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STABILITY AND BOUNDEDNESS IN VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Dedicated to Professor Michihiko Kikkawa on the occasion of his retirement

Abstract. Stability and boundedness of solutions of Volterra integro-differential equations are discussed. In particular, we show stability of the zero solution, boundedness and uniform boundedness of solutions by using suitable Liapunov functionals or functions. Moreover we give several examples to our theorems.

1. Introduction

Many results have been obtained for stability and boundedness in functional differential equations (for instance, [1-6] and references cited therein). In particular, concerning stability and boundedness in Volterra integro-differential equations, we can find many interesting results in the books [2,3] by Burton and many papers in their references.

In this paper, we discuss stability and boundedness of solutions of Volterra integro-differential equations. In §2, we discuss stability of the zero solution of a nonlinear Volterra integro-differential equation by using a Liapunov functional, and give an example. Finally in §3, we discuss boundedness and uniform boundedness of solutions of linear or nonlinear Volterra integro-differential equations by employing Liapunov functionals or functions, and give several examples to our theorems. In particular, we use the Liapunov-Razumikhin method to prove Theorems 3.4 and 3.5.

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2. Stability

In this section, we discuss stability of the zero solution of a nonlinear Volterra integro-differential equation. Consider the nonlinear system

$$x'(t) = a(x(t)) + \int_0^t C(t,s)f(x(s))ds, \tag{2.1}$$

in which

$$a(x) = Ax + b(x),$$

and $b, f : \mathbf{R}^n \to \mathbf{R}^n$ are continuous, $f(0) = 0$,

$$|b(x)| \leq \gamma |x|,$$

$$|f(x)| \leq \delta |x|$$

for some $\gamma,\ \delta > 0$, and A is a constant $n \times n$ matrix and C an $n \times n$ matrix of functions continuous for $0 \leq s \leq t < \infty$, where $|\cdot|$ is a norm of $\mathbf{R}^n$. We suppose that there is a symmetric matrix B with

$$A^T B + BA = -I, \tag{2.2}$$

where A^T denotes the transpose of A, and I denotes the $n \times n$ identity matrix.

Definition 2.1. *The zero solution of* (2.1) *is stable if, for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists $\delta > 0$ such that*

$$|\phi(s)| < \delta \quad \text{on} \quad [0, t_0] \qquad \text{and} \quad t \geq t_0$$

imply $|x(t)| < \varepsilon$.

Concerning stability of the zero solution of (2.1), first we obtain the following stability theorem.

Theorem 2.1. *Let* (2.2) *hold and suppose that there is a constant $M > 0$ with*

$$|B|\Big(\int_0^t |C(t,s)|ds + \delta^2 \int_t^\infty |C(u,t)|du + 2\gamma\Big) \leq M < 1. \tag{2.3}$$

If $x^T Bx > 0$ for each $x \neq 0$, then the zero solution of (2.1) *is stable.*

Proof. We define

$$V_1(t, x(\cdot)) = x(t)^T Bx(t) + |B| \int_0^t \int_t^\infty |C(u,s)|du|f(x(s))|^2 ds.$$

Then we have

$$\begin{aligned}
V'_{1(2.1)}(t,x(\cdot)) &= \{x^TA^T + b^T(x) + \int_0^t f^T(x(s))C^T(t,s)ds\}Bx \\
&\quad + x^TB\{Ax + b(x) + \int_0^t C(t,s)f(x(s))ds\} \\
&\quad + |B|\int_t^\infty |C(u,t)|du|f(x)|^2 - |B|\int_0^t |C(t,s)||f(x(s))|^2 ds \\
&= x^TA^TBx + x^TBAx + 2x^TBb(x) \\
&\quad + 2x^TB\int_0^t C(t,s)f(x(s))ds + |B|\int_t^\infty |C(u,t)|du|f(x)|^2 \\
&\quad - |B|\int_0^t |C(t,s)||f(x(s))|^2 ds \\
&\le -|x|^2 + 2|x||B||b(x)| + 2|x||B|\int_0^t |C(t,s)||f(x(s))|ds \\
&\quad + |B|\int_t^\infty |C(u,t)|du|f(x)|^2 - |B|\int_0^t |C(t,s)||f(x(s))|^2 ds \\
&\le -|x|^2 + 2\gamma|B||x|^2 + |B|\int_0^t |C(t,s)|\{|x|^2 + |f(x(s))|^2\}ds \\
&\quad + \delta^2|B|\int_t^\infty |C(u,t)|du|x|^2 - |B|\int_0^t |C(t,s)||f(x(s))|^2 ds \\
&\le -|x|^2 + 2\gamma|B||x|^2 + |B|\int_0^t |C(t,s)|ds|x|^2 \\
&\quad + \delta^2|B|\int_t^\infty |C(u,t)|du|x|^2 \\
&= \{-1 + |B|(2\gamma + \int_0^t |C(t,s)|ds + \delta^2\int_t^\infty |C(u,t)|du)\}|x|^2 \\
&\le \{-1 + M\}|x|^2 \stackrel{def}{=} -\alpha|x|^2, \quad \alpha > 0.
\end{aligned}$$

Now, if $x^TBx > 0$ for all $x \neq 0$, then V_1 is positive definite and $V'_{1(2.1)}$ is negative definite, so $x = 0$ is stable. This completes the proof.

