y)$$

in some normal neighbourhood of e . Here, $\tau_{e,x}$ denotes the parallel displacement of tangent vectors along the geodesic arc joining e to x. It has been proved that this local multiplication forms a local loop, called *geodesic local loop* at e ([3], also cf. [1], [14]). Denote by L_x the left translation by x. Non-associativity of μ is indicated by left inner maps $L_{x,y} := L^{-1}_{\mu(x,y)} \circ L_x \circ L_y$. Motivated by the concept of geodesic local loops on affinely connected manifolds, the algebraic concept of homogeneous loops and the concept of homogeneous Lie loops on manifolds have been introduced by the author in 1975 [4].

Let (G, μ) be an algebraic binary operation on a set G with the unit e such that all left translations $L_x : G \to G; L_x y := \mu(x, y)$, $x \in G$, are bijections of G. If the left translation L_x satisfies $(L_x)^{-1} = L_{x^{-1}}$ for $x^{-1} = L_x^{-1} e$, $x \in G$, it is called a *left loop* with the *left inverse property* . The left loop μ with the left inverse property is said to be *homogeneous* if all left inner maps $L_{x,y}$ are automorphisms of μ. The subgroup of the automorphism group $Aut(\mu)$ generated by all left inner maps is called the *left inner mapping group* of (G, μ). Moreover, if all right translations

1991 *Mathematics Subject Classification.* 17A40, 20N05, 53C05, 53C30.

Key words and phrases. Loop, Reductive homogeneous space, Connection.

of the left loop are bijection, it is called a homogeneous loop. *Homogeneous (left) Lie loop* is a homogeneous (left) loop defined on a differentiable manifold whose multiplication μ is differentiable.

2. Homogeneous left Lie loops as reductive homogeneous spaces

For any homogeneous left loop (G, μ), the concept of semi-direct product $A = G \times K$ of G by a group K is introduced, where K is a subgroup of $Aut(G, \mu)$ containing the left inner mapping group of (G, μ). That is, for any (x, α) and (y, β) in A, define their product by;

$$(x, \alpha)(y, \beta) := (\mu(x, \alpha y), L_{x,\alpha y} \circ \alpha \circ \beta).$$

Then, A forms a group which is called the *semi-direct product* of G and K.

Let (G, μ) be a homogeneous left Lie loop , K the closure of the left inner mapping group in the (differentiable) automorphism group of (G, μ). Then, the Lie group A of semi-direct product of G and K is called the *enveloping Lie group* of (G, μ). By using this, it is shown that any homogeneous left Lie loop is regarded as a reductive homogeneous space $G = A/K$. In fact, since any element $\alpha = (e, \alpha)$ is an automorphism of (G, μ), it is easy to check the following relations:

$$(e, \alpha)(x, \ \mathrm{id}) = (\alpha x, \ \mathrm{id})(e, \alpha)$$

for any $x \in G$ and $\alpha \in K$, which show that the submanifold $G = (G, \mathrm{id})$ of A is invariant under any inner automorphism of A by any element $(e, \alpha) \in \{e\} \times K = K$. Settle on A/K the canonical connection of the reductive homogeneous space of Nomizu [13]. Then the homogeneous left Lie loop G is said to be *geodesic* if the multiplication μ is coincident with the geodesic local loop of the canonical connection, in some neighbourhood of the unit e.

Especially, any Lie group (G, μ) is a geodesic homogeneous left Lie loop whose left inner maps are equal to the identity map. In this case, the canonical connection is reduced to the (-)-connection of E. Cartan.

On the other hand, assume that a homogeneous left Lie loop (G, μ) satisfies the relation;

$$\mu(x, y)^{-1} = \mu(x^{-1}, y^{-1}) \quad \text{for any } x, y \in G.$$

This relation means that the *inversion* $J : G \to G; J(x) := x^{-1}$, is an automorphism of (G, μ). Then, it is shown that the reductive homogeneous space $G = A/K$ is reduced to an affine symmetric space. So, it is called a *symmetric loop* (cf. [4]). It has been shown that any symmetric loop is geodesic ([4]).

