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AN OBSERVATION OF DYNAMIC PROGRAMMING WITH SET-VALUED TRANSLATE MAPS BY USING SOME DUALITY FORMULATIONS

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Abstract. A dynamic programming problem that each state at some stage is chosen from a set decided by the state and action at the last stage, in other words, translate maps are set-valued maps, is considered and investigated. To solve the problem, two roles of choice of next stage are introduced; one is to the player's advantage, and the other is disadvangate. Also, two duality formulations based on Fenchel-Rockafellar duality [7] and Kanniappan duality [6] for such dynamic programming problem are defined and observed.

1. Introduction

In discrete deterministic dynamic programming problem, the outcome at each stage is determined by the state and action. Ordinary, a next state is uniquely decided by the state and action at the last stage, see for example [3].

In this paper, we propose a dynamic programming problem that state at each stage is chosen from the set determined by the state and the action at the last stage. In other words, it is a dynamic programming problem whose translate map at each stage is presented by a set-valued map. We can regard ordinary deterministic dynamic programming problem as a specification of such dynamic programming problem, because each translate map of ordinary one can be regard as a singleton set-valued map.

First, we define a finite stage dynamic programming problem (DP) which is mentioned above. This problem is specified by nine elements

$$(\text{inf},\ \text{Opt},\ \{X_n\},\ \{Y_n\},\ \{A_n\},\ \{T_n\},\ \{r_n\},\ \beta,\ N):$$

where

(i) 'inf' means infimum;
(ii) 'Opt' means a certain rule of choice of next state;
(iii) X_n is a Banach space, the set of *state space* at the n-th stage;

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(iv) Y_n is a Banach space, the set of *action space* at the n-th stage;
(v) A_n is a set-valued map from X_n to Y_n, $A_n(x_n)$ is the *available action set* for each $x_n \in X_n$ at the n-th stage;
(vi) T_n is a set-valued map from $X_n \times Y_n$ to X_{n+1}, $T_n(x_n, y_n)$ is the *available next state set* for each $(x_n, y_n) \in X_n \times Y_n$ at the n-th stage;
(vii) r_n is an extended real-valued function on $X_n \times Y_n$, the *loss function* at the n-th stage;
(viii) $\beta \in [0, 1]$, the *discount factor*;
(ix) N is an integer, the *number of the stages.*

As saying above, each set-valued map T_n, the translate map at the n-th stage, is singleton map in ordinary deterministic dynamic programming problem.

For any initial state $x_1 \in X_1$, problem (DP) is represented by the following optimization problem (P):

$$\text{(P)} \quad \inf_{y_1} \operatorname*{Opt}_{x_2} \inf_{y_2} \operatorname*{Opt}_{x_3} \cdots \inf_{y_N} \sum_{n=1}^{N} \beta^{n-1} r_n(x_n, y_n)$$
$$\text{subject to} \quad \begin{cases} y_n \in A_n(x_n), \\ x_{n+1} \in T_n(x_n, y_n), \\ n = 1, 2, \dots, N. \end{cases}$$

Let $v_N(x_1)$ be the optimal value of problem (P).

Now, for $n = 1, 2, \dots, N + 1$, we define an extended real-valued function f_{N-n+1} on X_n, defined by

$$f_{N-n+1}(x_n) \equiv \inf_{y_n \in A_n(x_n)} \left\{ r_n(x_n, y_n) + \beta \operatorname*{Opt}_{x_{n+1} \in T_n(x_n, y_n)} f_{N-n}(x_{n+1}) \right\},$$
$$x_n \in X_n, \ n = 1, 2, \dots, N;$$
$$f_0(x_{N+1}) \equiv 0(x_{N+1}) \equiv 0, \quad x_{N+1} \in X_{N+1}.$$

Then, we can verify easily that $f_N(x_1) = v_N(x_1)$, for all $x_1 \in X_1$. Therefore, for $n = 1, 2, \dots, N+1$, we call the function f_{N-n+1} the $(N - n + 1)$*-th optimal loss function.*

An example of this dynamic programming is Ekeland type minimizing algorithm, which has been observed recently in nonlinear programming. The objective of this algorithm is to minimize a value of a real valued function f on a metric space X. When the initial state is x_1, we choose, by virtue of a certain rule, a state x_2 in a set $T(x_1, a_1) = \{x \in X | \, f(x) \le f(x_1) - a_1 d(x, x_1)\}$ decided by the state x_1 and the action a_1. By repeating this work, we get a sequence $\{f(x_n)\}_{n=1}^{\infty}$ which converges to $\inf_{x \in X} f(x)$.

Another example is zero-sum 2-person game. That is, if player 1 chooses an action a_n in a state x_n, player 2 chooses x_{n+1}, which is a next state of player 1, in order to making player 1 disadvantage.

Thus, there are various rules of choice of the next state. We consider two particular rules: one is a dynamic programming problem whose rule is the best for the player, i.e., Opt $=$ inf, the other is a dynamic programming problem whose rule is the worst for the player, i.e., Opt $=$ sup.

Our purpose is to investigate optimal loss functions for the two cases above and produce two sequences of actions $\{y_n\}$ and states $\{x_n\}$ attaining to each optimal loss function. It is, however, too difficult to find out these sequences without any strong conditions. In this paper, to solve this problem, we derive some duality formulations for each optimal loss function. There are two characteristic points for usage of duality formulation. One is that the values of a duality formulation is not more than (or less than) the value of the optimal loss function without any condition. The other is that there exists a dual action and a dual state attaining duality formulation under weaker conditions.

This paper is organized as follows. In Section 2, we discuss the optimal loss function when Opt = inf. Then, we derive a certain duality formulation for the optimal loss function in the normal Rockafellar-Fenchel sense [7]. Moreover we observe the case that the number of stages are infinite. In Section 3, we discuss the optimal loss function when Opt = sup. For this, for the optimal loss function, we derive two duality formulations in two senses: one is the sense of Rockafellar-Fenchel, the other is the sense of Kanniappan [6]. We also observe the case that the number of the stages is infinite.

2. A Duality Formulation Calculus When Opt is Infimum

In this section, we observe the optimal loss function when Opt=inf. First, we give a simplification of the $(N-n+1)$-th optimal loss function in problem (P) as the following. Let $\mathscr{F}(Z;\overline{\mathbf{R}}_+) = \{f : Z \to \overline{\mathbf{R}}_+\}$ and $\overline{\mathbf{R}}_+ = \{r \in \mathbf{R} | r \geq 0\} \cup \{+\infty\}$. For all $f \in \mathscr{F}(Z;\overline{\mathbf{R}}_+)$,

$$\varphi(f)(x) \equiv \inf_{y\in A(x)} \left\{ r(x,y) + \beta \inf_{z\in T(x,y)} f(z) \right\},\ x \in X$$

where X, Y, and Z are Banach spaces, A a set-valued map from X to Y, T a set-valued map from $X \times Y$ to Z, r an extended non-negative real-valued function on $X \times Y$, $\beta \in [0,1]$.

