

## SOME ASPECTS ON CIRCLES AND HELICES IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. In this note we study helices in a complex projective space. We characterize complex projective spaces among Hermitian symmetric spaces by the property that all holomorphic circles are closed. We also give examples of helices with multiple points.

### 1. Introduction.

A smooth curve  $\gamma = \gamma(s)$  parametrized by its arclength  $s$  is called a *helix* of order  $d$  if there exist orthonormal frame fields  $\{V_1 = \dot{\gamma} = \frac{d\gamma}{ds}, \dots, V_d\}$  along  $\gamma$  and positive numbers  $\kappa_1, \dots, \kappa_{d-1}$  which satisfy the following system of ordinary equations

$$(1.1) \quad \nabla_{\dot{\gamma}} V_j(s) = -\kappa_{j-1} V_{j-1}(s) + \kappa_j V_{j+1}(s), \quad j = 1, \dots, d,$$

where  $V_0 \equiv V_{d+1} \equiv 0$  and  $\nabla_{\dot{\gamma}}$  denotes the covariant differentiation along  $\gamma$ . The numbers  $\kappa_j$  ( $j = 1, \dots, d-1$ ) and the orthonormal frame  $\{V_1, \dots, V_d\}$  are called the *curvatures* and the *Frenet frame* of  $\gamma$ , respectively. A helix of order 2 is usually called a *circle*. That is, for a positive constant  $k$ , a circle of curvature  $k$  satisfies the following equations with a unit vector field  $Y = Y(s)$  along  $\gamma$ :

$$(1.2) \quad \begin{cases} \nabla_{\dot{\gamma}} \dot{\gamma} = kY \\ \nabla_{\dot{\gamma}} Y = -k\dot{\gamma}. \end{cases}$$

In a Euclidean space  $\mathbb{R}^3$ , helices of order 3 are ordinary helices and circles are nothing but circles in the sense of Euclidean geometry.

In the recent works [2], [3], [4] and [8], K. Mashimo, K. Tojo and the authors studied circles in details in a Riemannian symmetric space of rank one. We call a smooth curve  $\gamma$  parametrized by its arclength  $s$  *closed* if there exists  $s_0$  ( $\neq 0$ )

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with  $\gamma(s + s_0) = \gamma(s)$  for every  $s$ . The minimal positive  $s_0$  is called the length of this closed curve. A curve is said to be open if it is not closed. In these papers they showed that in symmetric spaces of rank one every circle is a simple curve, and that for arbitrary positive  $k$  there exist many open circles and many closed circles with curvature  $k$ .

In the first half of this paper we provide a characterization of a complex projective space  $\mathbb{C}P^n(c)$  of constant holomorphic sectional curvature  $c$  in the class of compact Hermitian symmetric spaces in terms of some particular circles. In a Kähler manifold  $(M, \langle \cdot, \cdot \rangle, J)$ , circles  $\gamma$  are naturally classified by the index  $\langle \dot{\gamma}, JY \rangle$ , which is constant along  $\gamma$  and is called the *complex torsion* of  $\gamma$ . In classes of circles in a Kähler manifold, the most important one is the class of holomorphic circles. A *holomorphic circle* is a circle satisfying the condition that  $\dot{\gamma}$  and  $Y$  span a holomorphic plane, that is,  $Y = J\dot{\gamma}$  or  $Y = -J\dot{\gamma}$ . For a holomorphic circle  $\gamma$ , the equations (1.2) reduce to

$$(1.3) \quad \nabla_{\dot{\gamma}} \dot{\gamma} = kJ\dot{\gamma} \quad \text{or} \quad \nabla_{\dot{\gamma}} \dot{\gamma} = -kJ\dot{\gamma}.$$

We can interpret such circles in terms of physics (see [1]). Our attempt is to study Kähler manifolds from the viewpoint of real Riemannian geometry.

**Theorem 1.** *Let  $M$  be an  $n$ -dimensional compact Hermitian symmetric space. Suppose that for some  $k > 0$ , every holomorphic circle of curvature  $k$  in  $M$  is closed. Then  $M$  is complex analytically isometric to a complex projective space  $\mathbb{C}P^n$  with Fubini-Study metric of constant holomorphic sectional curvature.*

In a real space form, that is a sphere, a Euclidean space or a hyperbolic space, every helix is a simple curve, that is a curve which does not have multiple points. But the aspect of helices is not the same in a complex projective space. In the latter half of this paper we give examples of closed helices of order 4 with multiple points in a complex projective space.

