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# DIRICHLET PROBLEM IN AN EXTERIOR DOMAIN WITH POTENTIAL IN A WEIGHTED LEBESGUE CLASS

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Abstract. Let $1 < p < \infty$ and let $w$ be a weight in the Muckenhoupt $A_p$ class. Suppose $\Omega$ is a smooth exterior domain in $\mathbb{R}^n$. Let $f \in L^{p,w}(\Omega) = \{f : \|f\|_{p,w;\Omega} = (\int_\Omega |f|^p w dx)^{1/p} < \infty\}$. We consider the Dirichlet problem: $-\Delta u = f$ on $\Omega$ and $u = 0$ on $\partial\Omega \cup \{\infty\}$ with $f \in L^{p,w}(\Omega)$. We give sufficient conditions for the Dirichlet problem to have a unique solution $u$ with estimate $\sum_{|\alpha|=2} \|D^\alpha u\|_{p,w;\Omega} \le c\|f\|_{p,w;\Omega}$.

## 1. Introduction

Let $\Omega$ be an exterior domain whose complement consists of finitely many $C^{1,1}$ bounded domains (cf. [4, p.94]). Without loss of generality we may assume that a ball $\{x : |x| \le r_0\}$ lies outside $\Omega$. Suppose $1 < p < \infty$ and $f \in L^{p,w}(\Omega) = \{f : \|f\|_{p,w;\Omega} < \infty\}$, where $\|f\|_{p,w;\Omega} = \left(\int_\Omega |f|^p w dx\right)^{1/p}$. Take $r_1$ so large that $\mathbb{R}^n \setminus \Omega \subset \{x : |x| < r_1\}$ and let $\Omega_1 = \{x \in \Omega : |x| < r_1\}$. Define the weighted Beppo Levi space $BL^{2,p,w}(\Omega) = \{u : \|u\|_{BL^{2,p,w}(\Omega)} < \infty\}$, where

$$\|u\|_{BL^{2,p,w}(\Omega)} = \|u\|_{p,w;\Omega_1} + \sum_{|\alpha|=1} \|D^\alpha u\|_{p,w;\Omega_1} + \sum_{|\alpha|=2} \|D^\alpha u\|_{p,w;\Omega}.$$

The weighted Beppo Levi space $BL^{2,p,w}(\Omega)$ and the weighted Sobolev space

$$W^{2,p,w}(\Omega) = \{u : \|u\|_{W^{2,p,w}(\Omega)} = \sum_{|\alpha|\le 2} \|D^\alpha u\|_{p,w;\Omega} < \infty\}.$$

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have the same local behavior. Besides the obvious implication $W^{2,p,w}(\Omega) \subset BL^{2,p,w}(\Omega)$, we have the opposite $BL^{2,p,w}(\Omega') \subset W^{2,p,w}(\Omega)$, if $\Omega'$ is a bounded subdomain of $\Omega$. At $\infty$, however, they behave in completely different ways.

Let us consider the Dirichlet problem

$$-\Delta u = f \quad \text{on } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Since $u$ must be "small" at $\infty$ in a certain sense for $u \in W^{2,p,w}(\Omega)$, it follows that (1)-(2) can have no solutions in $W^{2,p,w}(\Omega)$ if $f \in L^{p,w}(\Omega)$ is "large" at $\infty$. Nevertheless we shall see later in Theorem 3 that (1)-(2) always has a solution in $BL^{2,p,w}(\Omega)$. This is the main reason why we introduce the weighted Beppo Levi space. If $w \equiv 1$, then each function space reduces to a usual unweighted one and is denoted by a symbol without the superscript $w$.

Let us make the meaning of (2) clear. We take the boundary condition in the sense of $W^{1,q}(\Omega_1)$ for some $q > 1$, i.e. (2) holds if and only if there are continuous functions $u_j$ vanishing near the boundary $\partial\Omega$ and converging to $u$ in $W^{1,q}(\Omega_1)$ for some $q > 1$. Since $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$, it follows that $w^{-q/(p-q)}$ is locally integrable for some $q$ with $1 < q < p$. Hence the Hölder inequality yields that $L^{p,w}(\mathbb{R}^n) \subset L^q_{loc}(\Omega_1)$, and so

$$L^{p,w}(\Omega)\big|_{\Omega_1} \subset L^q(\Omega_1), \qquad BL^{2,p,w}(\Omega)\big|_{\Omega_1} \subset W^{2,q}(\Omega_1). \tag{*}$$

Thus the above interpretation of (2) is compatible with (1) for $f \in L^{p,w}(\Omega)$.

Let $h_2$ be the Riesz kernel

$$h_2(x) = \begin{cases} \gamma_2^{-1} \log \dfrac{1}{|x|} & \text{if } n = 2 \\ \gamma_n^{-1} |x|^{2-n} & \text{if } n \geq 3, \end{cases}$$

where $\gamma_2 = 2\pi$ and $\gamma_n = 4\pi^{n/2}/\Gamma(\frac{n}{2} - 1)$ if $n \geq 3$. In view of the logarithmic growth of $h_2$ for $n = 2$, the two dimensional case is somewhat different from the higher dimensional case. In the sequel we restrict ourselves to the case $n \geq 3$. We shall state the case $n = 2$ in the final section.

Let us consider first the case when $f$ has compact support. Consider the boundary condition at $\infty$:

$$\lim_{|x|\to\infty} u(x) = 0. \tag{3}$$

Then (1)-(3) has a unique solution.