Remark 2.1. *In [2; Theorem 8.2.6], we can find a boundedness result of Grimmer-Seifert for the linear equation*

(2.4) $$x'(t) = Ax(t) + \int_0^t C(t,s)x(s)ds + g(t).$$

Now we show an example to Theorem 2.1

Example 2.1. *Consider the scalar equation*

(2.5) $$x' = (-x(t) + b(x(t))) + \int_0^t e^{-3(t-s)}x(s)\sin x(s)ds,$$

where

$$b(x)=\begin{cases}\frac{1}{2}\log(x+1) & \text{if } x\geq 0\\ -\frac{1}{2}\log(-x+1) & \text{if } x<0.\end{cases}$$

Then we can take $\gamma = B = \frac{1}{2}$, $\delta = 1$, *and we have*

$$\begin{aligned}&|B|(\int_0^t |C(t,s)|ds + \delta^2 \int_t^\infty |C(u,t)|du + 2\gamma)\\ =&\frac{1}{2}(\int_0^t e^{-3(t-s)}ds + 1^2 \int_t^\infty e^{-3(u-t)}du + 2\cdot\frac{1}{2})\\ =&\frac{1}{2}(\frac{5}{3}-\frac{1}{3}e^{-3t}) < \frac{5}{6} = M < 1.\end{aligned}$$

Thus all conditions of Theorem 2.1 are satisfied, so that the zero solution of (2.5) *is stable.*

3. Boundedness

In this section, we discuss boundedness of solutions of Volterra integro-differential equations. First we consider a perturbed form of (2.1)

$$x'(t) = a(x(t)) + g(t,x(t)) + \int_0^t C(t,s)f(x(s))ds \tag{3.1}$$

with $a(x)$ and C, f as in (2.1), $g : [0,\infty)\times \mathbf{R}^n \to \mathbf{R}^n$ continuous, and

$$|g(t,x)| \leq \lambda(t)(|x|+1), \tag{3.2}$$

where $\lambda : [0,\infty) \to [0,\infty)$ is continuous,

$$\int_0^\infty \lambda(s)ds < \infty \quad \textit{and} \quad \lambda(t)\to 0 \textit{ as } t\to\infty. \tag{3.3}$$

Definition 3.1. *Solutions of* (3.1) *are uniformly bounded if, for each* $H > 0$ *there exists* $D > 0$ *such that*

$$t_0 \geq 0 \quad \textit{and} \quad |\phi(s)| < H \quad \textit{on} \quad [0,t_0] \quad \textit{and} \quad t\geq t_0$$

imply $|x(t)| < D$.

The following theorem is our first boundedness theorem.

Theorem 3.1. *Suppose that* (2.2), (2.3), (3.2) *and* (3.3) *hold. If* $x^T Bx > 0$ *for* $x \neq 0$, *then all solutions of* (3.1) *are bounded.*

Proof. In the proof of Theorem 2.1. we found $V'_{1(2.1)}(t,x(\cdot)) \leq -\alpha|x|^2, \alpha > 0$. Select $L > 0$ so that

$$-\alpha|x|^2 + 2|B||x|(|x|+1)\lambda(t) - L\lambda(t) \leq -\bar{\alpha}|x|^2,$$

for some $\bar{\alpha} > 0$ and all x when t is large enough, say, $t \geq S$. Next, define

$$V(t, x(\cdot)) = [x(t)^T Bx(t) + 1 + |B| \int_0^t \int_t^\infty |C(u, s)|du|f(x(s))|^2 ds] \times \exp[-L \int_0^t \lambda(s)ds],$$

so that

$$\begin{aligned} V'_{(3.1)}(t, x(\cdot)) &\leq -L\lambda(t)V + \exp[-L \int_0^t \lambda(s)ds] \\ &\quad \times \{V'_{1(2.1)}(t, x(\cdot)) + 2|B||x||g(t, x)|\} \\ &\leq -L\lambda(t)V + \exp[-L \int_0^t \lambda(s)ds] \\ &\quad \times \{-\alpha|x|^2 + 2|B||x|\lambda(t)(|x| + 1)\} \\ &\leq \exp[-L \int_0^t \lambda(s)ds]\{-\alpha|x|^2 + 2|B||x|\lambda(t)(|x| + 1) - L\lambda(t)\} \\ &\leq -\bar{\alpha}|x|^2 \exp[-L \int_0^t \lambda(s)ds] \\ &\stackrel{def}{=} -\beta|x|^2, \quad if \quad t \geq S. \end{aligned}$$

Suppose that $x^T Bx > 0$ for all $x \neq 0$. If $x(t)$ is any solution of (3.1), then by the growth condition of g, it can be continued for all future time. Hence, for $t \geq S$ we have $V(t, x(\cdot)) \leq V(S, x(\cdot))$, so that $x(t)$ is bounded. This completes the proof.