Next, we give a duality formulation for the above function in the sense of Fenchel-Rockafellar type duality [7]. For all $f \in \mathscr{F}(Z;\overline{\mathbf{R}}_+)$, we define

$$\psi(f)(x) \equiv -\inf_{(y^*,z^*)\in \mathrm{Gr}T(x,\cdot)^+} [\{r(x,\cdot) + \delta_{A(x)}\}^*(y^*) + (\beta f)^*(z^*)],$$

where

$$\delta_{A(x)}(y) = \begin{cases} 0 & y \in A(x)\ ; \\ +\infty & y \notin A(x)\ , \end{cases}$$

$\mathrm{Gr}T(x,\cdot) = \{(y,z) \in Y \times Z | z \in T(x,y)\}$,

$\mathrm{Gr}T(x,\cdot)^+ = \{(y^*,z^*) \in Y^* \times Z^* | \langle y,y^*\rangle + \langle z,z^*\rangle \geq 0,\ \forall (y,z) \in \mathrm{Gr}T(x,\cdot)\}$,

$$\{r(x,\cdot)+\delta_{A(x)}\}^*(y^*) = \sup_{y\in Y}[\langle y,y^*\rangle - \{r(x,y)+\delta_{A(x)}(y)\}] = \sup_{y\in A(x)} \{\langle y,y^*\rangle - r(x,y)\},$$

$$(\beta f)^*(z^*) = \sup_{z\in Z}\{\langle z,z^*\rangle - \beta f(z)\} = \beta \sup_{z\in Z} \left\{ \left\langle z, \frac{z^*}{\beta} \right\rangle - f(z) \right\} = \beta f^* \left(\frac{z^*}{\beta} \right).$$

Then we have the following inequality without any condition.

Proposition 2.1.

$$\psi(f)(x) \le \varphi(f)(x).$$

Proof. For any $y \in A(x)$, $z \in T(x,y)$, $(y^*, z^*) \in \mathrm{Gr}T(x,\cdot)^+$,

$$\begin{aligned}
&-[\{r(x,\cdot)+\delta_{A(x)}\}^*(y^*) + (\beta f)^*(z^*)] \\
\le& -\sup_{y\in Y}\{\langle y, y^*\rangle - r(x,y) - \delta_{A(x)}(y)\} - \sup_{z\in Z}\{\langle z, z^*\rangle - \beta f(z)\} \\
\le& -\langle y, y^*\rangle + r(x,y) - \langle z, z^*\rangle + \beta f(z) \\
=& \, r(x,y) + \beta f(z) - \langle (y,z),(y^*,z^*)\rangle \\
\le& \, r(x,y) + \beta f(z).
\end{aligned}$$

From the definitions of $\psi(f)(x)$ and $\varphi(f)(x)$, we obtain the conclusion of the proposition. □

We give a sufficient condition to attain the infimum in the dual formulation $\psi(f)(x)$. Let $\mathrm{int}C$ be the set of all interior points of C which is a subset of a topological space V, and let $\mathrm{dom}f \equiv \{z \in Z | f(z) < +\infty\}$.

Theorem 2.1. *If*

$$(\theta_Y, \theta_Z) \in \mathrm{int}[\mathrm{Gr}T(x,\cdot) - \{\mathrm{dom}\, r(x,\cdot) \cap A(x)\} \times \mathrm{dom}f]$$

holds, there exists $(\overline{y^*}, \overline{z^*}) \in [\mathrm{Gr}T(x,\cdot)]^+$ *satisfying*

$$\psi(f)(x) = -[\{r(x,\cdot)+\delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*})].$$

Proof. By the assumption and Proposition 2.1, $\psi(f)(x) < +\infty$. If $\psi(f)(x) = -\infty$, $-[\{r(x,\cdot)+\delta_{A(x)}\}^*(y^*) + (\beta f)^*(z^*)] = -\infty$ for any $(y^*, z^*) \in \mathrm{Gr}T(x,\cdot)^+$. Then, there exists $(y^*, z^*) \in \mathrm{Gr}T(x,\cdot)^+$ such that $\psi(f)(x) = -[\{r(x,\cdot)+\delta_{A(x)}\}^*(\overline{y^*}) +(\beta f)^*(\overline{z^*})]$. Hence, we may prove this theorem when $\psi(f)(x) > -\infty$. By the definition of $\psi(f)(x)$, for any $n \in \mathbf{N}$, we can choose $(y_n^*, z_n^*) \in \mathrm{Gr}T(x,\cdot)^+$ satisfying

$$\{r(x,\cdot)+\delta_{A(x)}\}^*(y_n^*) + (\beta f)^*(z_n^*) \le -\psi(f)(x) + \frac{1}{n}.$$

First, we show the sequence $\{(y_n^*, z_n^*)\}_{n=1}^{\infty}$ is bounded.

By the assumption of this theorem, there is a positive number δ such that

$$\delta B_Y \times \delta B_Z \subset \mathrm{Gr}T(x,\cdot) - [\mathrm{dom}\, r(x,\cdot) \cap A(x)] \times \mathrm{dom}f.$$

Then, for all $(u,v) \in B_Y \times B_Z$, there exist $(y_1, z_1) \in [\mathrm{dom}\, r(x,\cdot) \cap A(x)] \times \mathrm{dom}f$ and $(y_2, z_2) \in \mathrm{Gr}T(x,\cdot)$ such that $\delta u = y_1 - y_2$ and $\delta v = z_1 - z_2$. Hence, for all

$n \in \mathbf{N}$, we have

$$
\begin{aligned}
\delta\langle(u,v),(y_n^*,z_n^*)\rangle &= \langle y_1 - y_2, y_n^*\rangle + \langle z_1 - z_2, z_n^*\rangle \\
&= \langle y_1, y_n^*\rangle + \langle z_1, z_n^*\rangle - \langle y_2, y_n^*\rangle - \langle z_2, z_n^*\rangle \\
&= \langle y_1, y_n^*\rangle - [r(x,\cdot) + \delta_{A(x)}](y_1) + \langle z_1, z_n^*\rangle - \beta f(z_1) - \langle y_2, y_n^*\rangle \\
&\quad - \langle z_2, z_n^*\rangle + [r(x,\cdot) + \delta_{A(x)}](y_1) + \beta f(z_1) \\
&\le [r(x,\cdot) + \delta_{A(x)}]^*(y_n^*) + [\beta f]^*(z_n^*) - \langle (y_2,z_2),(y_n^*,z_n^*)\rangle \\
&\quad + [r(x,\cdot) + \delta_{A(x)}](y_1) + \beta f(z_1) \\
&\le [r(x,\cdot) + \delta_{A(x)}]^*(y_n^*) + [\beta f]^*(z_n^*) + [r(x,\cdot) + \delta_{A(x)}](y_1) + \beta f(z_1).
\end{aligned}
$$

Since the sequence $\{[r(x,\cdot) + \delta_{A(x)}]^*(y_n^*) + (\beta f)^*(z_n^*)\}_{n=1}^{\infty}$ converges to the finite number $-\psi(f)(x)$, the right-hand side of the above inequality is bounded with respect to n for all $(u,v) \in B_Y \times B_Z$. Using the uniform boundedness theorem, we conclude that $\{(y_n^*, z_n^*)\}_{n=1}^{\infty}$ is bounded.