Maremonti and Solonnikov [6, Theorems 1 and 2] proved the following theorem.

**Theorem A.** *Let $n \geq 3$ and $w \equiv 1$. Suppose $f \in L^p(\Omega)$ has compact support. Then* (1)-(3) *has a unique solution $u$. Moreover, the estimate $\|u\|_{BL^{2,p}(\Omega)} \leq c\|f\|_{p;\Omega}$ holds in each one of the following cases:*

(i) $1 < p < n/2$.

(ii) $n/2 \leq p < n$ *and $f$ satisfies*

$$\int_\Omega f(x)h_2(x)dx = 0. \tag{4}$$

(iii) $p \geq n$ *and $f$ satisfies* (4) *and*

$$\int_\Omega f(x)\frac{\partial h_2}{\partial x_j}dx = 0 \text{ for } j = 1, \ldots, n. \tag{5}$$

In [1] we have introduced a subclass $A_{p,k}$ of the Muckenhoupt $A_p$ class:

$$A_{p,k} = \left\{ w \in A_p : \int_{\mathbb{R}^n} (1 + |x|)^{(k-n)p/(p-1)} w(x)^{1/(1-p)} dx < \infty \right\}.$$

It is easy to see that

$$\emptyset = A_{p,n} \subset A_{p,n-1} \subset \cdots \subset A_{p,1} \subset A_{p,0} = A_p;$$

$$\begin{aligned} 1 \in A_{p,2} &\iff 1 < p < n/2, \\ 1 \in A_{p,1} \setminus A_{p,2} &\iff n/2 \leq p < n, \\ 1 \in A_p \setminus A_{p,1} &\iff p \geq n \end{aligned}$$

(cf. [1, §4]). In view of these facts, we shall show the following generalization of Theorem A.

**Theorem 1.** *Let $n \geq 3$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$ has compact support. Then* (1)-(3) *has a unique solution $u$. Moreover, the estimate*

$$\|u\|_{BL^{2,p,w}(\Omega)} \leq c\|f\|_{p,w;\Omega}. \tag{6}$$

*holds in each one of the following cases:*

(i) $w \in A_{p,2}$.

(ii) $w \in A_{p,1} \setminus A_{p,2}$ *and $f$ satisfies* (4).

(iii) $w \in A_p \setminus A_{p,1}$ *and $f$ satisfies* (4) *and* (5).

Let us consider next the case when $f$ does not necessarily have compact support. In this case we cannot expect a solution of (1)-(3). In order to obtain a solution of (1)-(2) small at $\infty$ in a sense, we define the subspace $BL_0^{2,p,w}(\Omega)$ of $BL^{2,p,w}(\Omega)$ by

$$BL_0^{2,p,w}(\Omega) = \{u \in BL^{2,p,w}(\Omega) : \text{there is } u_j \in BL^{2,p,w}(\Omega) \\ \text{such that } u_j(x) = 0 \text{ for } |x| > j \text{ and } \lim_{j\to\infty} \|u_j - u\|_{BL^{2,p,w}(\Omega)} = 0\}.$$

A solution $u \in BL_0^{2,p,w}(\Omega)$ of (1)-(2) may be considered to be small at $\infty$. In view of (*) and the usual trace argument, we see that the first derivatives of $u$ exist on $\partial\Omega$, and they are $q$-th integrable with respect to the surface measure $dS$ for some $q > 1$ (see the following Lemma 2). In particular, we can consider the integral of the normal derivative $\partial u/\partial n$ over $\partial\Omega$. We shall prove the following.

**Theorem 2.** *Let $n \geq 3$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$.*

(i) *If $w \in A_{p,2}$, then there exists a unique solution $u \in BL_0^{2,p,w}(\Omega)$ of* (1)-(2).

(ii) *If $w \in A_{p,1} \setminus A_{p,2}$, then there exists a unique solution $u \in BL_0^{2,p,w}(\Omega)$ of* (1)-(2) *satisfying the additional condition:*

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} h_2 dS = 0. \tag{7}$$

(iii) *If $w \in A_p \setminus A_{p,1}$, then there exists a unique solution $u \in BL_0^{2,p,w}(\Omega)$ of* (1)-(2) *satisfying* (7) *and*

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\frac{\partial h_2}{\partial x_j} dS = 0 \textit{ for } j = 1, \ldots, n. \tag{8}$$

*In each case the solution $u$ satisfies* (6).

Theorem 2 is a generalization of Maremonti and Solonnikov [6, Theorem 3]. However, there is a significant difference between [6] and ours. A layer potential method and a certain approximation property for $L^p(\Omega)$ were the main tools in [6]. Since $w \in A_p$ needs to be neither continuous nor isotropic, the arguments in [6] are not applicable to our weighted case. We shall make use of the same technique as in [1], e.g. modified Riesz potentials, an approximation of polynomials in $BL^{2,p,w}(\Omega)$ and so on. We shall, in fact, give an explicit representation of the solutions $u$ of (1)-(2). Although the solutions are not unique, among them there exists a canonical solution which will be written as the difference of a modified Riesz potential and its Poisson integral with respect to $\Omega$. We shall show that this canonical solution satisfies (6). Conditions (4), (5), (7) and (8) will imply that the solution considered in each statement must coincide with the canonical one.