Example 3.1. *Consider the perturbed form of the scalar equation* (2.5)

$$x'(t) = -x(t) + b(x(t)) + g(t, x(t)) + \int_0^t e^{-3(t-s)} x(s) \sin x(s) ds, \tag{3.4}$$

where $b(x)$ is the function given in Example 2.1. If we suppose that

$$g(t, x) = \begin{cases} \frac{1}{6}\sqrt{x}e^{-t} & if \;\; x \geq 0 \\ -\frac{1}{6}\sqrt{-x}e^{-t} & if \;\; x < 0, \end{cases}$$

then

$$|g(t, x)| \leq \frac{1}{12}e^{-t}(|x| + 1),$$

so that we take

$$\lambda(t) = \frac{1}{12}e^{-t}.$$

Thus

$$\int_0^\infty \lambda(s)ds = \int_0^\infty \frac{1}{12}e^{-s}ds = \frac{1}{12}[-e^{-s}]_0^\infty = \frac{1}{12} < \infty$$
$$\lambda(t) \to 0 \quad as \quad t \to \infty.$$

Hence, the condition for g holds.

We select $L > 0$ so that there exists $\bar{\alpha} > 0$ such that when $t \geq S$, for all x

$$-\frac{1}{6}x^2 + \frac{1}{12}|x|(|x|+1)e^{-t} - \frac{1}{12}Le^{-t} \leq -\bar{\alpha}\, x^2.$$

In fact, we suppose that $\bar{\alpha} = \frac{1}{24}$ and we have

$$\begin{aligned}
-\frac{1}{6}x^2 + \frac{1}{12}|x|(|x|+1)e^{-t} - \frac{1}{12}Le^{-t} &\leq -\frac{1}{24}x^2 \\
-4x^2 + 2x^2e^{-t} + 2|x|e^{-t} - 2Le^{-t} &\leq -x^2 \\
(3-2e^{-t})x^2 - 2|x|e^{-t} + 2Le^{-t} &\geq 0 \\
(3-2e^{-t})(|x| - \frac{e^{-t}}{3-2e^{-t}})^2 - \frac{e^{-2t}}{3-2e^{-t}} + 2Le^{-t} &\geq 0.
\end{aligned}$$

Therefore we can select $L > 0$ satisfying $-\frac{e^{-2t}}{3-2e^{-t}} + 2Le^{-t} \geq 0$ for $t \geq S$, say,

$$L \geq \frac{1}{2(3e^t - 2)} \quad \text{for} \quad t \geq S.$$

Now we take $L = \frac{1}{2(3e^S-2)}$. Thus all conditions of Theorem 3.1 are satisfied and all solutions of (2.5) *are bounded.*

The next result is a boundedness theorem for an equation with a variable coefficient. We consider a system of Volterra equations

$$x'(t) = A(t)x(t) + \int_0^t C(t,s)x(s)ds + F(t), \tag{3.5}$$

where $A(t) = A + P(t)$ and A is a matrix such that there are positive constants r, k and K with

$$A^TB + BA = -I \tag{3.6}$$

$$|Bx| \leq K[x^TBx]^{\frac{1}{2}} \tag{3.7}$$

$$|x| \geq 2k[x^TBx]^{\frac{1}{2}} \tag{3.8}$$

$$[x^TBx]^{\frac{1}{2}} \geq r|x|. \tag{3.9}$$

We ask that $C(t,s)$ is an $n \times n$ matrix continuous function for $0 \leq s \leq t < \infty$ and that $F : [0,\infty) \to \mathbf{R}^n$ is continuous and bounded, $P(t)$ is an $n \times n$ continuous matrix with

$$|P(t)| \leq \rho \quad \text{for} \quad t \geq 0, \tag{3.10}$$

where ρ is a constant with $0 < \rho < 1$.

Theorem 3.2. *Let* (3.6) - (3.10) *hold and suppose that*
(a) there is $\bar{K} > K$ with

$$|x|[k - K\rho - \bar{K}\int_t^\infty |C(u,t)|du] \geq (\bar{K} - K)[|Ax| + |x|],$$

(b) for some D with $0 < D < \bar{K} - K + r$,

$$\int_0^t \int_t^\infty \bar{K}|C(u,s)|duds \le D.$$

Then all solutions of (3.5) *are bounded.*

Proof. We define

$$V(t, x(\cdot)) = [x(t)^T Bx(t)]^{\frac{1}{2}} + \bar{K} \int_0^t \int_t^\infty |C(u,s)|du|x(s)|ds.$$

A calculation yields

$$V'_{(3.5)}(t, x(\cdot)) \le [-k + K\rho + \bar{K} \int_t^\infty |C(u,t)|du]|x| + K|F(t)| + [-\bar{K} + K] \int_0^t |C(t,s)||x(s)|ds.$$

By assumption (a),

$$\begin{aligned} V'_{(3.5)}(t, x(\cdot)) &\le -(\bar{K} - K)[|Ax| + |x|] + K|F(t)| \\ &\quad - (\bar{K} - K) \int_0^t |C(t,s)||x(s)|ds \\ &= -(\bar{K} - K)[|Ax| + |P(t)||x| + \int_0^t |C(t,s)||x(s)|ds \\ &\quad + |F(t)|] - (\bar{K} - K)|x| + \bar{K}|F(t)| + (\bar{K} - K)|P(t)||x| \\ &\le -(\bar{K} - K)|x'| - (\bar{K} - K)(1 - \rho)|x| + \bar{K}|F(t)|. \end{aligned}$$