By Alaoglu's theorem, $\{(y_n^*, z_n^*)\}_{n=1}^{\infty}$ is relatively weak* compact. Then there exists a subsequence of $\{(y_n^*, z_n^*)\}_{n=1}^{\infty}$ converging to some $(\overline{y^*}, \overline{z^*}) \in Y^* \times Z^*$ in weak* topology. We rewrite the subsequence as $\{(y_n^*, z_n^*)\}_{n=1}^{\infty}$. Since $\mathrm{Gr}T(x,\cdot)^+$ is weak* closed in $Y^* \times Z^*$, $(\overline{y^*}, \overline{z^*}) \in \mathrm{Gr}T(x,\cdot)^+$.

Since $\{r(x,\cdot)+\delta_{A(x)}\}^*$ and $(\beta f)^*$ are weak* lower semicontinuous (weak* l.s.c., shortly) on Y^* and Z^*, respectively, then we have

$$
\begin{aligned}
\{r(x,\cdot) + \delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*}) &\le \liminf_{n\to\infty}[\{r(x,\cdot) + \delta_{A(x)}\}^*(y_n^*) + (\beta f)^*(z_n^*)] \\
&\le \liminf_{n\to\infty}\left\{-\psi(f)(x) + \frac{1}{n}\right\} \\
&= -\psi(f)(x).
\end{aligned}
$$

The converse inequality is obvious, and hence we have

$$\psi(f)(x) = -[\{r(x,\cdot) + \delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*})].$$

This completes the proof. □

Next, we derive a sufficient condition for $\varphi(f)(x)$ to be equivalent with $\psi(f)(x)$. We define a set-valued map $\Phi : \big(\mathrm{dom}\, r(x,\cdot)^* \cap A(x)\big) \times \mathrm{dom}(\beta f)^* \to \mathbf{R} \times 2^{Z^*}$ as follows: for all $(y^*, z^*) \in \big(\mathrm{dom}\, r(x,\cdot)^* \cap A(x)\big) \times \mathrm{dom}(\beta f)^*$,

$$\Phi(y^*, z^*) \equiv \big(\{r(x,\cdot) + \delta_{A(x)}\}^*(y^*) + (\beta f)^*(z^*),\ K(y^*) - z^*\big)$$

where $K(y^*) = \{z^* \in Z^* | (y^*, z^*) \in \mathrm{Gr}T(x,\cdot)^+\}$. Let $\mathrm{Im}\Phi$ be the image of the set-valued map φ, i.e., $\mathrm{Im}\Phi \equiv \cup\{\Phi(y^*, z^*) | (y^*, z^*) \in \big(\mathrm{dom}\, r(x,\cdot)^* \cap A(x)\big) \times \mathrm{dom}(\beta f)^*\}$.

Theorem 2.2. *If* $(-\varphi(f)(x), \theta_{Z^*}) \in \mathrm{Im}\Phi + \big(\mathbf{R}_+ \times \{\theta_{Z^*}\}\big)$ *holds, then*

$$\varphi(f)(x) = \psi(f)(x),$$

and there exists $(\overline{y^*}, \overline{z^*}) \in \mathrm{Gr}T(x,\cdot)^+$ *satisfying that*

$$\psi(f)(x) = -[\{r(x,\cdot) + \delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*})].$$

Moreover, if there exist $\overline{y} \in A(x)$ *and* $\overline{z} \in T(x, \overline{y})$ *such that* $\varphi(f)(x) = r(x, \overline{y}) + \beta f(\overline{z})$*, then we obtain*

$$\langle (\overline{y^*}, \overline{z^*}), (\overline{y}, \overline{z}) \rangle = \langle \overline{y}, \overline{y^*} \rangle + \langle \overline{z}, \overline{z^*} \rangle = 0.$$

Proof. If $(-\varphi(f)(x), \theta_{Z^*}) \in \mathrm{Im}\Phi + \big(\mathbf{R}_+ \times \{\theta_{Z^*}\}\big)$, there exists $(\overline{y^*}, \overline{z^*}) \in \mathrm{dom}[r(x,\cdot) + \delta_{A(x)}]^* \times \mathrm{dom}(\beta f)^*$ such that

$$(-\varphi(f)(x), \theta_{Z^*}) \in \big([r(x,\cdot) + \delta_{A(x)}]^*(\overline{y^*}) + (\beta f)^*(\overline{z^*}), K(\overline{y^*}) - \overline{z^*}\big) + (\overline{r}, \theta_{Z^*}).$$

So, $-\varphi(f)(x) = \{r(x,\cdot) + \delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*}) + \overline{r}$ and $(\overline{y^*}, \overline{z^*}) \in \mathrm{Gr}T(x,\cdot)^+$. Then

$$\begin{aligned} -\varphi(f)(x) &= \{r(x,\cdot) + \delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*}) + \overline{r} \\ &\geq -\psi(f)(x). \end{aligned}$$

The converse inequality is obvious. Hence, there exists $(\overline{y^*}, \overline{z^*}) \in \mathrm{Gr}T(x,\cdot)^+$ such that $\psi(f)(x) = -[\{r(x,\cdot) + \delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*})]$.

Moreover, if there exist $\overline{y} \in A(x)$ and $\overline{z} \in T(x, \overline{y})$ such that $\varphi(f)(x) = r(x, \overline{y}) + \beta f(\overline{z})$, we have

$$\begin{aligned} 0 &= \varphi(f)(x) - \psi(f)(x) \\ &= r(x, \overline{y}) + \beta f(\overline{z}) + \{r(x,\cdot) + \delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*}) \\ &\geq r(x, \overline{y}) + \beta f(\overline{z}) + \langle \overline{y}, \overline{y^*} \rangle - r(x, \overline{y}) + \langle \overline{z}, \overline{z^*} \rangle - \beta f(\overline{z}) \\ &= \langle \overline{y}, \overline{y^*} \rangle + \langle \overline{z}, \overline{z^*} \rangle \\ &= \langle (\overline{y}, \overline{z}), (\overline{y^*}, \overline{z^*}) \rangle \\ &\geq 0. \end{aligned}$$

Hence, we obtain that all inequality signs of the above inequality are equality. We have thus proved the theorem. □

From the following theorem, we see that the condition of Theorem 2.2,

$$(-\varphi(f)(x), \theta_{Z^*}) \in \mathrm{Im}\Phi + \big(\mathbf{R}_+ \times \{\theta_{Z^*}\}\big),$$

is not so strong. Let $\Gamma_0(V)$ be the set of all functions which are proper lower semicontinuous convex on a normed space V.