## 2. Preliminaries

In this section we collect some basic $L^p$ estimates which are essentially based on singular integrals and hence applicable to our weighted version in a straightforward fashion. We shall give no proofs since they may be well-known or easy to prove. We denote by $c$ for a positive constant depending only on $n, p, w$ and domains whose value may change from one occurrence to the next.

Let us recall first an extension property of $BL^{2,p,w}(\Omega)$. Since the weighted Beppo Levi space $BL^{2,p,w}(\Omega)$ and the weighted Sobolev space $W^{2,p,w}(\Omega)$ have the same local behavior, it follows from Calderón's extension theorem (see [7, Chapter VI, 4.8]) that a function in $BL^{2,p,w}(\Omega)$ extends to $\mathbb{R}^n$. We have

**Lemma 1.** *If $u \in BL^{2,p,w}(\Omega)$, then there exists $u^* \in BL^{2,p,w}(\mathbb{R}^n)$ such that $u^* = u$ on $\Omega$ and $\|u^*\|_{BL^{2,p,w}(\mathbb{R}^n)} \le c\|u\|_{BL^{2,p,w}(\Omega)}$.*

We need also a result for the restriction of elements of $BL^{2,p,w}(\Omega)$ to $\partial\Omega$. Since $w$ may degenerate on $\partial\Omega$, it is, in general, impossible to define an appropriate weighted space over $\partial\Omega$ associated with $BL^{2,p,w}(\Omega)$. We can, however, give a coarse result, which is essentially unweighted. In view of (*) and [7, Chapter VI, 4.2] we have

**Lemma 2.** (i) *Let $1 < q < \infty$. If $f \in W^{1,q}(\Omega_1)$, then the trace of $f$ on $\partial\Omega$ is defined and*

$$\|f\|_{L^q(\partial\Omega)} \le c\|f\|_{W^{1,q}(\Omega_1)}.$$

(ii) *Let $w \in A_p$. Then there exists $q > 1$ such that if $u \in BL^{2,p,w}(\Omega)$, then the traces of $u$ and $\nabla u$ on $\partial\Omega$ are defined and*

$$\|u\|_{L^q(\partial\Omega)} + \|\nabla u\|_{L^q(\partial\Omega)} \le c\|u\|_{BL^{2,p,w}(\Omega)}.$$

Let us recall $L^p$ estimates for solutions of the Dirichlet problem in a bounded $C^{1,1}$ domain.

**Lemma 3.** (cf. [4, Theorem 9.13 and Lemma 9.17]) *Let $D$ be a bounded domain in $\mathbb{R}^n$ with a $C^{1,1}$ boundary portion $T \subset \partial D$. Let $u \in W^{2,p,w}(D)$ be a solution of $\Delta u = f$ in $D$ with $u = 0$ on $T$ in the sense of $W^{1,q}(D)$ for some $q > 1$. Then for any domain $D' \Subset D \cup T$,*

$$\|u\|_{W^{2,p,w}(D')} \le c(\|u\|_{p,w;D} + \|f\|_{p,w;D}).$$

*Moreover, if $T = \partial D$, then $\|u\|_{W^{2,p,w}(D')} \le c\|f\|_{p,w;D}$.*

Note that the boundary condition in Lemma 3 is weaker than that in [4, Theorem 9.13]. However, we infer from a careful observation of [4, Lemma 9.12] that the conclusion remains true.

## 3. Modified Riesz potentials

Observe that that $-\Delta h_2$ is the Dirac measure at the origin, so that $-\Delta(h_2 * f) = f$ if the convolution $h_2 * f$ exists. Therefore the first attempt to solve (1)-(2) begins with consideration on the Riesz potential $h_2 * f$. However, a problem arises: the convolution $h_2 * f$ does not necessarily exist for all $f \in L^{p,w}(\Omega)$. This difficulty can be overcome by means of modified Riesz potentials. Observe that if $y \neq 0$, then $h_2(x-y)$ has a multiple power series expansion in $x_1, x_2, \dots, x_n$, convergent in a neighborhood of the origin. We write

$$h_2(x-y) = \sum_{\nu=0}^{\infty} a_\nu(x,y),$$

where, for fixed $\nu$ and $y \neq 0, a_\nu(x,y) = \sum_{|\beta|=\nu} \frac{x^\beta}{\beta!} D^\beta h_2(-y)$ is a homogeneous polynomial in $x_1$ to $x_n$ of degree $\nu$ and continuous in $x, y$ jointly for $y \neq 0$ (see [5, Chapter 4]). We set

$$h_{2,k}(x,y) = \sum_{\nu=k}^{\infty} a_\nu(x,y)$$

for $k \geq 0$ and write

$$I_{2,k}(f) = \int_{\mathbb{R}^n} h_{2,k}(\cdot, y) f(y) dy,$$

whenever the right hand side has a meaning. For notational convenience we let $I_{2,0}(f) = I_2(f)$. By definition $I_2(f)$ coincides with the convolution $h_2 * f$.