Because $|F|$ is bounded, there is $U > 0$ with $V' \le -(\bar{K} - K)|x'|$ if $|x(t)| \ge 2kU$. We define

$$[x^T Bx]^{\frac{1}{2}} = Q(x),$$

so that $Q = U$ implies $U^2 = x^T Bx \ge r^2|x|^2$, or

$$|x| \le \frac{U}{r} \stackrel{def}{=} L.$$

Because $r|x| \le [x^T Bx]^{\frac{1}{2}}$, we define

$$W_1(|x|) = r|x|.$$

Since we also have

$$V(t, x(\cdot)) \le Q(x) + \sup_{0 \le s \le t} |x(s)| \int_0^t \int_t^\infty \bar{K}|C(u,s)|duds,$$

define

$$W_2(p) = Dp, \quad p \ge 0.$$

Then we obtain

$$W_1(|x(t)|) \le V(t,x(\cdot)) \le Q(x(t)) + W_2(\|x\|^{[0,t]}),$$
$$Q(x) = U \text{ implies } |x| \le L,$$
$$Q(x(t)) \ge U \text{ implies } V'_{(3.5)}(t,x(\cdot)) \le -(\bar{K}-K)|x'(t)|,$$

where $\|x\|^{[0,t]} \stackrel{def}{=} \sup_{0\le s\le t} |x(s)|$.

Suppose that $x(t)$ is an unbounded solution, i.e, there exists $\{u_n\} \to \infty$ such that $|x(u_n)| \to \infty$ as $n \to \infty$. If $Q > U$, then

$$\begin{aligned} 0 < V(u_n, x(\cdot)) &\le V(u_1, x(\cdot)) - (\bar{K}-K)\int_{u_1}^{u_n} |x'(s)|ds \\ &= V(u_1, x(\cdot)) - (\bar{K}-K)|x[u_1,u_n]| \\ &\le V(u_1, x(\cdot)) - (\bar{K}-K)(|x(u_n)| - |x(u_1)|) \\ &\to -\infty \quad as \quad n \to \infty, \end{aligned}$$

and so there is a sequence $\{s_n\} \to \infty$ as $n \to \infty$ with $Q(x(s_n)) = U$.

We therefore find $t_0 \ge 0$ and $R > 0$ with $Q(x(t_0)) = U$ and $\|x\|^{[0,t_0]} < L + R$. Because $|x(t)|$ is unbounded, there is the first $t_2 > t_0$ with $|x(t_2)| = L + R$, and therefore, there is $t_1 \ge t_0$ with $Q(x(t_1)) = U$ and $Q(x(t)) > U$ on $(t_1, t_2]$. Now on $[t_1, t_2]$ we have $V'_{(3.5)}(t, x(\cdot)) \le -(\bar{K}-K)|x'(t)|$, and so

$$\begin{aligned} W_1(|x(t)|) \le V(t,x(\cdot)) &\le V(t_1, x(\cdot)) - (\bar{K}-K)|x[t_1,t]| \\ &\le Q(x(t_1)) + W_2(\|x\|^{[0,t_1]}) - (\bar{K}-K)|x[t_1,t]| \\ &\le U + D(L+R) - (\bar{K}-K)|x[t_1,t]|, \end{aligned}$$

so that at $t = t_2$ we have

$$r(L+R) \le U + D(L+R) - (\bar{K}-K)R$$

or

$$r + \bar{K} - K - D \le 0,$$

a contradiction. This completes the proof.

Remark 3.1. *Theorem 3.2 for* (3.5) *with* $A(t) \equiv A$ *corresponds to Theorem 8.4.2 in [2].*

Example 3.2. *Consider the scalar equation*

$$x'(t) = (-2 + \frac{1}{3}\sin t)x(t) + \int_0^t e^{-2(t-s)}x(s)ds + \sin t. \tag{3.11}$$

Then we can take $B = \frac{1}{4},\ K = k = r = \frac{1}{2},\ \rho = \frac{1}{3}$, *and we have*

$$|x|[k - K\rho - \bar{K}\int_t^\infty |C(u,t)|du] \geq (\bar{K} - K)[|Ax| + |x|]$$
$$\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{3} - \bar{K}\int_t^\infty e^{-2(u-t)}du \geq (\bar{K} - \frac{1}{2})(2+1)$$

$$\frac{1}{3} - \frac{1}{2}\bar{K} \geq 3\bar{K} - \frac{3}{2}$$
$$\bar{K} \leq \frac{11}{21},$$

so that we can take $\bar{K} = \frac{11}{21}$, *and we obtain*

$$\int_0^t\int_t^\infty \bar{K}|C(u,s)|duds = \frac{11}{21}\int_0^t\int_t^\infty e^{-2(u-s)}duds$$
$$= \frac{11}{84}(1 - e^{-2t}) \leq \frac{11}{84} = D.$$

Since all conditions of Theorem 3.2 are satisfied, all solutions of (3.11) *are bounded.*

The next result is a theorem of uniform boundedness for (3.5).