3. Canonical connections

For any left Lie loop (G, μ) with the left inverse property, let $\eta : G \times G \times G \to G$;

$$\eta(x, y, z) := L_x \mu(L_x^{-1} y, L_x^{-1} z)$$

be the homogeneous system associated with (G, μ) (cf. [7]). It has been shown in [10] that the *canonical connection* ∇ of the left Lie loop (G, μ) is given explicitly by the formula:

$$(\nabla_X Y)_x = X_x Y - \eta(x, X_x, Y_x) \quad \text{at} \quad x \in M, \tag{3.1}$$

for any vector fields X, Y on G. Here, we denote

$$X_x Y := X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial}{\partial x^i}$$

and, if u, v, w are considered as independent coordinate variables in the $3n$-variable functions $\eta(u, v, w)^k$, $k = 1, \cdots, n$, on the manifold G of dimension n, we denote

$$\eta(x, X_x, Y_x) := X^i Y^j \frac{\partial^2 \eta^k}{\partial v^i \partial w^j}_{|u=v=w=x} \frac{\partial}{\partial x^k}_{|x}$$

in some coordinate neighborhood of the unit e.

Especially, this is valid for homogeneous left Lie loops. In this case, the canonical connection given above is coincident with the canonical connection of reductive homogeneous space A/K for the semi-direct product $A = G \times K$ mentioned in the previous section. That is;

Theorem 3.1. *Let (G, μ) be a homogeneous left Lie loop, $A = G \times K$ the semi-direct product of G by the closure K of the left inner mapping group of (G, μ). Let ∇ be the canonical connection of (G, μ) defined by the formula (3.1). Then, the connection ∇ is an invariant connection of the reductive homogeneous space $G = A/K$ which is coincident with the canonical connection of A/K.*

Proof. Let α be any (differentiable) automorphism of (G, μ). For any two vector fields X, Y on G, we can see

$$d\alpha(X_x Y - \eta(x, X_x, Y_x)) = (d\alpha X)_{\alpha x} d\alpha Y - \eta(\alpha x, (d\alpha X)_{\alpha x}, (d\alpha Y)_{\alpha x}).$$

So the connection ∇ is invariant under the group K. Let $t_{(x,\alpha)} : G \to G$, $(x, \alpha) \in A$ denote the action of A on the manifold G. By regarding the semi-direct product as $A = (G, \mathrm{id}) \times (e, K)$, the action of (x, id) on $y = (y, K) \in G = \pi A$ is expressed by $t_{(x,\mathrm{id})}(y, K) = (\mu(x, y), K)$, for any $x, y \in G$. This means that the element (x, id) of A acts on G as the left translation L_x. For each tangent vector $X \in \mathfrak{G} = T_e(G)$ at the origin $e = \pi(e, K)$, consider a vector field X^* on some neighborhood U of e given by

$$X^*_x := dL_x X, \ x \in U.$$

By the original definition of K.Nomizu [13], the canonical connection ∇^* (of 2nd kind) of a reductive homogeneous space is an invariant connection satisfying

$$(\nabla^*_{X^*} Y^*)_e = O, \tag{3.2}$$

for any $X, Y \in \mathfrak{G}$. So, it is sufficient to show that the connection given by (3.1) satisfies the relation (3.2). Since $\mu(x, y) = \eta(e, x, y)$, the vector field X^* can be described in terms of the homogeneous system η as $X^*_x = \eta(e, x, X)$. Then, we have

$$\begin{aligned}\eta(x, X^*, Y^*) &= \eta(x, \eta(e, x, X), \eta(e, x, Y)) \\ &= \eta(e, x, \eta(e, X, Y)).\end{aligned}$$