We collect some properties of modified Riesz potentials $I_{2,k}(f)$. No proofs will be given. We refer to [1]. Note that if $f \in L^{p,w}(\mathbb{R}^n)$, $0 \leq k \leq 2$ and $I_{2,k}(f)$ exists, then $D^\alpha I_{2,k}(f) = (D^\alpha h_2) * f$ for $|\alpha| \geq k$ and

$$\sum_{|\alpha|=2} \|D^\alpha I_{2,k}(f)\|_{p,w;\Omega} \leq c\|f\|_{p,w;\Omega} \tag{9}$$

(cf. [2, Theorems I and III] and [1, Lemma 8]). In particular, $u = I_{2,2}(f)$ satisfies (1). We see that

$$\begin{aligned} |h_2(x-y) - h_{2,k}(x,y)| &\leq c \sum_{\nu=0}^{k-1} |x|^\nu |y|^{2-n-\nu}; \\ |h_{2,k}(x,y)| &\leq c|x|^k |y|^{2-n-k} \qquad \text{for } 2|x| > |y| \end{aligned} \tag{10}$$

(cf. [5, Lemmas 4.1 and 4.2] and [1, Lemma 6]). We infer from Hölder's inequality that $w \in A_{p,k}$ if and only if

$$\int_{\mathbb{R}^n} (1+|x|)^{2-n-k} |f(x)| dx \leq c\|f\|_{p,w;\mathbb{R}^n} \tag{11}$$

(cf. [1, Theorem 5]). Hence we obtain from (10) and (11) that $I_{2,k}(f)$ exists for every $f \in L^{p,w}(\Omega)$ if and only if $w \in A_{p,2-k}$. In particular, $I_{2,2}(f)$ exists for every $f \in L^{p,w}(\Omega)$. Therefore we have

**Lemma 4.** ([1, Theorem 4]) *Let* $w \in A_p$. *If* $u \in BL^{2,p,w}(\mathbb{R}^n)$, *then there are constants* $a$ *and* $b_j$ *such that* $u = I_{2,2}(-\Delta u) + a + \sum_{j=1}^n b_j x_j$.

Let us write

$$BL_0^{2,p,w}(\mathbb{R}^n) = \{u \in BL^{2,p,w}(\mathbb{R}^n) : \\ \text{there is } u_j \in C_0^\infty(\mathbb{R}^n) \text{ such that } \lim_{j\to\infty} \|u_j - u\|_{BL^{2,p,w}(\mathbb{R}^n)} = 0\}.$$

We have given a characterization of $BL_0^{2,p,w}(\mathbb{R}^n)$.

**Lemma 5.** ([1, Corollary]) *Let* $w \in A_p$.

(i) $BL_0^{2,p,w}(\mathbb{R}^n) = \{I_2(g) : g \in L^{p,w}(\mathbb{R}^n)\} \iff w \in A_{p,2}$.
(ii) $BL_0^{2,p,w}(\mathbb{R}^n) = \{I_{2,1}(g) + a : g \in L^{p,w}(\mathbb{R}^n), a \in \mathbb{R}\} \iff w \in A_{p,1} \setminus A_{p,2}$.
(iii) $BL_0^{2,p,w}(\mathbb{R}^n) = \{I_{2,2}(g) + a + \sum_{j=1}^n b_j x_j : g \in L^{p,w}(\mathbb{R}^n), a, b_j \in \mathbb{R}\} \iff w \in A_p \setminus A_{p,1}$.

In the proof of Lemma 5 we have used the following approximation property. This may be regarded as an alternative of [6, Lemma 2].

**Lemma 6.** (cf. [1, Lemma 18]) *Let* $w \in A_p$. *Suppose* $\varepsilon > 0$ *and* $R > 0$.

(i) *If* $w \in A_{p,1} \setminus A_{p,2}$, *then there is* $g \in L^{p,w}(\mathbb{R}^n)$ *with compact support such that* $g(x) = 0$ *for* $|x| < R$, $\|g\|_{p,w;\mathbb{R}^n} < \varepsilon$ *and* $I_2(g)(0) = 1$.
(ii) *If* $w \in A_p \setminus A_{p,1}$, *then there are* $g_j \in L^{p,w}(\mathbb{R}^n)$, $j = 1, \ldots, n$, *with compact support such that* $g_j(x) = 0$ *for* $|x| < R$, $\|g_j\|_{p,w;\mathbb{R}^n} < \varepsilon$ *and*

$$\frac{\partial}{\partial x_i} I_2(g_j)(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

## 4. Representation of solutions

In this section we shall give an explicit representation of solutions of (1)-(2). We have observed in the last section that $I_{2,2}(f)$ satisfies (1). In order to obtain a solution of (1)-(2), we subtract from $I_{2,2}(f)$ a harmonic function in $\Omega$ having the same boundary values. This harmonic function will be given by the Poisson integral. Let $G(x, y)$ be the Green function for $\Omega$, that is, $G(x, y)$ satisfies

(i) $G(\cdot, y) - h_2(\cdot - y)$ is harmonic on $\Omega$;
(ii) $G(\cdot, y)$ vanishes on $\partial\Omega$;
(iii) if $n \geq 3$, then $G(\cdot, y)$ tends to zero at $\infty$; if $n = 2$, then $G(\cdot, y)$ is bounded at $\infty$.