## 2. Characterization of a complex projective space.

In order to prove Theorem 1 we investigate holomorphic circles in  $M = \mathbb{C}P^1(c_1) \times \mathbb{C}P^1(c_2) (= S^2(c_1) \times S^2(c_2))$  which is the simplest example of a Hermitian symmetric space of rank 2. Here we denote by  $S^n(c)$  an  $n$ -dimensional standard sphere of curvature  $c$ . The following is a key for Theorem 1.

**Proposition 1.** *Let  $\gamma = \gamma(s)$  be a holomorphic circle of curvature  $k$  in  $M = \mathbb{C}P^1(c_1) \times \mathbb{C}P^1(c_2)$  with the initial unit vector  $\dot{\gamma}(0) = (X_1, X_2) \in T\mathbb{C}P^1(c_1) \times T\mathbb{C}P^1(c_2)$ . Then the following hold:*

- (1) *When  $X_1 = 0$  (resp.  $X_2 = 0$ ), the circle  $\gamma$  is a simple closed curve with length  $\frac{2\pi}{\sqrt{k^2+c_2}}$  (resp.  $\frac{2\pi}{\sqrt{k^2+c_1}}$ ).*
- (2) *When  $X_1 \neq 0$  and  $X_2 \neq 0$ , we find*

- (i) If  $\sqrt{\frac{k^2+c_2\|X_2\|^2}{k^2+c_1\|X_1\|^2}}$  is rational, then the circle  $\gamma$  is a simple closed curve. Its length is the least common multiple of  $\frac{2\pi}{\sqrt{k^2+c_1\|X_1\|^2}}$  and  $\frac{2\pi}{\sqrt{k^2+c_2\|X_2\|^2}}$ .
- (ii) If  $\sqrt{\frac{k^2+c_2\|X_2\|^2}{k^2+c_1\|X_1\|^2}}$  is irrational, then the circle  $\gamma$  is a simple open curve.

*Proof.*

(1) Since in  $\mathbb{C}P^1(c) = S^2(c)$  circles of curvature  $k$  are small circles with length  $\frac{2\pi}{\sqrt{k^2+c}}$ , the assertion is obvious (see [11] or [3]).

(2) Let  $\gamma$  be a holomorphic circle in  $M$ . We just treat the case  $\nabla_{\dot{\gamma}}\dot{\gamma} = kJ\dot{\gamma}$ . In this case we find  $\nabla_{\dot{\gamma}_i}\dot{\gamma}_i(s) = kJ\dot{\gamma}_i(s)$  ( $i = 1, 2$ ), where  $\gamma(s) = (\gamma_1(s), \gamma_2(s)) \in \mathbb{C}P^1(c_1) \times \mathbb{C}P^1(c_2)$ . We first get that  $\|\dot{\gamma}_i(s)\| = \|X_i\|$  for every  $s$  ( $i = 1, 2$ ). In fact, we see

$$\begin{aligned} \nabla_{\dot{\gamma}_i}\langle\dot{\gamma}_i(s), \dot{\gamma}_i(s)\rangle &= 2\langle\nabla_{\dot{\gamma}_i}\dot{\gamma}_i(s), \dot{\gamma}_i(s)\rangle \\ &= 2k\langle J\dot{\gamma}_i(s), \dot{\gamma}_i(s)\rangle = 0. \end{aligned}$$

We here set  $\sigma_i(s) = \gamma_i\left(\frac{s}{\|X_i\|}\right)$  so that  $\|\dot{\sigma}_i\| = 1$  ( $i = 1, 2$ ). We then have

$$\begin{aligned} \nabla_{\dot{\sigma}_i}\dot{\sigma}_i(s) &= \frac{1}{\|X_i\|^2}\nabla_{\dot{\gamma}_i}\dot{\gamma}_i\left(\frac{s}{\|X_i\|}\right) \\ &= \frac{k}{\|X_i\|^2}J\dot{\gamma}_i\left(\frac{s}{\|X_i\|}\right) = \frac{k}{\|X_i\|}J\dot{\sigma}_i(s). \end{aligned}$$

This implies that the curve  $\sigma_i = \sigma_i(s)$  is a holomorphic circle of curvature  $\frac{k}{\|X_i\|}$  in  $\mathbb{C}P^1(c_i)$ . Hence, the curve  $\sigma_i$  is a closed curve with length  $\frac{2\pi}{\sqrt{\frac{k^2}{\|X_i\|^2}+c_i}}$  ( $i = 1, 2$ ).