Theorem 2.3. *Assume that* $r(x,\cdot) \in \Gamma_0(Y)$*,* $f \in \Gamma_0(Z)$*,* $A(x)$ *is a nonempty closed convex set,* $\mathrm{Gr}T(x,\cdot)$ *is a nonempty closed convex cone,* $\psi(f)(x)$ *is finite, and that the assumption of Theorem 2.1 is fulfilled. Then the condition of Theorem 2.2 is satisfied.*

Proof. We define two functionals F, $G : Y \times Z \to \overline{\mathbf{R}}$ by

$$F(y, z) = r(x, y) + \delta_{A(x)} + \beta f(z),$$

$$G(y, z) = \delta_{\mathrm{Gr}T(x,\cdot)}(y, z).$$

It is verified that F, $G \in \Gamma_0(Y \times Z)$ from the assumptions of this theorem. By using Corollary 1 in [7],

$$\inf_{(y,z) \in Y \times Z} \{F(y, z) + G(y, z)\} = -\min_{(y^*,z^*) \in Y^* \times Z^*} \{F^*(y^*, z^*) + G^*(y^*, z^*)\}.$$

We can verify easily that

$$\inf_{y\in A(x)}\{r(x,y)+\beta\inf_{z\in T(x,y)}f(z)\} = -\min_{(y^*,z^*)\in \mathrm{Gr}T(x,\cdot)^+}[\{r(x,\cdot)+\delta_{A(x)}\}^*(y^*)+(\beta f)^*(z^*)].$$

Hence, there exists $(\overline{y^*},\overline{z^*}) \in \mathrm{Gr}T(x,\cdot)^+$ such that

$$-\varphi(f)(x) = \{r(x,\cdot)+\delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*}).$$

Thus,

$$\Phi(\overline{y^*},\overline{z^*}) = \big(\{r(x,\cdot)+\delta_{A(x)}\}^*(\overline{y^*}) + (\beta f)^*(\overline{z^*}),\, K(\overline{y^*}) - \overline{z^*}\big) \ni (-\varphi(f)(x),\theta_{Z^*}).$$

This completes the proof. □

Next, we consider an infinite stage dynamic programming problem (DP) which is specified by eight elements

$$(\inf,\ \mathrm{Opt},\ X,\ Y,\ A,\ T,\ r,\ \beta).$$

Here, 'Opt' means infimum in this section, X and Y are Banach spaces, A is a set-valued map from X to Y, T is a set-valued map from $X\times Y$ to X, r is an extended non-negative real-valued function on $X\times Y$, and $\beta\in[0,1]$.

For any initial state $x_1 \in X_1$, (DP) is represented by the following optimization problem (Q):

$$\text{(Q)}\quad \inf_{y_1,x_2,y_2,x_3,\dots} \sum_{n=1}^{\infty}\beta^{n-1}r(x_n,y_n)$$
$$\text{subject to}\quad \begin{cases} y_n\in A(x_n), & n=1,2,\dots\,;\\ x_{n+1}\in T(x_n,y_n), & n=1,2,\dots\,. \end{cases}$$

When an initial state is $x_1\in X$, we denote $v_\infty(x_1)$ the optimal value of problem (Q).

By Theorem 2.2, optimization problem (Q) can be solved by the duality formulation.

Corollary 2.1. *Suppose that for each $n=1,2,\dots$ and for all $x\in X$, $\varphi^n(0)(x)$ is satisfied the assumption Theorem 2.2. Then there exist $\{p_n\}_{n=1}^\infty \subset \mathscr{F}(X;Y^*)$ and $\{q_n\}_{n=1}^\infty\subset\mathscr{F}(X;Z^*)$ such that for all initial state $x_1\in X$,*

$$v_\infty(x_1) = \lim_{n\to\infty} f_n(x_1),$$

where the sequence of functions $\{f_n\}_{n=1}^\infty\subset\mathscr{F}(X;\overline{\mathbf{R}}_+)$ is generated as follows:

$$f_0\equiv 0,\ \ f_{n+1}(x)\equiv -[\{r(x,\cdot)+\delta_{A(x)}\}^*(p_n(x)) + (\beta f_n)^*(q_n(x))].$$

Proof. It is obvious to show that $\lim_{n\to\infty}\varphi^n(0)(x) = v_\infty(x)$. Using Theorem 2.2, we conclude the proof of the corollary. □

3. Calculus of Two Duality Formulations When Opt is Supremum

In this section, we observe the optimal loss function of (DP) when Opt=sup. In the similar way of the previous section, we give a simplification of the $(N-n+1)$-th optimal loss function in problem (DP) as follows. For all $f \in \mathscr{F}(Z; \overline{\mathbf{R}}_+)$,

$$\varphi(f)(x) \equiv \inf_{y \in A(x)} \left\{ r(x,y) + \beta \sup_{z \in T(x,y)} f(z) \right\}, \ x \in X.$$

We define two duality formulations for $\varphi(f)(x)$. One is defined in the sense of Fenchel-Rockafellar type duality [7] and the other in the sense of Kanniappan type duality [6]. For all $f \in \mathscr{F}(Z; \overline{\mathbf{R}}_+)$, we set

$$\psi^\flat(f)(x) \equiv \sup_{y^* \in Y^*} \left[-\{r(x,\cdot) + \delta_{A(x)}\}^*(y^*) - \left\{ \beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (-y^*) \right],$$

$$\psi^\sharp(f)(x) \equiv \inf_{y^* \in Y^*} \left[-\{r(x,\cdot) + \delta_{A(x)}\}^*(y^*) + \left\{ -\beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (y^*) \right].$$

Then we have the following inequalities without any condition.