For $g \in L^1(\partial\Omega)$ we define the Poisson integral by

$$PI(g) = \frac{1}{\gamma_n} \int_{\partial\Omega} \frac{\partial G(x, y)}{\partial n_y} g(y) dS(y),$$

where $n_y$ is the inward normal unit vector on $\partial\Omega$ and $dS(y)$ stands for the surface element. We observe that $PI(g)$ is harmonic in $\Omega$, and that if $g$ is continuous, then $PI(g) = g$ on $\partial\Omega$. Note that $PI(g)$ tends to zero at $\infty$ if $n \geq 3$; $PI(g)$ is bounded at $\infty$ if $n = 2$. We remark that

$$\begin{aligned} &PI(1) < 1 \quad \text{and} \quad PI(h_2) \equiv h_2 \quad \text{if } n \geq 3; \\ &PI(1) \equiv 1 \quad \text{and} \quad PI(h_2) < h_2 \quad \text{if } n = 2. \end{aligned} \tag{12}$$

By Lemma 2 the trace of $I_{2,k}(f)$ on $\partial\Omega$ belongs to $L^q(\partial\Omega)$. We write $PI(I_{2,k}(f))$ for the Poisson integral of the trace $I_{2,k}(f)$ on $\partial\Omega$. One may expect that $I_{2,2}(f) - PI(I_{2,2}(f))$ is a solution of (1)-(2). This is, in fact, the case.

**Theorem 3.** *Let $n \geq 3$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$. Then* (1)-(2) *has a solution in $BL^{2,p,w}(\Omega)$. Every solution $u$ of (1)-(2) in $BL^{2,p,w}(\Omega)$ is represented as*

$$u = I_{2,2}(f) - PI(I_{2,2}(f)) + a(1 - PI(1)) + \sum_{j=1}^{n} b_j(x_j - PI(x_j)),$$

*where $a$ and $b_j$ are constants. Moreover, $u = I_{2,2}(f) - PI(I_{2,2}(f))$ satisfies* (6).

Let us consider first a special case of Theorem 3 when $f \equiv 0$.

**Lemma 7.** *Let $n \geq 3$. If $h = a(1 - PI(1)) + \sum_{j=1}^{n} b_j(x_j - PI(x_j))$, then $h$ is a harmonic function in $BL^{2,p,w}(\Omega)$ satisfying* (2), *and vice versa. Moreover, $\|h\|_{BL^{2,p,w}(\Omega)} \leq c(|a| + \sum_{j=1}^{n} |b_j|)$.*

*Proof.* It is easy to see that $h = a(1 - PI(1)) + \sum_{j=1}^{n} b_j(x_j - PI(x_j))$ is a harmonic function vanishing continuously on $\partial\Omega$, and hence satisfying (2). Since $D^\alpha PI(g)(x) = O(|x|^{2-n-|\alpha|})$ as $|x| \to \infty$ for $g \in L^1(\partial\Omega)$, it follows from (11) that

$$\sum_{|\alpha|=2} \|D^\alpha h\|_{p,w;\Omega\setminus\Omega_1} \leq c(|a| + \sum_{j=1}^{n} |b_j|).$$

We have from Lemma 3

$$\|h\|_{W^{2,p,w}(\Omega_1)} \leq c(|a| + \sum_{j=1}^{n} |b_j|),$$

whence $h \in BL^{2,p,w}(\Omega)$ and the required norm estimate follows.

Conversely, suppose $h \in BL^{2,p,w}(\Omega)$ is a harmonic function in $\Omega$ such that $h = 0$ on $\partial\Omega$ in the sense of $W^{1,q}(\Omega_1)$ for some $q > 1$. In view of [4, Lemma 9.16] we see that $h = 0$ on $\partial\Omega$ in the sense of $W^{1,q}(\Omega_1)$ for *any* $q > 1$, and hence $h = 0$ on $\partial\Omega$ continuously. By Lemma 1 we extend $h$ to $\mathbb{R}^n$ so that the extension $h^*$ belongs to $BL^{2,p,w}(\mathbb{R}^n)$. From Lemma 4 we can find $a$ and $b_j$ such that $h^* = I_{2,2}(-\Delta h^*) + a + \sum_{j=1}^n b_j x_j$. Since $\Delta h^*$ is concentrated on $\mathbb{R}^n \setminus \Omega$, it follows $I_2(-\Delta h^*)$ exists and $h^* = I_2(-\Delta h^*) + a + \sum_{j=1}^n b_j x_j$ with different constants $a$ and $b_j$. Observe that $I_2(-\Delta h^*)$, $PI(1)$ and $PI(x_j)$ tend to zero at $\infty$, and hence

$$\lim_{|x|\to\infty,\, x\in\Omega} h(x) - (a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j))) = 0.$$

Therefore the maximum principle yields $h = a(1-PI(1))+\sum_{j=1}^n b_j(x_j-PI(x_j))$.

*Proof of Theorem 3.* Let $u_0 = I_{2,2}(f) - PI(I_{2,2}(f))$. We have seen that $u_0$ satisfies (1). In order to prove (2) let $\{f_\varepsilon\}$ be a regularization of $f$ such that $f_\varepsilon \to f$ in $L^{p,w}(\Omega)$. Since $I_{2,2}(f_\varepsilon)$ is continuous, it follows that the continuous function $u_\varepsilon = I_{2,2}(f_\varepsilon) - PI(I_{2,2}(f_\varepsilon))$ satisfies

$$\begin{aligned} -\Delta u_\varepsilon &= f_\varepsilon \quad \text{on } \Omega, \\ u_\varepsilon &= 0 \quad\ \text{on } \partial\Omega. \end{aligned}$$

Let $1 < q < n/(n-1)$. We have from (10)

$$\int_{\Omega_1} |h_{2,2}(x,y)|^q dx \le c(1+|y|)^{-nq}.$$

Hence Minkowski's inequality for integrals (see [7, p.271]) and (11) yield

$$\|I_{2,2}(f_\varepsilon)\|_{q;\Omega_1} \le c\|f_\varepsilon\|_{p,w;\Omega}. \tag{13}$$