Take a point  $x = (x_1, x_2)$  on the circle  $\gamma$  in  $M$ . If  $x$  moves along  $\gamma$  with velocity 1, then the point  $x_i$  moves along the curve  $\sigma_i$  with velocity  $\|X_i\|$ . Hence the circle  $\gamma$  is closed if and only if the ratio  $\frac{2\pi}{\|X_1\|\sqrt{\frac{k^2}{\|X_1\|^2}+c_1}} : \frac{2\pi}{\|X_2\|\sqrt{\frac{k^2}{\|X_2\|^2}+c_2}} = \sqrt{\frac{k^2+c_2\|X_2\|^2}{k^2+c_1\|X_1\|^2}}$  is rational. Thus we get the conclusion.  $\square$

As an immediate consequence of Proposition 1 we find the following:

**Corollary.** *If a Kähler manifold  $M$  admits  $\mathbb{C}P^1 \times \mathbb{C}P^1$  as a totally geodesic Kähler submanifold, then for arbitrary positive  $k$  there exist closed holomorphic circles and open holomorphic circles of curvature  $k$  in  $M$ .*

We are now in a position to prove Theorem 1.

*Proof of Theorem 1.* It is known that any compact Hermitian symmetric space of rank  $d(\geq 2)$  admits  $\mathbb{C}P^1 \times \mathbb{C}P^1$  as a totally geodesic Kähler submanifold (see, [10]). Corollary tells us that our manifold  $M$  satisfying the hypothesis of Theorem

1 is a compact Hermitian symmetric space of rank one, namely  $M$  is complex analytically isometric to  $\mathbb{C}P^n$ . Every holomorphic circle of curvature  $k(> 0)$  in  $\mathbb{C}P^n(c)$  is a closed curve with length  $\frac{2\pi}{\sqrt{k^2+c}}$  (see, [3]).  $\square$

Motivated by Theorem 1, we pose the following problems:

**Problem 1.** *Let  $M$  be a simply connected compact Kähler homogeneous manifold (i.e.  $M$  is a Kähler C-space). Suppose for some  $k > 0$  every holomorphic circle of curvature  $k$  in  $M$  is closed. Then, is  $M$  complex analytically isometric to  $\mathbb{C}P^n$  with Fubini-Study metric of constant holomorphic sectional curvature?*

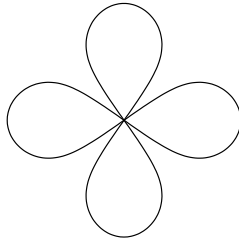
**Problem 2.** *Let  $M$  be a compact Kähler manifold and  $k$  be a positive number. Suppose that all holomorphic circles in  $M$  with curvature  $k$  are closed with the same length. Under what conditions can we conclude that  $M$  is a complex projective space?*

These are in some sense complex versions of the L.W. Green's theorem on closed geodesics (see [7] and [6]).

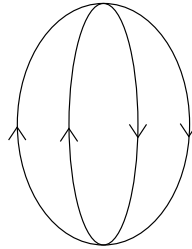
### 3. Helices with multiple points in a complex projective space.