Theorem 3.3. *Let* (3.6) - (3.10) *hold and suppose that* $|F(t)| \leq \xi$ *and there exists* $d > 0$ *with*

$$k - K\rho - \frac{K}{2kr}\int_t^\infty |C(u,t)|du \geq d.$$

If there exists $m < 1$ *with*

$$\frac{K}{r}\int_0^t\int_t^\infty |C(u,s)|duds \leq m,$$

then solutions of (3.5) *are uniformly bounded.*

Proof. We define

$$V(t, x(\cdot)) = [x(t)^T Bx(t)]^{\frac{1}{2}} + \frac{K}{r}\int_0^t\int_t^\infty |C(u,s)|du[x^T(s)Bx(s)]^{\frac{1}{2}}ds$$

and obtain

$$V'_{(3.5)}(t, x(\cdot)) \leq -[k - K\rho - \frac{K}{2kr}\int_t^\infty |C(u,t)|du]|x| + K|F(t)|$$
$$\leq -2kd[x^T Bx]^{\frac{1}{2}} + K\xi.$$

If

$$[x^T Bx]^{\frac{1}{2}} \geq \frac{K\xi}{2kd} \stackrel{def}{=} U,$$

then $V' \leq 0$.

If we define $Q(x) = [x^T Bx]^{\frac{1}{2}}$, then we have

$$\frac{K}{r}\int_0^t \int_t^\infty |C(u,s)|du[x^T(s)Bx(s)]^{\frac{1}{2}}ds \\ \leq m \sup_{0\leq s\leq t} Q(x(s)).$$

Thus there hold that

$$Q(x(t)) \leq V(t, x(\cdot)) \leq Q(x(t)) + m \sup_{0\leq s\leq t} Q(x(s)),$$

$$r|x| \leq Q(x) \leq \frac{1}{2k}|x|,$$

$$Q(x) \geq U \text{ implies } V' \leq 0.$$

Let $H > 0$ be given. We must find $D > 0$ such that

$$t_0 \geq 0, \quad t \geq t_0, \quad \|\phi\|^{[0,t_0]} \leq H$$

imply $|x(t, t_0, \phi)| \leq D$.

For the given $H > 0$, if $\|\phi\|^{[0,t_0]} < H$, then $t \in [0, t_0]$ yields $Q(\phi(t)) \leq \frac{1}{2k}|\phi(t)| < \frac{H}{2k}$ and we can find $M > \max\{\frac{U}{1-m}, \frac{H}{2k}\}$.

We shall show that all solutions are bounded, and so they are continuable. For $x(t) = x(t, t_0, \phi)$ either

(a) $Q(x(t)) < M$ for all $t \geq t_0$, or
(b) there is the first $t^* > t_0$ with $Q(x(t^*)) = M$.

If (b) holds, then either

(b1) there is the first $t_1 > t^*$ with $Q(x(t_1)) = U$, or
(b2) $Q(x(t)) > U$ for $t \geq t^*$.

Let $\bar{t} \in [t^*, t_1)$ be a number such that $Q(x(\bar{t}))$ is the maximum of $Q(x(t))$ on $[0, t_1]$. If (b1) holds, then we claim that $Q(x(\bar{t}))$ is the maximum of $Q(x(t))$ on $[0, \infty)$. If not, then there is an interval past t_1, say, $[t_2, t_3]$ with $Q(x(t)) \leq Q(x(\bar{t}))$ on $[t_2, t_3]$ and with $Q(x(t)) \geq U$ on $[t_2, t_3]$, $Q(x(t_2)) = U$ and $Q(x(t_3)) = Q(x(\bar{t}))$. This is impossible because $V'_{(3.5)}(t, x(\cdot)) \leq 0$ on $[t_2, t_3]$, and so

$$\begin{aligned} Q(x(t_3)) \leq V(t_3, x(\cdot)) &\leq V(t_2, x(\cdot)) \\ &< U + m \sup_{0\leq s\leq t_2} Q(x(s)) \\ &\leq U + mQ(x(\bar{t})) < Q(x(\bar{t})). \end{aligned}$$

Next, we find a bound of $Q(x(\bar{t}))$. We have $V' \leq 0$ on $[t^*, \bar{t}]$, so

$$\begin{aligned} Q(x(\bar{t})) \leq V(\bar{t}, x(\cdot)) &\leq V(t^*, x(\cdot)) \\ &\leq Q(x(t^*)) + m \sup_{0\leq s\leq t^*} Q(x(s)) \\ &\leq M + mM = M(1 + m) \end{aligned}$$

and this is the bound of $Q(x(t))$ if case(b1) holds.

In the case of (b2) we have $Q(x(t)) > U$ for $t \geq t^*$, so that $V' \leq 0$ for $t \geq t^*$ implies

$$\begin{aligned} r|x(t)| \leq Q(x(t)) &\leq V(t, x(\cdot)) \\ &\leq V(t^*, x(\cdot)) \\ &\leq Q(x(t^*)) + m \sup_{0 \leq s \leq t^*} Q(x(s)) \\ &= M + mM = M(1+m). \end{aligned}$$

Certainly, if case(a) holds we have $Q(x(t)) < M < M(1+m)$. Thus, in all cases,

$$r|x(t)| \leq Q(x(t)) \leq M(1+m),$$

and so

$$|x(t)| \leq \frac{M}{r}(1+m) \stackrel{def}{=} D.$$

This shows uniform boundedness and the proof is complete.