Similarly $\|\frac{\partial}{\partial x_j} I_{2,2}(f_\varepsilon)\|_{q;\Omega_1} \le c\|f_\varepsilon\|_{p,w;\Omega}$. By Lemma 2 the trace of $I_{2,2}(f_\varepsilon)$ on $\partial\Omega$ satisfies

$$\|I_{2,2}(f_\varepsilon)\|_{L^1(\partial\Omega)} \le c\|f_\varepsilon\|_{p,w;\Omega}.$$

By the estimate $\partial G(x,y)/\partial n_y \le c\delta(x)|x-y|^{-n}$ with $\delta(x) = \operatorname{dist}(x,\partial\Omega)$ for $x, y$ near the boundary (cf. [8]), we have $\sup_{y\in\partial\Omega} \int_{\Omega_1} |\partial G(x,y)/\partial n_y|^q \, dx < \infty$. Hence Minkowski's inequality for integrals yields

$$\|PI(I_{2,2}(f_\varepsilon))\|_{q;\Omega_1} \le c\|I_{2,2}(f_\varepsilon)\|_{L^1(\partial\Omega)} \le c\|f_\varepsilon\|_{p,w;\Omega}.$$

Therefore we infer from (13) and Lemma 3 that

$$\|u_\varepsilon\|_{W^{2,q}(\Omega_1)} \le c\|f_\varepsilon\|_{p,w;\Omega}.$$

In particular, $u_\varepsilon \to u_0$ in $W^{1,q}(\Omega_1)$, so that $u_0 = 0$ in the sense of $W^{1,q}(\Omega_1)$. Thus (2) holds. Consequently, $u = I_{2,2}(f) - PI(I_{2,2}(f)) + a(1 - PI(1)) + \sum_{j=1}^{n} b_j(x_j - PI(x_j))$ is a solution of (1)-(2) and vice versa by Lemma 7.

Finally let us prove the norm estimate for $u_0$. To this end let $r_2 > r_1$, $\Omega_2 = \{x \in \Omega : |x| < r_2\}$ and take $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \eta \le 1$ on $\mathbb{R}^n$, $\eta = 1$ on $\{|x| \le r_1\}$ and $\eta = 0$ on $\{|x| \ge r_2\}$. Let $u_1 = \eta u_0$ and $u_2 = (1-\eta)u_0$. An elementary calculation shows that

$$\|\Delta u_1\|_{p,w;\Omega} \le c(\|f\|_{p,w;\Omega} + \|\nabla u_0\|_{p,w;\Omega_2\setminus\Omega_1} + \|u_0\|_{p,w;\Omega_2\setminus\Omega_1}).$$

By (10) we can compare $D^\alpha I_{2,2}(f)$ ($|\alpha| \le 1$) and the maximal function $Mf$ locally, and obtain

$$\sum_{|\alpha|\le 1} \|D^\alpha I_{2,2}(f)\|_{p,w;\Omega_2} \le c\|f\|_{p,w;\Omega}.$$

By the estimate of the Green function we have

$$\sup_{x\in\Omega_2\setminus\Omega_1} |D^\alpha(PI(I_{2,2}(f)))(x)| \le c\|I_{2,2}(f)\|_{L^1(\partial\Omega)} \le c\|f\|_{p,w;\Omega}$$

for $|\alpha| \le 1$. Hence

$$\|\Delta u_1\|_{p,w;\Omega} \le c\|f\|_{p,w;\Omega}. \tag{14}$$

Since $u_1$ vanishes on $\{|x| = r_2\}$ continuously and on $\partial\Omega_1$ in the sense of $W^{1,q}(\Omega_2)$ for $1 < q < n/(n-1)$, it follows from Lemma 3 that

$$\|u_1\|_{W^{2,p,w}(\Omega_2)} \le c\|f\|_{p,w;\Omega}.$$

By Lemma 1 we extend $u_2$ to $\mathbb{R}^n$ and represent it on $\Omega$ as $u_2 = I_{2,2}(-\Delta u_2) + a + \sum_{j=1}^{n} b_j x_j$. Then by (9) and (14)

$$\sum_{|\alpha|=2} \|D^\alpha u_2\|_{p,w;\Omega} \le c\|\Delta u_2\|_{p,w;\Omega} = c\|\Delta u - \Delta u_1\|_{p,w;\Omega} \le c\|f\|_{p,w;\Omega}.$$

Consequently, (6) holds for $u = u_0$. The proof is complete.

Since $u \in BL_0^{2,p,w}(\Omega)$ extends to $u^* \in BL_0^{2,p,w}(\mathbb{R}^n)$ by Lemma 1, we have from Lemma 5 the following theorem.

**Theorem 4.** *Let $n \geq 3$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$. Then every solution $u$ of* (1)-(2) *in $BL_0^{2,p,w}(\Omega)$ is represented as follows:*

(i) *If $w \in A_{p,2}$, then $u = I_2(f) - PI(I_2(f))$.*
(ii) *If $w \in A_{p,1} \setminus A_{p,2}$, then $u = I_{2,1}(f) - PI(I_{2,1}(f)) + a(1 - PI(1))$, where $a$ is a constant.*
(iii) *If $w \in A_p \setminus A_{p,1}$, then $u = I_{2,2}(f) - PI(I_{2,2}(f)) + a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j))$, where $a$ and $b_j$ are constants.*

*In each case the canonical solution $I_{2,k}(f) - PI(I_{2,k}(f))$, ($k = 0, 1, 2$ for* (i), (ii), (iii), *respectively), satisfies* (6).