In this section we study how to get examples of closed helices with multiple points in a complex projective space. We shall call a point  $x = \gamma(s_*) \in M$  of a closed curve  $\gamma$  an  $n$ -recurrent point of  $\gamma$  if there exist  $n$  positive numbers  $t_1 < t_2 < \cdots < t_n$  satisfying the following three conditions;

- (i)  $x = \gamma(s_*) = \gamma(s_* + t_1) = \cdots = \gamma(s_* + t_n)$ .
- (ii)  $t_n$  is the length of the closed curve  $\gamma$ .
- (iii) For any  $t \in (0, t_n]$ , if  $t \neq t_i$  ( $i = 1, \dots, n$ ), then  $\gamma(s_* + t) \neq x$ .



one 4-recurrent point



two 2-recurrent points

Note that a closed curve  $\gamma$  is simple if and only if each point on  $\gamma$  is 1-recurrent. When a closed curve  $\gamma$  has an  $n$ -recurrent point  $x = \gamma(s_*)$ , we call  $t_{i-1,i} = t_i - t_{i-1}$  the *recurrent time* from the point  $\gamma(s_* + t_{i-1})$  to the point  $\gamma(s_* + t_i)$  ( $i = 1, \dots, n$ ), where  $t_0 = 0$ . An  $n$ -recurrent point ( $n \geq 2$ ) is also called a multiple point.

In order to get our examples of closed helices with multiple points in a complex projective space, we study a circle of curvature  $k$  in a flat torus  $T^2 = S^1(r_1) \times S^1(r_2)$ . Here  $S^1(r)$  denotes a circle (i.e. a 1-dimensional sphere) of radius  $r$ . We

denote by  $\tilde{\gamma}$  a covering circle of  $\gamma$  in  $\mathbb{R}^2$ . Note that the circle  $\tilde{\gamma}$  is a circle of radius  $1/k$  in the sense of Euclidean Geometry. This implies that every circle  $\gamma$  of curvature  $k$  in  $T^2$  is a closed curve of length  $2\pi/k$ .

Here we study multiple points of circles  $\gamma$  in a flat torus  $T^2 = S^1(r_1) \times S^1(r_2)$ . We may set without loss of generality the following initial conditions on a covering circle  $\tilde{\gamma}$  in  $\mathbb{R}^2$ :

$$\tilde{\gamma}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (\in \mathbb{R}^2), \quad X_0 = \dot{\tilde{\gamma}}(0) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (\in U_{\tilde{\gamma}(0)}\mathbb{R}^2 = S^1(1)).$$

The principal normal unit vector then satisfies  $Y_0 = J \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  or  $Y_0 = -J \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ , where  $J$  denotes the natural complex structure on  $\mathbb{R}^2 = \mathbb{C}$  which is given by  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . In the following, we only discuss the case that  $Y_0 = J\dot{\tilde{\gamma}}(0)$ . When  $Y_0 = -J\dot{\tilde{\gamma}}(0)$ , the circle  $\sigma(s) = \gamma(-s)$  satisfies  $Y_0 = J\dot{\sigma}(0)$ .

A direct computation yields that the circle  $\tilde{\gamma}(s) = \begin{pmatrix} \tilde{\gamma}_1(s) \\ \tilde{\gamma}_2(s) \end{pmatrix}$  is expressed as

$$(3.1) \quad \begin{cases} \tilde{\gamma}_1(s) = \frac{1}{k}(v_2 \cdot \cos ks + v_1 \cdot \sin ks) - \frac{v_2}{k}, \\ \tilde{\gamma}_2(s) = \frac{1}{k}(-v_1 \cdot \cos ks + v_2 \cdot \sin ks) + \frac{v_1}{k}. \end{cases}$$

We here prove the following:

**Proposition 2.** *Every circle  $\gamma$  of curvature  $k$  in a flat torus  $T^2 = S^1(r_1) \times S^1(r_2)$  is a closed curve of length  $2\pi/k$ . Moreover, the following hold:*

- (i) *When  $k > \frac{1}{\pi \cdot \min\{r_1, r_2\}}$ ,  $\gamma$  is a simple curve.*
- (ii) *When  $k \leq \frac{1}{\pi \cdot \min\{r_1, r_2\}}$ ,  $\gamma$  has a multiple point.*

*Proof.* For a given circle  $\gamma$  we denote by  $\tilde{\gamma}$  its covering circle in  $\mathbb{R}^2$ .

(i) In this case, since the circle  $\tilde{\gamma}$  lies in the interior of a fundamental region  $\mathfrak{F} = \{(x_1, x_2) \mid 0 \leq x_1 < 2\pi r_1, 0 \leq x_2 < 2\pi r_2\}$  of  $T^2 = S^1(r_1) \times S^1(r_2)$ , the assertion (i) is obvious.