Remark 3.2. *Theorem 3.2 for* (3.5) *with* $A(t) \equiv A$ *corresponds to Theorem 8.4.4 in [2].*

Example 3.3. *Consider the scalar equation*

$$x'(t) = (-2 + \frac{1}{3}\sin t)x(t) + \int_0^t e^{-4(t-s)} x(s)ds + \sin t. \tag{3.12}$$

Then we can take $B = \frac{1}{4}, K = k = r = \frac{1}{2}, \rho = \frac{1}{3}$, *and we have*

$$\begin{aligned} k - K\rho - \frac{K}{2kr}\int_t^\infty |C(u,t)|du &= \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{3} - \frac{\frac{1}{2}}{2\cdot\frac{1}{2}\cdot\frac{1}{2}}\int_t^\infty e^{-4(u-t)}du \\ &= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \end{aligned}$$

so that we can take $d = \frac{1}{12}$. *As we have*

$$\begin{aligned} \frac{K}{r}\int_0^t \int_t^\infty |C(u,s)|duds &= \int_0^t \int_t^\infty e^{-4(u-s)}duds \\ &= \frac{1}{16}(1 - e^{-4t}) < \frac{1}{16}, \end{aligned}$$

we can take $m = \frac{1}{16}$. *Thus all conditions of Theorem 3.3 are satisfied, so that solutions of* (3.12) *are uniformly bounded.*

The next result is a boundedness theorem obtained by using a Liapunov function instead of a Liapunov functional.

Again we consider the linear system (3.5), where $A(t) = A + P(t)$, A constant, all characteristic roots of A have negative real parts. Select $B = B^T$ with (3.6), and let α^2 and β^2 be the smallest and largest (respectively) characteristic roots of B. By using the Liapunov-Razumikhin method, we obtain the following theorem, which is deeply related to Theorem 8.2.6 in [2] for (2.4).

Theorem 3.4. *Let the above stated conditions hold and suppose that there is $M > 0$ with*

$$\int_0^t |BC(t,s)|ds \le M, \qquad t \ge 0,$$

where $\frac{2\beta M}{\alpha} + 2\rho|B| < 1$. If, in addition, F is bounded, then all solutions of (3.5) *are bounded.*

Proof. Define $V(t,x) = x^T Bx$, so that $\alpha^2|x|^2 \le V(t,x) \le \beta^2|x|^2$, yielding $W_1(|x|) \le V(t,x) \le W_2(|x|)$, where $W_1(|x|) = \alpha^2|x|^2$ *and* $W_2(|x|) = \beta^2|x|^2$. Then we have

$$\begin{aligned} V'_{(3.5)}(t,x) &= -|x|^2 + 2x^T BP(t)x + 2x^T B\int_0^t C(t,s)x(s)ds + 2x^T BF(t) \\ &\le -|x|^2 + 2|B||P(t)||x|^2 + 2|x|\int_0^t |BC(t,s)||x(s)|ds \\ &\quad + 2|x||B|\|F\|^{[0,\infty)} \\ &\le -|x|^2 + 2\rho|B||x|^2 + 2|x|\int_0^t |BC(t,s)||x(s)|ds \\ &\quad + 2|x||B|\|F\|^{[0,\infty)} \\ &= (2\rho|B|-1)|x|^2 + 2|x|\int_0^t |BC(t,s)||x(s)|ds \\ &\quad + 2|x||B|\|F\|^{[0,\infty)}. \end{aligned}$$

Now, if $h^2V(t,x(t)) > V(s,x(s))$ for $0 \le s \le t$, where $h > 1$ is a constant to be determined, then

$$\begin{aligned} h^2\beta^2|x(t)|^2 &\ge h^2V(t,x(t)) \ge V(s,x(s)) \\ &\ge \alpha^2|x(s)|^2 \end{aligned}$$

and

$$\frac{h\beta}{\alpha}|x(t)| \ge |x(s)|, \quad s \le t.$$

Thus,

$$V'_{(3.5)}(t,x) \le (2\rho|B|-1)|x|^2 + \frac{h\beta}{\alpha}|x(t)|^2\int_0^t |BC(t,s)|ds + 2|B|\|F\|^{[0,\infty)}|x|$$

and, because $\frac{2\beta M}{\alpha} + 2\rho|B| < 1$, h may be chosen so that $h > 1$ and $\frac{2h\beta M}{\alpha} + 2\rho|B| < 1$ yielding

$$V'_{(3.5)}(t,x) \le [\frac{2h\beta M}{\alpha} + 2\rho|B| - 1]|x|^2 + 2|B|\|F\|^{[0,\infty)}|x| \le 0$$

if $|x| \ge \dfrac{2|B|\|F\|^{[0,\infty)}}{1 - (\frac{2h\beta M}{\alpha} + 2\rho|B|)} \overset{def}{=} K.$

Thus we have

(a) $W_1(|x|) \le V(t,x) \le W_2(|x|)$,

(b) there exists $K > 0$ so that if $x(t)$ is a solution of (3.5) with $|x(t)| \geq K$ for some $t \geq 0$ and $V(s, x(s)) < v(V(t, x(t)))$ for $0 \leq s \leq t$ and $v(r) > r$, then $V'_{(3.5)}(t, x) \leq 0$, where $v(r) = h^2 r$.