*Proof.* Only (6) may require a proof. We have observed (6) for (iii) in Theorem 3. Suppose $w \in A_{p,2}$. Writing

$$
\begin{aligned}
I_{2,2}(f) &= I_2(f) - (\int_\Omega h_2(-y)f(y)dy + \sum_{j=1}^n x_j \int_\Omega \frac{\partial h_2}{\partial x_j}(-y)f(y)dy) \\
&= I_2(f) - (a' + \sum_{j=1}^n b'_j x_j),
\end{aligned}
$$

we obtain from (11) that $|a'| \leq c\|f\|_{p,w;\Omega}$, $|b'_j| \leq c\|f\|_{p,w;\Omega}$. Hence from Theorem 3 and Lemma 7 we have

$$
\begin{aligned}
&\|I_2(f) - PI(I_2(f))\|_{BL^{2,p,w}(\Omega)} \\
&\leq \|I_{2,2}(f) - PI(I_{2,2}(f))\|_{BL^{2,p,w}(\Omega)} + \left\| a'(1 - PI(1)) + \sum_{j=1}^n b'_j(x_j - PI(x_j)) \right\|_{BL^{2,p,w}(\Omega)} \\
&\leq c\|f\|_{p,w;\Omega}.
\end{aligned}
$$

Thus (6) follows for (i). We can prove (6) similarly for (ii).

## 5. Proof of Theorems 1 and 2

*Proof of Theorem 1.* Since $f$ is of compact support, it follows that $I_2(f)$ exists, $\lim_{|x|\to\infty} I_2(f)(x) = 0$ and

$$
\begin{aligned}
I_2(f) &= I_{2,1}(f) + \int_\Omega h_2(-y)f(y)dy \\
&= I_{2,2}(f) + \int_\Omega h_2(-y)f(y)dy + \sum_{j=1}^n x_j \int_\Omega \frac{\partial h_2}{\partial x_j}(-y)f(y)dy.
\end{aligned}
$$

Hence Theorem 3 and the maximum principle say that $u = I_2(f) - PI(I_2(f))$ is a unique solution of (1)-(3) in all cases of (i)-(iii). Conditions (4) and (5) imply that $u$ coincides with the canonical solutions $I_{2,1}(f) - PI(I_{2,1}(f))$ and $I_{2,2}(f) - PI(I_{2,2}(f))$ in the cases of (ii) and (iii), respectively. Therefore the norm estimate follows from Theorem 4.

*Proof of Theorem 2.* Since (i) has been proved in Theorem 4, we shall prove (ii) and (iii). Let us prove first the unicity. To this end it is sufficient to show that if $v = 1 - PI(1), v_i = x_i - PI(x_i)$, then

$$\int_{\partial\Omega} \frac{\partial v}{\partial n} h_2 dS = 1, \tag{15}$$
$$\int_{\partial\Omega} \frac{\partial v_i}{\partial n} \frac{\partial h_2}{\partial x_j} dS = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $R > 0$ be sufficiently large. Then the Green formula yields

$$\int_{\partial\Omega} \frac{\partial v}{\partial n} h_2 dS = \int_{|x|=R} \left( \frac{\partial v}{\partial n} h_2 - \frac{\partial h_2}{\partial n} v \right) dS.$$

Since $PI(1) = O(|x|^{2-n})$ and $\nabla PI(1) = O(|x|^{1-n})$, it follows that

$$\int_{\partial\Omega} \frac{\partial v}{\partial n} h_2 dS = - \lim_{R\to\infty} \int_{|x|=R} \frac{\partial h_2}{\partial n} dS = 1.$$

A similar calculation shows the second assertion.

What remains is to prove that if $w \in A_{p,1} \setminus A_{p,2}$, then $u = I_{2,1}(f) - PI(I_{2,1}(f))$ satisfies (7); and that if $w \in A_p \setminus A_{p,1}$, then $u = I_{2,2}(f) - PI(I_{2,2}(f))$ satisfies (7) and (8). Suppose $w \in A_{p,1} \setminus A_{p,2}$. Let $u = I_{2,1}(f) - PI(I_{2,1}(f))$. Take $\varepsilon > 0$. Split $u$ into $u_1 + u_2$, where $u_j = I_{2,1}(f_j) - PI(I_{2,1}(f_j))$, $f_1 = \chi_{|x|<R} f$ and $f_2 = \chi_{|x|>R} f$. Take $R > 0$ so large that $\|f_2\|_{p,w;\Omega} < \varepsilon$ and $\|D^\alpha I_{2,1}(f_2)\|_{\infty;\Omega_1} < \varepsilon$ for $|\alpha| \le 2$. Then

$$\left| \int_{\partial\Omega} \frac{\partial u_2}{\partial n} h_2 dS \right| < c\varepsilon.$$