(ii) By using the expression (3.1) we shall show that the circle  $\gamma$  has a multiple point. We set  $\gamma(t_0) = \gamma(0)$ , which implies that  $\tilde{\gamma}_1(t_0) = 2\pi r_1 p$  and  $\tilde{\gamma}_2(t_0) = 2\pi r_2 q$  for some integers  $p$  and  $q$ . These, together with the equations (3.1), yield

$$(3.2) \quad \begin{cases} \cos kt_0 = 2\pi k(r_1 v_2 p - r_2 v_1 q) + 1, \\ \sin kt_0 = 2\pi k(r_1 v_1 p + r_2 v_2 q). \end{cases}$$

Hence, when  $pq \neq 0$ , the equations (3.2) give

$$(3.3) \quad \pi k = \frac{-r_1 v_2 p + r_2 v_1 q}{r_1^2 p^2 + r_2^2 q^2}.$$

We here note the following: If we take another pair  $(p', q')$  of integers which is different from the pair  $(p, q)$  and satisfies (3.3), then we find by an easy computation that  $t'_0 \neq t_0$ , where  $t'_0$  is defined in the equations (3.2) by replacing  $(p, q)$  by  $(p', q')$ . Therefore the point  $\gamma(0)$  is 1-recurrent if and only if there does not exist a pair of integers  $(p, q) (\neq (0, 0))$  satisfying (3.3). Hence our discussion shows that the circle  $\gamma$  is simple if and only if for any  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (\in S^1(1))$  there does not exist a pair of integers  $(p, q) (\neq (0, 0))$  satisfying (3.3).

Due to this criterion we can conclude that our circle is not simple. In fact, when  $r_1 \leq r_2$ , we set  $v_2 = -\pi k r_1 (\geq -1)$  and  $v_1 = \pm \sqrt{1 - (\pi k r_1)^2}$ . Then the pair  $(p, q) = (1, 0)$  is a solution of the equation (3.3). And also, when  $r_1 > r_2$ , we set  $v_1 = \pi k r_2 (< 1)$  and  $v_2 = \pm \sqrt{1 - (\pi k r_2)^2}$ . Then the pair  $(p, q) = (0, 1)$  is a solution of the equation (3.3).  $\square$

By virtue of the discussion in the proof of Proposition 2 we can establish the following:

**Proposition 3.** *Let  $\gamma$  be a circle of curvature  $k$  in a flat torus  $T^2 = S^1(r_1) \times S^1(r_2)$ . A point  $\gamma(s_*) (\in T^2)$  is an  $n$ -recurrent point of  $\gamma$  if and only if there exist  $(n - 1)$  pairs of integers  $(p, q) (\neq (0, 0))$  satisfying the following equation ;*

$$(3.4) \quad \pi k = \frac{-r_1 v_2 p + r_2 v_1 q}{r_1^2 p^2 + r_2^2 q^2},$$

where  $(v_1, v_2) \in \mathbb{R}^2$  denotes the tangent vector  $\dot{\tilde{\gamma}}(s_*)$  of a covering circle  $\tilde{\gamma}$  in  $\mathbb{R}^2$  for  $\gamma$ .

From now, by using Proposition 3 we give a class of helices with multiple points in a complex projective plane  $\mathbb{C}P^2$  with the aid of a well-known isometric parallel imbedding of a 2-dimensional flat torus into  $\mathbb{C}P^2$  (see [9] for detail). We consider a Riemann surface  $N = (S^1 \times S^1)/\varphi$ . Here by representing the first component by  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$  and the second component by  $S^1 = \{(a_1, a_2) \in \mathbb{R}^2 \mid (a_1)^2 + (a_2)^2 = 1\}$ , we define the identification  $\varphi$  by  $\varphi((e^{i\theta}, (a_1, a_2))) = (-e^{i\theta}, (-a_1, -a_2))$ . The Riemannian metric on  $N$  is given by  $\langle A + \xi, B + \eta \rangle = \frac{2}{9} \langle A, B \rangle_{S^1} + \frac{2}{3} \langle \xi, \eta \rangle_{S^1}$  for tangent vectors  $A, B \in TS^1$  of the first component and tangent vectors  $\xi, \eta \in TS^1$  of the second component, where  $\langle \cdot, \cdot \rangle_{S^1}$  denotes the canonical metric of  $S^1$ . We define an isometric imbedding of  $N$  into  $\mathbb{C}P^2(4)$  by