Proposition 3.1.

$$\psi^\flat(f)(x) \le \varphi(f)(x) \le \psi^\sharp(f)(x).$$

Proof. Recall the fact that $g^{**}(v) \le g(v)$ for all $v \in V$ and for an extended real-valued functional g defined on a normed space V. Then

$$\begin{aligned}
&\psi^\flat(f)(x) \\
&= \sup_{y^* \in Y^*} \left[- \left\{ r(x,\cdot) + \delta_{A(x)} \right\}^* (y^*) - \left\{ \beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (-y^*) \right] \\
&= \inf_{y \in A(x)} \sup_{y^* \in Y^*} \left[\langle y, y^* \rangle - \left\{ r(x,\cdot) + \delta_{A(x)} \right\}^* (y^*) + \langle y, -y^* \rangle - \left\{ \beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (-y^*) \right] \\
&\le \inf_{y \in A(x)} \left[\left\{ r(x,\cdot) + \delta_{A(x)} \right\}^{**} (y) + \left\{ \beta \sup_{z \in T(x,\cdot)} f(z) \right\}^{**} (y) \right] \\
&\le \inf_{y \in A(x)} \left[\left\{ r(x,\cdot) + \delta_{A(x)} \right\} (y) + \left\{ \beta \sup_{z \in T(x,\cdot)} f(z) \right\} (y) \right] \\
&= \varphi(f)(x) \\
&\le \inf_{y \in A(x)} \left[\left\{ r(x,\cdot) + \delta_{A(x)} \right\} (y) - \left\{ -\beta \sup_{z \in T(x,\cdot)} f(z) \right\}^{**} (y) \right] \\
&= \inf_{y \in A(x)} \inf_{y^* \in Y^*} \left[\left\{ r(x,\cdot) + \delta_{A(x)} \right\} (y) - \langle y, y^* \rangle + \left\{ -\beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (y^*) \right]
\end{aligned}$$

$$= \inf_{y^* \in Y^*} \inf_{y \in A(x)} \left[-\langle y, y^* \rangle + \left\{ r(x,\cdot) + \delta_{A(x)} \right\}(y) + \left\{ -\beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (y^*) \right]$$

$$= \inf_{y^* \in Y^*} \left[- \left\{ r(x,\cdot) + \delta_{A(x)} \right\}^* (y^*) + \left\{ -\beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (y^*) \right]$$

$$= \psi^\sharp(f)(x).$$

This completes the proof. □

Next we give sufficient conditions to attain the supremum and the infimum in the duality formulations $\psi^\flat(f)(x)$ and $\psi^\sharp(f)(x)$ in Theorems 3.1 and 3.2, respectively.

Theorem 3.1. *If*

$$\theta_Y \in \operatorname{int} \left(\operatorname{dom} \left\{ r(x,\cdot) - \delta_{A(x)} \right\} - \operatorname{dom} \Big(\sup_{z \in T(x,\cdot)} f(z) \Big) \right),$$

holds, there exists $\overline{y^*} \in Y^*$ *satisfying that*

$$\psi^\flat(f)(x) = - \left\{ r(x,\cdot) + \delta_{A(x)} \right\}^* (\overline{y^*}) - \left\{ \beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (-\overline{y^*}).$$

Proof. By the assumption, it is clear that $\varphi(f)(x) < \infty$. Then we have $\psi^\flat(f)(x) < +\infty$ by Proposition 3.1. If $\psi^\flat(f)(x) = -\infty$, then

$$- \left\{ r(x,\cdot) + \delta_{A(x)} \right\}^* (y^*) - \left\{ \beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (-y^*) = -\infty$$

for all $y^* \in Y^*$. Hence we may prove this theorem when $\psi^\flat(f)(x) > -\infty$. By the definition of $\psi^\flat(f)(x)$, for all integer n, there exists $y_n^* \in Y^*$ such that

$$\psi^\flat(f)(x) - \frac{1}{n} \le - \left\{ r(x,\cdot) + \delta_{A(x)} \right\}^* (y_n^*) - \left\{ \beta \sup_{z \in T(x,\cdot)} f(z) \right\}^* (-y_n^*).$$

First, we show that the sequence $\{y_n^*\}_{n=1}^\infty$ is bounded. By the assumption, there is a positive number δ such that

$$\delta B_Y \subset \operatorname{dom} \left\{ r(x,\cdot) + \delta_{A(x)} \right\} - \operatorname{dom} \left\{ \sup_{z \in T(x,\cdot)} f(z) \right\}.$$

Hence, for all $u \in B_Y$, there exist

$$y_1 \in \operatorname{dom} \left\{ r(x,\cdot) + \delta_{A(x)} \right\} \text{ and } y_2 \in \operatorname{dom} \left\{ \sup_{z \in T(x,\cdot)} f(z) \right\}$$

such that $\delta u = y_1 - y_2$ Then, for all $n \in \mathbf{N}$,

$$\begin{aligned}
\delta\langle u, y_n^*\rangle &= \langle y_1, y_n^*\rangle - \langle y_2, y_n^*\rangle \\
&= \langle y_1, y_n^*\rangle - \left\{r(x,\cdot) + \delta_{A(x)}\right\}(y_1) - \langle y_2, y_n^*\rangle - \left\{\beta \sup_{z\in T(x,\cdot)} f(z)\right\}(y_2) \\
&\quad + \left\{r(x,\cdot) + \delta_{A(x)}\right\}(y_1) + \left\{\beta \sup_{z\in T(x,\cdot)} f(z)\right\}(y_2) \\
&\leq \left\{r(x,\cdot) + \delta_{A(x)}\right\}^*(y_n^*) + \left\{\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(-y_n^*) \\
&\quad + \left\{r(x,\cdot) + \delta_{A(x)}\right\}(y_1) + \left\{\beta \sup_{z\in T(x,\cdot)} f(z)\right\}(y_2) \\
&\leq -\psi^\flat(f)(x) + \frac{1}{n} + r(x, y_1) + \beta \sup_{z\in T(x,y_2)} f(z).
\end{aligned}$$

The last term of the inequalities above is bounded with respect to n for all $u \in B_Y$, since the sequence with respect to n converges to a certain finite number. Using the uniform boundedness theorem, we conclude that $\{y_n^*\}_{n=1}^\infty$ is bounded.

By Alaoglu's theorem, $\{y_n^*\}_{n=1}^\infty$ is relatively weak* compact. Then there exists a subsequence of $\{y_n^*\}_{n=1}^\infty$ converging for some $\overline{y^*} \in Y^*$ in weak* topology. We rewrite the subsequence $\{y_n^*\}_{n=1}^\infty$.

Since $\left\{r(x,\cdot) + \delta_{A(x)}\right\}^*$ and $\{\beta \sup_{z\in T(x,\cdot)} f(z)\}^*$ are weak* l.s.c. on Y^*,

$$\begin{aligned}
&\left\{r(x,\cdot) + \delta_{A(x)}\right\}^*(\overline{y^*}) + \left\{\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(\overline{y^*}) \\
&\qquad\leq \liminf_{n\to\infty}\left[\left\{r(x,\cdot) + \delta_{A(x)}\right\}^*(y_n^*) + \left\{\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(y_n^*)\right] \\
&\qquad\leq \liminf_{n\to\infty}\left\{-\psi^\flat(f)(x) + \frac{1}{n}\right\} \\
&\qquad= -\psi^\flat(f)(x).
\end{aligned}$$

Hence,

$$\psi^\flat(f)(x) \leq -\left\{r(x,\cdot) + \delta_{A(x)}\right\}^*(\overline{y^*}) - \left\{\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(\overline{y^*}).$$

Conversely, it is easy to show that the inequality

$$\psi^\flat(f)(x) \geq -\left\{r(x,\cdot) + \delta_{A(x)}\right\}^*(\overline{y^*}) - \left\{\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(\overline{y^*}).$$