Observe that

$$u_1 = I_2(f_1) - PI(I_2(f_1)) - a(1 - PI(1)),$$

where $a = \int_\Omega h_2 f_1 dx$. By Lemma 6 we find a compactly supported function $g \in L^{p,w}(\Omega)$ such that

$$g(x) = 0 \text{ for } |x| < r_2 \text{ with } r_2 > r_1,$$
$$\|g\|_{p,w;\Omega} < \varepsilon,$$
$$I_2(g)(0) = \int h_2 g dx = 1,$$
$$|I_2(g) - 1| < \varepsilon \text{ on } \Omega_1,$$
$$|u_1 - u_3| < \varepsilon \text{ on } \Omega_1,$$

where $u_3 = I_2(f_1 - ag) - PI(I_2(f_1 - ag))$. Observe that $u_1 - u_3$ is harmonic on $\Omega_2 = \{x \in \Omega : |x| < r_2\}$ and vanishes continuously on $\partial\Omega$. Hence,

$$\|\nabla(u_1 - u_3)\|_{\infty;\Omega_1} \le c\|u_1 - u_3\|_{\infty;\Omega_2} \le c\varepsilon$$

(see [8, Theorem 2.4 and its proof]); in particular

$$\left\| \frac{\partial u_1}{\partial n} - \frac{\partial u_3}{\partial n} \right\|_{\infty;\partial\Omega} \le c\varepsilon.$$

Since $u_3$ tends to zero at $\infty$, it follows from the Green formula that

$$\int_{\partial\Omega} \frac{\partial u_3}{\partial n} h_2 dS = \int_\Omega \Delta u_3 h_2 dx = -\int_\Omega (f_1 - ag) h_2 dx = 0.$$

Therefore

$$\left| \int_{\partial\Omega} \frac{\partial u_1}{\partial n} h_2 dS \right| < c\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (7) follows. In the case of (iii) we can prove (7) and (8) by using the functions $g_j$ appearing in Lemma 6. Details are left to the reader. The proof is complete.

## 6. Two dimensional case

Now let us consider the two dimensional case. Hereafter we let $n = 2$. In view of the Phragmén-Lindelöf principle, the boundary condition at $\infty$ becomes

(3’) $$\limsup_{|x|\to\infty} |u(x)| < \infty.$$

We see from (12) that $1 - PI(1)$ for $n \ge 3$ is replaced by $h_2 - PI(h_2)$. Note also that if $f$ has compact support, then

$$I_{2,1}(f) - PI(I_{2,1}(f)) = I_2(f) - PI(I_2(f));$$

$$\lim_{|x|\to\infty} (I_2(f)(x) - ch_2(x)) = 0 \quad \text{with } c = \int_\Omega f dx.$$

Moreover, since $PI(h_2)$ is harmonic at $\infty$, it follows from [3, (2.73) Proposition] that

$$\lim_{R\to\infty} \int_{|x|=R} \frac{\partial}{\partial n} PI(h_2) dS = 0.$$

Hence we have an alternative of (15):

$$\int_{\partial\Omega} \frac{\partial}{\partial n} (h_2 - PI(h_2)) dS = 1.$$

Using these facts, we obtain the following counterparts of Theorems 1-4. Since $A_{p,2} = \emptyset$, we have two cases in each theorem.

**Theorem 1'.** *Let $n = 2$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$ has compact support. Then* (1)-(2) *and* (3') *have a unique solution $u$. Moreover,* (6) *holds in each one of the following cases:*

(i) *$w \in A_{p,1} \setminus A_{p,2}$ and $f$ satisfies*

(4')
$$\int_\Omega f(x)dx = 0.$$

(ii) *$w \in A_p \setminus A_{p,1}$ and $f$ satisfies* (4') *and* (5).

**Theorem 2'.** *Let $n = 2$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$.*

(i) *If $w \in A_{p,1} \setminus A_{p,2}$, then there exists a unique solution $u \in BL_0^{2,p,w}(\Omega)$ of* (1)-(2) *satisfying the additional condition:*

(7')
$$\int_{\partial\Omega} \frac{\partial u}{\partial n} dS = 0.$$

(ii) *If $w \in A_p \setminus A_{p,1}$, then there exists a unique solution $u \in BL_0^{2,p,w}(\Omega)$ of* (1)-(2) *satisfying* (7') *and* (8).

*In each case the solution $u$ satisfies* (6).

**Theorem 3'.** *Let $n = 2$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$. Then* (1)-(2) *has a solution in $BL^{2,p,w}(\Omega)$. Every solution $u$ of* (1)-(2) *in $BL^{2,p,w}(\Omega)$ is represented as*

$$u = I_{2,2}(f) - PI(I_{2,2}(f)) + c(h_2 - PI(h_2)) + \sum_{j=1}^{n} b_j(x_j - PI(x_j)),$$

*where $c$ and $b_j$ are constants. Moreover, $u = I_{2,2}(f) - PI(I_{2,2}(f))$ satisfies* (6).

**Theorem 4'.** *Let $n = 2$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$. Then every solution $u$ of* (1)-(2) *in $BL_0^{2,p,w}(\Omega)$ is represented as follows:*

(i) *If $w \in A_{p,1} \setminus A_{p,2}$, then $u = I_{2,1}(f) - PI(I_{2,1}(f)) + c(h_2 - PI(h_2))$, where $c$ is a constant.*

(ii) *If $w \in A_p \setminus A_{p,1}$, then $u = I_{2,2}(f) - PI(I_{2,2}(f)) + c(h_2 - PI(h_2)) + \sum_{j=1}^n b_j(x_j - PI(x_j))$, where $c$ and $b_j$ are constants.*

*In each case the canonical solution $I_{2,k}(f) - PI(I_{2,k}(f))$, ($k = 1, 2$ for* (i), (ii) *respectively), satisfies* (6).

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