On the other hand, the equalities

$$X^*_e Y^* = X\eta(e, x, Y) = \eta(e, X, Y)$$

hold so that we have

$$(\nabla_{X^*} Y^*)_e = X^*_e Y^* - \eta(e, X^*_e, Y^*_e) = O.$$

□

4. Geodesic reductive homogeneous spaces

Let $G = \tilde{G}/K$ be a reductive homogeneopus space of a Lie group $\tilde{G} = (\tilde{G}, \tilde{\mu})$ by a closed subgroup K, with the projection $\pi : \tilde{G} \to G$ and the decomposition of the Lie algebra

$$\tilde{\mathfrak{G}} = \mathfrak{G} \oplus \mathfrak{K}, \qquad \mathrm{ad}K\ \mathfrak{G} \subset \mathfrak{G},$$

where $\mathfrak{G}$ is a subspace of $\tilde{\mathfrak{G}}$ which is regarded as the tangent space to G at the origin $e = \pi(K)$, and $\mathfrak{K}$ is the Lie algebra of K. Let exp be the exponential map of the Lie algebra $\tilde{\mathfrak{G}}$ into $\tilde{G}$. Consider the geodsic local loop (U, μ) on a normal neighborhood U of e, with respect to the canonical connection of the reductive homogeneous space. We introduce the new concept of geodesic reductive homogeneous spaces:

Definition 4.1. The reductive homogeneous space $G = \tilde{G}/K$ will be said to be *geodesic* if the diagram

$$\begin{array}{ccc} \tilde{U} \times \tilde{U} & \xrightarrow{\tilde{\mu}} & \tilde{G} \\ {\scriptstyle \pi\times\pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ U \times U & \xrightarrow{\mu} & G \end{array} \tag{4.1}$$

commutes for some normal neighborhood U of the origin e, where $\tilde{U} = \pi^{-1}(U) \cap \exp(\mathfrak{G})$.

Theorem 4.1. *Let (G, μ) be a homogeneous left Lie loop on a connected differentiable manifold G, $A = G \times K$ the enveloping Lie group by the closure K of the left inner mapping group in* $\mathrm{Aut}(G, \mu)$. *Then, (G, μ) is geodesic if and only if the reductive homogeneous space A/K is geodesic.*

Proof. Assume that the homogeneous left Lie loop (G, μ) is geodesic. For any tangent vector $X \in T_e(G)$ at the unit e, the geodesic curve $x(t)$, $t \in \mathbf{R}$, tangent to X at $e = x(0)$ is a one-parameter subgroup of the left loop (G, μ). Moreover, it satisfies the relation $L_{x(t),x(s)} = \mathrm{id}$ for any $t, s \in \mathbf{R}$ (cf. [4] pp. 163 –168) .

Consider a curve given by $\tilde{x}(t) := (x(t), \mathrm{id}), t \in \mathbf{R}$, in the Lie group $A = G \times K$. Then, it is a one-parameter subgroup of A. In fact we have

$$(x(t), \mathrm{id})(x(s), \mathrm{id}) = (\mu(x(t), x(s)), L_{x(t),x(s)}) = (x(t+s), \mathrm{id}),\ t, s \in \mathbf{R}.$$

For some normal neighborhood U at e, set $\tilde{U} := \pi^{-1}(U) \cap \exp(\mathfrak{G})$ in the Lie group A whose Lie algebra is assumed to be decomposed into $\mathfrak{A} = \mathfrak{G} \oplus \mathfrak{K}$ by the tangent Lie triple algebra $\mathfrak{G}$ of (G, μ) and the Lie algebra $\mathfrak{K}$ of K. Then, any two elements $\tilde{x}, \tilde{y} \in \tilde{U}$ are given by some one-parameter subgroups $\tilde{x}(t), \tilde{y}(s),\ t, s \in \mathbf{R}$, in A as $\tilde{x} = \tilde{x}(t_0), \tilde{y} = \tilde{y}(s_0)$ for some $t_0, s_0 \in \mathbf{R}$. The group multiplication of $\tilde{x}$ and $\tilde{y}$ is given by

$$\tilde{\mu}(\tilde{x}, \tilde{y}) = (x(t_0), \mathrm{id})(y(s_0), \mathrm{id}) = (\mu(x(t_0), y(s_0)), L_{x(t_0),y(s_0)}).$$

Thus we get

$$\pi\tilde{\mu}(\tilde{x}, \tilde{y}) = \mu(\pi\tilde{x}, \pi\tilde{y}),$$

that is, the diagram (4.1) commutes so that the reductive homogeneos space $G = A/K$ is geodesic.