Now, we choose any solution $x(t)$ such that $|\phi(t)| < H$ for $0 \leq t \leq t_0$ for some $H > 0$. We suppose that $L > \max\{H, K\}$ and choose $D > 0$ with $W_2(L) < W_1(D)$. If this solution is unbounded, then there is $t_1 > 0$ such that

$$|x(t_1)| = D, \qquad |x(t)| < D \ \text{ for } \ 0 < t < t_1.$$

If $V(t_1, x(t_1)) \leq V(t_0, \phi(t_0))$, then we would have

$$\begin{aligned} W_1(|x(t_1)|) \leq V(t_1, x(t_1)) \leq V(t_0, \phi(t_0)) &\leq W_2(|\phi(t_0)|) \\ &< W_2(L) < W_1(D), \end{aligned}$$

and we get $|x(t_1)| < D$, a contradiction. So $V(t_1, x(t_1)) > V(t_0, \phi(t_0))$. Since $V(t, x(t))$ is continuous in t, there exists t_2, $0 < t_2 \leq t_1$, such that $V(t, x(t)) < V(t_2, x(t_2)) = V(t_1, x(t_1))$ for $0 \leq t < t_2$. Clearly there exists a sequence $\{\tau_j\}$, $t_0 < \tau_j < t_2$, such that $\tau_j \to t_2$ as $j \to \infty$ and

$$V'_{(3.5)}(\tau_j, x(\tau_j)) > 0, \quad j = 1, 2, \cdots . \tag{3.13}$$

Now we have

$$v(V(t_2, x(t_2))) - V(t_2, x(t_2)) = \varepsilon > 0, \tag{3.14}$$

since $V(t_2, x(t_2)) > 0$. We claim that there exists an integer j such that

$$v(V(\tau_j, x(\tau_j))) > V(s, x(s)) \ \text{ for } \ 0 \leq s \leq \tau_j. \tag{3.15}$$

If this were not so, then for each integer j there would exist s_j, $s_j \leq \tau_j$, such that $v(V(\tau_j, x(\tau_j))) \leq V(s_j, x(s_j))$. From this, it follows easily that for some $s_0 < t_2$ ($s_j \to s_0$ as $j \to \infty$),

$$v(V(t_2, x(t_2))) \leq V(s_0, x(s_0)). \tag{3.16}$$

But from (3.14) and (3.16), we obtain

$$V(t_2, x(t_2)) + \varepsilon = v(V(t_2, x(t_2))) \leq V(s_0, x(s_0)) < V(t_2, x(t_2)),$$

or

$$\varepsilon < 0,$$

which contradicts to our choice of ε. So we conclude that (3.15) holds for some integer j. But from (b) we must then have $V'_{(3.5)}(\tau_j, x(\tau_j)) \leq 0$, contradicting to (3.13). This completes the proof.

Example 3.4. *Consider the scalar equation*

$$x'(t) = (-2 + \sin t)x(t) + \int_0^t \frac{3}{4} e^{-(t-s)} x(s) ds + \sin t. \tag{3.17}$$

Then we can take $B = \frac{1}{4}$ *and* $\rho = 1$*, and we obtain*

$$\int_0^t |BC(t,s)|ds = \int_0^t \frac{1}{4} \cdot \frac{3}{4} e^{-(t-s)} ds$$
$$= \frac{3}{16}(1 - e^{-t}) \le \frac{3}{16},$$

so that $M = \frac{3}{16}$*. As we have*

$$2M + 2\rho|B| = 2 \cdot \frac{3}{16} + 2 \cdot \frac{1}{4} = \frac{7}{8} < 1$$

and $\sin t$ *is bounded, all conditions of Theorem 3.4 are satisfied, so that all solutions of* (3.17) *are bounded.*

Our last result is a boundedness theorem for a nonlinear system. We consider a nonlinear system

$$x'(t) = a(x(t)) + \int_0^t C(t,s) f(x(s)) ds + F(t), \tag{3.18}$$

where

$$a(x) = Ax + b(x),$$
$$f(x) = Dx + h(x),$$

A, D and C are $n \times n$ matrices, A and D constant, all characteristic roots of A have negative real parts, C continuous for $0 \le s \le t < \infty$, $b(x)$ and $h(x)$ continuous satisfying

$$|b(x)| \le \gamma |x|,$$
$$|h(x)| \le \eta |x|$$

for some $\gamma, \eta > 0$, and $F : [0, \infty) \to \mathbf{R}^n$ is continuous. Select $B = B^T$ with (3.6), and let α^2 and β^2 be the smallest and largest characteristic roots of B respectively. By employing the Liapunov-Razumikhin method, we obtain our last theorem.