$$(3.5) \quad f(e^{i\theta}, (a_1, a_2)) = \pi \left( \frac{1}{3} (e^{-\frac{2i\theta}{3}} + 2a_1 e^{\frac{i\theta}{3}}), \frac{\sqrt{2}}{3} (e^{-\frac{2i\theta}{3}} - a_1 e^{\frac{i\theta}{3}}), \frac{2}{\sqrt{6}} i a_2 e^{\frac{i\theta}{3}} \right),$$

where  $\pi : S^5(1) \rightarrow \mathbb{C}P^2(4)$  is the Hopf fibration. This isometric imbedding  $f$  is parallel, that is  $f$  has the parallel second fundamental form  $\sigma_f$ , and totally real.

We now study images of circles in  $N$  under this isometric imbedding. As we see in [9], the imbedding  $f$  maps each geodesic on  $N$  to a circle of curvature  $\frac{1}{\sqrt{2}}$  in  $\mathbb{C}P^2(4)$ . This circle does not have multiple points, but it is not necessarily closed in  $\mathbb{C}P^2(4)$ . By direct calculations we obtain the following (for details, see [5]).

**Fact.** *For a circle  $\gamma$  of curvature  $k(> 0)$  on  $N$ , the curve  $f \circ \gamma$  is a helix of order 4 in  $\mathbb{C}P^2(4)$ . More precisely,*

(1) *when  $k = \frac{1}{2}$ , it is a helix of proper order 3 with curvatures*

$$\kappa_1 = \frac{\sqrt{3}}{2}, \kappa_2 = \sqrt{\frac{3}{2}},$$

(2) *when  $k \neq \frac{1}{2}$ , it is a helix of proper order 4 with curvatures*

$$\kappa_1 = \sqrt{k^2 + \frac{1}{2}}, \kappa_2 = \frac{3k}{\sqrt{2k^2+1}}, \kappa_3 = \frac{|4k^2-1|}{\sqrt{2(2k^2+1)}}.$$

*Moreover the helix  $f \circ \gamma$  is closed of length  $2\pi/k$  and has a multiple point if and only if  $k \leq \frac{3}{\sqrt{2}\pi}$ . The number of multiple points is greater than 2.*

Combining Fact and Proposition 3, we obtain the following examples.

**Examples.** (1) When  $k = \frac{3}{\sqrt{2}\pi}$ , the helix  $f \circ \gamma$  has three 2-recurrent points with the same recurrent time  $\frac{\sqrt{2}}{3}\pi^2$ . Every point on this helix except these is simple.

(2) When  $k = \frac{3\sqrt{6}}{4\pi}$ , the helix  $f \circ \gamma$  has two 3-recurrent points with the same recurrent time  $\frac{4\sqrt{6}}{27}\pi^2$ . Every point on this helix except these is simple.

(3) When  $k = \frac{3\sqrt{2}}{4\pi}$  the helix  $f \circ \gamma$  has a 6-recurrent point with the same recurrent time  $\frac{2\sqrt{2}}{9}\pi^2$ . Every point on this helix except these is simple.

(4) When  $k = \frac{\sqrt{6}}{2\pi}$  the helix  $f \circ \gamma$  has eight 2-recurrent points.

#### 4. Remark.

As an immediate consequence of Proposition 2 we establish the following:

**Theorem 2.** *If a Riemannian manifold  $M$  admits a flat torus  $T^2 = S^1(r_1) \times S^1(r_2)$  as a totally geodesic submanifold, then for any  $k \leq 1/(\pi \cdot \min\{r_1, r_2\})$  we have a closed circle of curvature  $k$  with multiple point. Hence, in particular any compact symmetric space  $M$  of rank  $d(\geq 2)$  has many closed circles with multiple points.*

The authors have an impression that holomorphic circles in Kähler geometry may play a similar role of geodesics in real Riemannian geometry.

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