Conversely, assume that the reductive homogeneous space A/K is geodesic. For some normal neighborhood U at e, any element $\tilde{x} = (x, \alpha) \in \tilde{U} = \pi^{-1}(U) \cap \exp(\mathfrak{G})$ can be regarded as an element $\tilde{x} = \tilde{x}(t_0)$ of some one-parameter subgroup $\tilde{x}(t), t \in \mathbf{R}$, in the Lie group A. Theorem 3.1 implies that the one-parameter subgroup $\tilde{x}(t)$ acts on G by some parallel displacement $\tau_{e,x(t)} : T_e(G) \to T_{x(t)}(G)$ of tangent vectors along the geodesic curve $x(t) = \pi\tilde{x}(t)$ passing through $e = x(0)$. So, if we set $\tilde{x}(t) = (x(t), \alpha(t))$, we get

$$\tau_{e,x(t)} = dL_{x(t)} \circ d\alpha(t), t \in \mathbf{R}. \tag{4.2}$$

On the other hand, by the assumption on the reductive homogeneous space A/K, $\alpha(t)$ acts on U as the identity transformation. In fact, for any $y \in U$, set $\pi^{-1}(y) = (y, \beta) \in \tilde{U}$. Then we get

$$\pi[(x(t), \alpha(t))(y, \beta)] = \pi(L_{x(t)} \circ \alpha(t)y, L_{x(t),\alpha(t)y} \circ \alpha(t) \circ \beta) = L_{x(t)} \circ \alpha(t)y$$

and

$$\mu(\pi\tilde{x}(t), \pi\tilde{y}) = L_{x(t)}y.$$

These must coincide with each other so that $\alpha(t)y = y$ for $y \in U$. Hence, the parallel displacement $\tau_{e,x(t)}$ of tangent vectors along the geodesic curve $x(t)$ comes from the left translation $L_{x(t)}$ of the left loop (G, μ). Since $L_{x(t)}y = \eta(e, x(t), y)$, the left translation $L_{x(t)}$ is an affine transformation of the canonical connection. Thus we see that $L_{x(t)}$ brings any geodesic $y(s)$ passing through $e = y(0)$ to the geodesic $z(s) = \mu(x(t), y(s))$ through $x(t) = z(0)$. This means that the multiplication μ is coincident with that of the geodesic local loop on U.

$\square$

References

[1] M. Akivis, *Geodesic loops and local triple systems in affinely connected spaces* (Russian), Sibir. Mat. Z. **19** (1978), 243–253.

[2] O.Chein - H.O.Pflugfelder - J.D.H.Smith, *Quasigroups and Loops: Theory and Applications*, Heldermann Verlag 1990.

[3] M. Kikkawa, *On local loops in affine manifolds*, J. Sci. Hiroshima Univ. **A-I, 28**(1964), 199–207.

[4] ______,*Geometry of homogeneous Lie loops*, Hiroshima Math. J. **5** (1975), 141–179.

[5] ______, *A note on subloops of a homogeneous Lie loop and subsystems of its Lie triple algebra*, Hiroshima Math. J. **5**(1975), 439–446.

[6] ______, *On some quasigroups of algebraic models of symmetric spaces III*, Mem. Fac. Lit. Sci. Shimane Univ. Nat. Sci. **9**(1975), 7–12.

[7] ______, *On the left translations of homogeneous loops*, Mem. Fac. Lit. Sci. Shimane Univ. Nat. Sci. **10**(1976), 19–25.

[8] ______, *Totally geodesic imbeddings of homogeneous systems into their enveloping Lie groups*, Mem. Fac. Sci. Shimane Univ. **18**(1984), 1–8.

[9] ______, *Canonical connections of homogeneous Lie loops and 3-webs*, Mem. Fac. Sci. Shimane Univ. **19**(1985), 37–55.

[10] ______, *Remarks on canonical connections of loops with the left inverse property*, Mem. Fac. Sci. Shimane Univ. **20**(1986), 9–19.

[11] ______, *Projectivity of homogeneous left loops*, Nonassociative Algebras and Related Topics, World Scientific 1991, 77–99.

[12] O.Loos, Symmetric Spaces I, Benjamin 1969.

[13] K.Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76**(1954), 33–65.

[14] L.V.Sabinin, *Geometry of loops*(Russian), Mat. Zametki **12**(1972), 605–616.

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