Theorem 3.5. *Let the above stated conditions hold and suppose that there is* $M > 0$ *with*

$$\int_0^t |BC(t,s)| ds \le M, \quad t \ge 0,$$

where $\frac{2\beta M(|D| + \eta)}{\alpha} + 2\gamma|B| < 1$*. If, in addition,* F *is bounded, then all solutions of* (3.18) *are bounded.*

Proof. As in the proof of Theorem 3.4, we define $V(t, x) = x^T Bx$, so that $\alpha^2|x|^2 \le V(t,x) \le \beta^2|x|^2$, yielding $W_1(|x|) \le V(t,x) \le W_2(|x|)$, where $W_1(|x|) =$

$\alpha^2|x|^2$ *and* $W_2(|x|) = \beta^2|x|^2$. Here we have

$$\begin{aligned} V'_{(3.18)}(t,x) &= -|x|^2 + 2x^T Bb(x)x + 2x^T B\int_0^t C(t,s)[Dx(s) + h(x(s))]ds \\ &\quad + 2x^T BF(t) \\ &\le -|x|^2 + 2|B||b(x)||x| \\ &\quad + 2|x|\int_0^t |BC(t,s)|[|D||x(s)| + |h(x(s))|]ds + 2|x||B|\|F\|^{[0,\infty)} \end{aligned}$$

$$\begin{aligned} &\le -|x|^2 + 2\gamma|B||x|^2 + 2|x|\int_0^t |BC(t,s)|[|D| + \eta]|x(s)|ds \\ &\quad + 2|x||B|\|F\|^{[0,\infty)} \\ &= (2\gamma|B| - 1)|x|^2 + 2(|D| + \eta)|x|\int_0^t |BC(t,s)||x(s)|ds \\ &\quad + 2|x||B|\|F\|^{[0,\infty)}. \end{aligned}$$

Now, if $h^2 V(t,x(t)) > V(s,x(s))$ for $0 \le s \le t$, where $h > 1$ is a constant to be determined, then

$$h^2\beta^2|x(t)|^2 \ge h^2 V(t,x(t)) \ge V(s,x(s)) \ge \alpha^2|x(s)|^2$$

and

$$\frac{h\beta}{\alpha}|x(t)| \ge |x(s)|, \quad 0 \le s \le t.$$

Thus,

$$\begin{aligned} V'_{(3.18)}(t,x) &\le (2\gamma|B| - 1)|x|^2 + \frac{2h\beta}{\alpha}(|D| + \eta)|x(t)|^2\int_0^t |BC(t,s)|ds \\ &\quad + 2|x||B|\|F\|^{[0,\infty)} \end{aligned}$$

and, because $\frac{2\beta M(|D|+\eta)}{\alpha} + 2\gamma|B| < 1$, h may be chosen so that $h > 1$ and $\frac{2h\beta M(|D|+\eta)}{\alpha} + 2\gamma|B| < 1$ yielding

$$\begin{aligned} V'_{(3.18)}(t,x) &\le [\frac{2h\beta M(|D| + \eta)}{\alpha} + 2\gamma|B| - 1]|x|^2 + 2|B|\|F\|^{[0,\infty)}|x| \\ &\le 0 \end{aligned}$$

if $|x| \ge \dfrac{2|B|\|g\|^{[0,\infty)}}{1 - (\frac{2h\beta M(|D|+\eta)}{\alpha} + 2\gamma|B|)} \overset{def}{=} K.$

Thus there hold that

(a) $W_1(|x|) \le V(t,x) \le W_2(|x|)$,
(b) there exists $K > 0$ so that if $x(t)$ is a solution of (3.5) with $|x(t)| \ge K$ for some $t \ge 0$ and $V(s,x(s)) < v(V(t,x(t)))$ for $0 \le s \le t$ and $v(r) > r$, then $V'_{(3.5)}(t,x) \le 0$, where $v(r) = h^2 r$.

The remaining part can be proved by a similar method to the one used in the proof of Theorem 3.4. Thus all solutions of (3.18) are bounded. This completes the proof.

Finally we give an example to Theorem 3.5.

Example 3.5. *Consider the scalar nonlinear equation*

$$x'(t) = (-x(t) + b(x(t))) + \int_0^t e^{-(t-s)}[\frac{1}{4}x(s) + h(x(s))]ds + \sin t, \tag{3.19}$$

where

$$b(x) = \begin{cases} \frac{1}{2}\log(x+1) & \text{if } x \geq 0 \\ -\frac{1}{2}\log(-x+1) & \text{if } x < 0 \end{cases}$$

and

$$h(x) = \begin{cases} \frac{1}{8}\sin x \log(x+1) & \text{if } x \geq 0 \\ -\frac{1}{8}\sin x \log(-x+1) & \text{if } x < 0. \end{cases}$$

Then we can take $B = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $\eta = \frac{1}{8}$, *and we obtain*

$$\int_0^t |BC(t,s)|ds = \int_0^t \frac{1}{2}e^{-(t-s)}ds = \frac{1}{2}(1 - e^{-t}) \leq \frac{1}{2},$$

so that we can take $M = \frac{1}{2}$. *As we have*

$$2M(|D| + \eta) + 2\gamma|B| = 2 \cdot \frac{1}{2}(\frac{1}{2} + \frac{1}{8}) + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{8} < 1$$

and $\sin t$ *is bounded, all conditions of Theorem 3.5 are satisfied, so that all solutions of* (3.19) *are bounded.*

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