We have completed the proof. □

Theorem 3.2. *If $r(x,\cdot)$ is l.c.s. on Y, $A(x)\subset Y$ is compact, and*

$$A(x)\subset \operatorname{int}\{y\in Y|\exists M \text{ s.t. } f(z)\le M,\ \forall z\in T(x,y)\},$$

then there exists $\widehat{y^*}\in Y^*$ such that

$$\psi^\sharp(f)(x) = -\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(\widehat{y^*}) + \left\{-\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(\widehat{y^*}).$$

Proof. Since $r(x,\cdot)\ge 0$ and $f\ge 0$, it is clear that $0\le \varphi(f)(x)$. Then we have $0\le \psi^\sharp(f)(x)$ by Proposition 3.1. If $\psi^\sharp(f)(x)=+\infty$, then $-\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(y^*)+\left\{-\beta\sup_{z\in T(x,\cdot)} f(z)\right\}^*(y^*)=+\infty$ for all $y^*\in Y^*$. Hence, we may prove this theorem when $\psi^\sharp(f)(x)<+\infty$. By the definition of $\psi^\sharp(f)(x)$, for all $n\in\mathbf{N}$, there exists $y_n^*\in Y^*$ such that

$$-\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(y_n^*) + \left\{-\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(y_n^*) \le \psi^\sharp(f)(x)+\frac{1}{n}.$$

We show that the sequence $\{y_n^*\}_{n=1}^\infty$ is bounded under the assumption of this theorem. Since

$$A(x)\subset \operatorname{int}\{y\in Y|\exists M \text{ s.t. } f(z)\le M,\ \forall z\in T(x,y)\} = \operatorname{int}\operatorname{dom}\left\{\sup_{y\in T(x,\cdot)} f(z)\right\}$$

and $A(x)$ is compact, there is a positive number $\delta>0$ such that

$$A(x)+\delta B\subset \operatorname{dom}\left\{\sup_{y\in T(x,\cdot)} f(z)\right\}.$$

For all $n\in\mathbf{N}$, there exists $y_n\in A(x)$ such that

$$\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(y_n^*) = -r(x,y_n)+\langle y_n, y_n^*\rangle.$$

We may write y_n converges to some vector $y\in A(x)$ since $A(x)$ is compact. For all $u\in B_Y$ and for all $n\in\mathbf{N}$, there is $v_n\in\operatorname{dom}\left\{\sup_{y\in T(x,\cdot)} f(z)\right\}$ satisfying with $y_n+\delta u = v_n$. Then, for all $n\in\mathbf{N}$,

$$\begin{aligned}
\delta\langle u, y_n^*\rangle &= \langle v_n - y_n, y_n^*\rangle \\
&= \langle v_n, y_n^*\rangle - \langle y_n, y_n^*\rangle \\
&\le \langle v_n, y_n^*\rangle + \beta \sup_{z\in T(x,v_n)} f(z) - \langle y_n, y_n^*\rangle + r(x,y_n) \\
&\le \left\{-\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(y_n^*) - \left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(y_n^*) \\
&\le \psi^\sharp(f)(x)+\frac{1}{n}.
\end{aligned}$$

Since the last term of the inequalities above is bounded with respect to n for all $u\in B_Y$, by using the uniform boundedness theorem, we conclude that the sequence $\{y_n^*\}_{n=1}^\infty$ is bounded.

By Alaoglu's theorem, the sequence $\{y_n^*\}_{n=1}^{\infty}$ is relatively weak* compact. Then there exists a subsequence of $\{y_n^*\}_{n=1}^{\infty}$ converging to some $\widehat{y^*} \in Y^*$ in weak* topology. We rewrite the subsequence $\{y_n^*\}_{n=1}^{\infty}$, that is, y_n^* converges to $\widehat{y^*}$ in weak* topology.

It is easy to verify that the function $\left\{-\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*$ is weak* l.s.c. on Y^*. By the assumption, we show that the function $-\left\{r(x,\cdot)+\delta_{A(x)}\right\}^* = \inf_{y\in A(x)}\{r(x,y)-\langle y,\cdot\rangle\}$ is also weak* l.s.c. on Y^*. For $y^*\in Y^*$ define $\alpha(y^*) \equiv \inf_{y\in A(x)}\{r(x,y)-\langle y,y^*\rangle\}$. If α is not weak* l.s.c. on Y^*, there exists a sequence $\{y_n^*\}$ which converges to some y^* in weak* topology such that $\alpha(y^*) > \liminf_{n\to\infty}\alpha(y_n^*)$. Then, there exist a real number γ and a subsequence $\{y_{n'}^*\}$ such that $\alpha(y^*) > \gamma > \alpha(y_{n'}^*)$, and $\lim_{n'\to\infty}\alpha(y_{n'}^*) = \gamma$. Since $A(x)$ is compact, there exists $y_{n'} \in A(x)$ such that $\alpha(y_{n'}^*) = r(x,y_{n'}) - \langle y_{n'}, y_{n'}^*\rangle$. We may write $y_{n'} \to y \in A(x)$. We can verify that $\lim_{n'\to\infty}\langle y_{n'}, y_{n'}^*\rangle = \langle y, y^*\rangle$ because every weak* converging sequence is bounded. Then

$$\alpha(y^*) \le r(x,y) - \langle y,y^*\rangle \le \liminf_{n'\to\infty}\left\{r(x,y_{n'}) - \langle y_{n'}, y_{n'}^*\rangle\right\} = \liminf_{n'\to\infty}\alpha(y_{n'}^*) = \gamma,$$

which is a contradiction. Hence we conclude that $-\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*$ is weak* l.s.c. on Y^*.

Therefore, we derive

$$-\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(\widehat{y^*}) + \left\{-\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(\widehat{y^*}) \le \psi^\sharp(f)(x).$$

Conversely, the inequality $-\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(\widehat{y^*}) + \left\{-\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(\widehat{y^*}) \ge \psi^\sharp(f)(x)$ is clearly, then the proof is completed. □

As stated the previous section, we consider an infinite stage dynamic programming (DP). For any initial state $x_1 \in X_1$, this (DP) is represented by the following optimization problem (R):

$$\text{(R)}\quad \inf_{y_1}\sup_{x_2}\inf_{y_2}\sup_{x_3}\cdots\inf_{y_n}\sup_{x_{n+1}}\cdots \sum_{n=1}^{\infty}\beta^{n-1}r(x_n,y_n).$$

$$\text{subject to}\quad \begin{cases} y_n \in A(x_n), & n=1,2,\dots; \\ x_{n+1}\in T(x_n,y_n), & n=1,2,\dots. \end{cases}$$

We write $v_\infty(x_1)$ the optimal value of problem (R).

In the previous section, Theorem 2.2 guarantees the equivalence of the primal problem $\varphi(f)(x)$ and the dual problem $\psi^\flat(f)(x)$. However, the equivalence is not true in the case of Opt = sup.

Then, the optimal value $v_\infty(x_1)$ whose initial state is x_1 is characterized by the two duality formulations $\psi^\flat$ and $\psi^\sharp$ in the following theorem.

Theorem 3.3. *For all $n=1,2,\dots$ and $x\in X$, $(\psi^\flat)^n(0)(x)$ and $(\psi^\sharp)^n(0)(x)$ are fulfilled the assumptions of Theorem 3.1 and Theorem 3.2, respectively. Then there exist sequences $\{p_n\}_{n=1}^{\infty} \subset \mathscr{F}(X;Y^*)$ and $\{q_n\}_{n=1}^{\infty} \subset \mathscr{F}(X;Y^*)$ such that* *for any initial state* $x_1 \in X$,

$$\limsup_{n\to\infty} f_n(x_1) \le v_\infty(x_1) = \lim_{n\to\infty} g_n(x_1) \le \liminf_{n\to\infty} h_n(x_1)$$

where three sequences of functions $\{f_n\}_{n=1}^\infty$, $\{g_n\}_{n=1}^\infty$, $\{h_n\}_{n=1}^\infty \subset \mathscr{F}(X;\overline{\mathbf{R}}_+)$ *are generated as follows:* $f_0 \equiv 0$, $g_0 \equiv 0$, $h_0 \equiv 0$, *and for all* $x \in X$

$$f_{n+1}(x) \equiv -\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(p_n(x)) - \left\{\beta \sup_{z\in T(x,\cdot)} f_n(z)\right\}^*(-p_n(x)),$$

$$g_{n+1}(x) \equiv \varphi(g_n)(x),$$

$$h_{n+1}(x) \equiv -\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(q_n(x)) + \left\{-\beta \sup_{z\in T(x,\cdot)} h_n(z)\right\}^*(q_n(x)).$$

Proof. It is clear that $\lim_{n\to\infty} g_n(x) = v_\infty(x)$. By Theorem 3.1 and Theorem 3.2, for all $n \in \mathbf{N}$, there exist $\overline{y_n^*}(\equiv p_n(x)) \in Y^*$ and $\widehat{y_n^*}(\equiv q_n(x)) \in Y^*$ such that

$$\psi^\flat(f_n)(x) = -\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(p_n(x)) - \left\{\beta \sup_{z\in T(x,\cdot)} f_n(z)\right\}^*(-p_n(x))$$

and

$$\psi^\sharp(h_n)(x) = -\left\{r(x,\cdot)+\delta_{A(x)}\right\}^*(q_n(x)) + \left\{-\beta \sup_{z\in T(x,\cdot)} h_n(z)\right\}^*(q_n(x)).$$

Next, we show that $f_n \le g_n \le h_n$ for all $n \in \mathbf{N}$ by induction. The inequalities $f_0 \le g_0 \le h_0$ are obvious. We assume that $f_n \le g_n \le h_n$. By the definition of $\varphi(g_n)$ and Proposition 3.1,

$$\psi^\flat(f_n) \le \varphi(f_n) \le \varphi(g_n) \le \varphi(h_n) \le \psi^\sharp(h_n),$$

hence, we have $f_{n+1} \le g_{n+1} \le h_{n+1}$. Then $f_n \le g_n \le h_n$ hold for all $n \in \mathbf{N}$. Thus

$$\limsup_{n\to\infty} f_n(x_1) \le \lim_{n\to\infty} g_n(x_1) = v_\infty(x_1) \le \liminf_{n\to\infty} h_n(x_1).$$

This completes the proof. □

In the above theorem, $\limsup_{n\to\infty} f_n(x_1)$, $v_\infty(x_1)$, and $\liminf_{n\to\infty} h_n(x_1)$ may be infinite. Moreover it is not known how far it is from $\limsup_{n\to\infty} f_n(x_1)$ to $\liminf_{n\to\infty} h_n(x_1)$. Hence, we are interested in the distance between the upper limit of $f_n(x_1)$ and the lower limit of $h_n(x_1)$.

For the following theorem, we define some notations. Let $B(X)$ be the set of all real-valued functions which are bounded on a bounded set in X. Now, we derive a norm $\|\cdot\|_\infty$ to $B(X)$. For $f \in B(X)$,

$$\|f\|_\infty \equiv \sum_{n=1}^\infty \frac{1}{2^n}\frac{\|f\|_n}{1+\|f\|_n},$$

where

$$\|f\|_n \equiv \sup_{\|x\|\le n} |f(x)| \quad \text{for all } n \in \mathbf{N}.$$

It is well known that this normed space $(B(X), \|\cdot\|_\infty)$ is a Banach space.

Theorem 3.4. *Assume that* $\beta < 1$, $\psi^\flat : B(X) \to B(X)$, $\psi^\sharp : B(X) \to B(X)$, *and for all* $x \in X$,

$$\|z\| \leq \|x\|, \ \forall z \in T(x,y), \ \forall y \in A(x).$$

Then there exist $\widehat{f} \in B(X)$, $\widehat{h} \in B(X)$ *and* $\{p_n\}_{n=1}^\infty \subset \mathscr{F}(X;Y^*)$ *such that, for each initial state* $x_1 \in X$,

$$\lim_{n\to\infty} f_n(x_1) = \widehat{f}(x_1) \leq v_\infty(x_1) \leq \widehat{h}(x_1),$$

where the sequence of functions $\{f_n\}_{n=1}^\infty \subset \mathscr{F}(X;\overline{\mathbf{R}}_+)$ *is generated as follows:*

$$f_0 \equiv 0, \ f_{n+1}(x) \equiv -\left\{r(x,\cdot) + \delta_{A(x)}\right\}^* (p_n(x)) - \left\{\beta \sup_{z\in T(x,\cdot)} f_n(z)\right\}^* (-p_n(x))$$

for each $x \in X$. *Moreover, if* $r(x,\cdot)$ *is l.s.c. and* $A(x)$ *is compact for all* $x \in X$, *there exists* $\{q_n\}_{n=1}^\infty \subset \mathscr{F}(X;Y^*)$ *such that for each initial state* $x_1 \in X$,

$$\lim_{n\to\infty} h_n(x) = \widehat{h}(x),$$

where the sequence of functions $\{h_n\}_{n=1}^\infty \subset \mathscr{F}(X;\overline{\mathbf{R}}_+)$ *is generated as follows: for any* $x \in X$,

$$h_0 \equiv 0, \ h_{n+1}(x) \equiv -\left\{r(x,\cdot) + \delta_{A(x)}\right\}^* (q_n(x)) + 0\left\{-\beta \sup_{z\in T(x,\cdot)} h_n(z)\right\}^* (q_n(x))$$

Proof. First, we show that for all f, $g \in B(X)$,

$$\|\psi^\flat(f) - \psi^\flat(g)\|_\infty \leq \beta\|f-g\|_\infty$$

and

$$\|\psi^\sharp(f) - \psi^\sharp(g)\|_\infty \leq \beta\|f-g\|_\infty.$$

For all f, $g \in B(X)$, and for all $x \in X$, we have

$$\begin{aligned}
&\psi^\flat(f)(x) - \psi^\flat(g)(x) \\
&\qquad = \sup_{y^*\in Y^*}\left[-\{r(x,\cdot)+\delta_{A(x)}\}^*(y^*) - \left\{\beta \sup_{z\in T(x,\cdot)} f(z)\right\}^*(-y^*)\right] \\
&\qquad\quad - \sup_{y^*\in Y^*}\left[-\{r(x,\cdot)+\delta_{A(x)}\}^*(y^*) - \left\{\beta \sup_{z\in T(x,\cdot)} g(z)\right\}^*(-y^*)\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{y^*\in Y^*}\left[-\left\{\beta\sup_{z\in T(x,\cdot)} f(z)\right\}^*(y^*)+\left\{\beta\sup_{z\in T(x,\cdot)} g(z)\right\}^*(y^*)\right]\\
&= \sup_{y^*\in Y^*}\left[-\sup_{y\in Y}\left\{\langle y,y^*\rangle-\beta\sup_{z\in T(x,\cdot)} f(z)\right\}+\sup_{y\in Y}\left\{\langle y,y^*\rangle-\beta\sup_{z\in T(x,\cdot)} g(z)\right\}\right]\\
&\le \sup_{y^*\in Y^*}\sup_{y\in Y}\left\{\beta\sup_{z\in T(x,\cdot)} f(z)-\beta\sup_{z\in T(x,\cdot)} g(z)\right\}\\
&\le \beta\sup_{y\in Y}\sup_{z\in T(x,\cdot)}\{f(z)-g(z)\}\\
&\le \beta\|f-g\|_{\|x\|}.
\end{aligned}
$$

By replacing the situations of f and g each other,

$$|\psi^\flat(f)(x)-\psi^\flat(g)(x)|\le\beta\|f-g\|_{\|x\|}.$$

Hence

$$\|\psi^\flat(f)-\psi^\flat(g)\|_n\le\beta\|f-g\|_n.$$

By the definition of the norm $\|\cdot\|_\infty$,

$$\|\psi^\flat(f)-\psi^\flat(g)\|_\infty\le\beta\|f-g\|_\infty.$$

Again, for all f, $g\in B(X)$, and for all $x\in X$, we have

$$
\begin{aligned}
&\psi^\sharp(f)(x)-\psi^\sharp(g)(x)\\
&\qquad= \inf_{y^*\in Y^*}\left[-\{r(x,\cdot)+\delta_{A(x)}\}^*(y^*)+\left\{-\beta\sup_{z\in T(x,\cdot)} f(z)\right\}^*(y^*)\right]\\
&\qquad\quad- \inf_{y^*\in Y^*}\left[-\{r(x,\cdot)+\delta_{A(x)}\}^*(y^*)+\left\{-\beta\sup_{z\in T(x,\cdot)} g(z)\right\}^*(y^*)\right]\\
&\qquad\le \sup_{y^*\in Y^*}\left[\left\{-\beta\sup_{z\in T(x,\cdot)} f(z)\right\}^*(y^*)-\left\{-\beta\sup_{z\in T(x,\cdot)} g(z)\right\}^*(y^*)\right]\\
&\qquad= \sup_{y^*\in Y^*}\left[\sup_{y\in Y}\left\{\langle y,y^*\rangle+\beta\sup_{z\in T(x,\cdot)} f(z)\right\}-\sup_{y\in Y}\left\{\langle y,y^*\rangle+\beta\sup_{z\in T(x,\cdot)} g(z)\right\}\right]\\
&\qquad\le \sup_{y\in Y}\left\{\beta\sup_{z\in T(x,\cdot)} f(z)-\beta\sup_{z\in T(x,\cdot)} g(z)\right\}\\
&\qquad\le \beta\sup_{y\in Y}\sup_{z\in T(x,\cdot)}\{f(z)-g(z)\}\\
&\qquad\le \beta\|f-g\|_{\|x\|}.
\end{aligned}
$$

In the same way, we have

$$\|\psi^\sharp(f)-\psi^\sharp(g)\|_\infty\le\beta\|f-g\|_\infty.$$

By the Banach-Picard Theorem, there exist $\widehat{f}$ and $\widehat{h}\in B(X)$ such that $\psi^\flat(\widehat{f})=\widehat{f}$ and $\psi^\sharp(\widehat{h})=\widehat{h}$. Then we have $\widehat{f}(x)=\lim_{n\to\infty}(\psi^\flat)^n(0)(x)$ and $\widehat{h}(x)=$

$\lim_{n\to\infty}(\psi^\sharp)^n(0)(x)$. By the assumption $\psi^\flat : B(X) \to B(X)$, we have $(\psi^\flat)^n(0) \in B(X)$ for any $n \in \mathbf{N}$. Using Theorem 3.1, there exists $\{p_n\}_{n=1}^\infty \subset \mathscr{F}(X; Y^*)$ such that

$$(\psi^\flat)^{n+1}(0)(x) = -\left\{r(x,\cdot) + \delta_{A(x)}\right\}^* (p_n(x)) - \left\{\beta \sup_{z\in T(x,\cdot)} (\psi^\flat)^n(0)(z)\right\}^* (-p_n(x))$$

for all $x \in X$. Moreover, if $r(x,\cdot)$ is l.s.c. and $A(x)$ is compact for all $x \in X$, then by using Theorem 3.2, there exists $\{q_n\}_{n=1}^\infty \subset \mathscr{F}(X; Y^*)$ such that

$$(\psi^\sharp)^{n+1}(0)(x) = -\left\{r(x,\cdot) + \delta_{A(x)}\right\}^* (q_n(x)) + \left\{-\beta \sup_{z\in T(x,\cdot)} (\psi^\sharp)^n(0)(z)\right\}^* (q_n(x))$$

for each $x \in X$. Thus we have the proof of this theorem. □

4. Conclusions

In this paper, we have established some duality theorems for (DP). When Opt = inf, the duality formulation ψ has some attaining elements (say dual state and dual action) and ψ is equivalent to φ under a certain weak condition. Then we may solve the duality formulation ψ instead of the optimal loss function φ. When Opt = sup, however, the Fenchel-Rockafellar duality $\psi^\flat$ is not equivalent to the primal problem in general. Hence we derive the Kanniappan duality $\psi^\sharp$ and we have the inequalities $\psi^\flat \le \varphi \le \psi^\sharp$ without any condition. Also, their duality formulations $\psi^\flat$ and $\psi^\sharp$ attain their solutions under some weaker condition than a condition attains the primal problem φ. Hence we may solve the two duality formulations $\psi^\flat$ and $\psi^\sharp$ instead of the optimal loss function φ.

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