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CHARACTERIZATION THEOREM OF 4-VALUED DE MORGAN LOGIC

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Abstract. In this paper we give an axiom system of a non-linear 4-valued logic which we call a de Morgan logic (*ML*), whose Lindenbaum algebra is the de Morgan algebra with implication (*MI*-algebra), and show that

(1) For every *MI*-algebra L, there is a quotient *MI*-algebra $L^\sharp$ such that it is embeddable to the simplest 4-valued *MI*-algebra $\mathbf{M} = \{0, a, b, 1\}$;
(2) The Lindenbaum algebra of *ML* is the *MI*-algebra;
(3) The completeness theorem of *ML* is established;
(4) *ML* is decidable.

1. Introduction

It is well known the relation between logics and algebras through the property of Lindenbaum algebras. For example, the Lindenbaum algebra for the classical propositional logic (CPL) is a Boolean algebra and that of the intuitionistic propositional logic (IPL) is a Heyting one. On the other hand, CPL is characterized by the simplest non-trivial Boolean algebra $2 = \{0, 1\}$, that is, a formula A is provable in CPL if and only if (iff) $\tau(A) = 1$ for every function $\tau : \Pi \to 2$. In [3] or [4], we have an axiom system of Kleene logic (KL) and show in [3] that KL is characterized by the simplest non-trivial Kleene algebra $3 = \{0, 1/2, 1\}$. Now the next question arises:

(Question) What is a logic characterized by 4-valued algebra?

Of course, Rasiowa [5] has already given the axiom system of n-valued logic which is determined by n-valued Post-algebras. But Post-algebras are linear and so that these logics are not determined by 4-valued non-linear de Morgan algebra $\{0, a, b, 1\}$ below. In [1], it is proved that the class of de Morgan algebras with no implication is functional free for the algebra $\{0, a, b, 1\}$. Hence there is a question about the functional freeness of de Morgan algebras with implication.

Moreover, in the relevance logic investigated in [2], the set of first-degree formulas of the logic is determined by 4-valued de Morgan algebra $\{0, a, b, 1\}$, that is, for the zero-degree formulas (i.e., formulas with no implication simbols) A

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and B, $A \to B$ is provable in the relevance logic iff $\tau(A) \le \tau(B)$ for every $\tau : \Pi \to \{0, a, b, 1\}$. But the algebra $\{0, a, b, 1\}$ dose not essentially include the implication. Hence, from the view point of relevance logics, it is also interesting to find logics characterized by 4-valued de Morgan algebras with implication.

In this paper, we give an ansewr to these questions. That is, we shall define de Morgan algebras with implication (*MI*-algebras) and give an axiom system of some logic called de Morgan logic (*ML*), which is characterized by 4-valued de Morgan algebra **M** with implication. More precisely, we show that

(1) For every *MI*-algebra L there exists a quotient *MI*-algebra $L^\sharp$ such that it is embeddable to the simplest *MI*-algebra $\mathbf{M} = \{0, a, b, 1\}$;
(2) The Lindenbaum algebra of *ML* is a *MI*-algebra;
(3) The completeness theorem of *ML* is established;
(4) *ML* is decidable.

2. de Morgan algebra with implication

In this section we define a de Morgan algebra with implication. In this paper we call it an *MI*-algebra. By an *MI*-algebra, we mean an algebraic structure $L = (L; \wedge, \vee, \to, N, 0, 1)$ of type (2,2,2,1,0,0) such that
(1) $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice;
(2) $N : L \to L$ is a map satisfying the following conditions
(N1) $N0 = 1, N1 = 0$
(N2) $x \le y$ implies $Ny \le Nx$
(N3) $N(Nx) = x$, $N(Nx)$ is denoted by N^2x
(3) the implication $\to$ satisfies
(I1) $y \le x \to y$
(I2) $x \le y$ implies $x \to y = 1$
(I3) $Nx \wedge Ny \le (x \to y) \vee y$
(I4) $Nx \wedge y \wedge N(x \to y) = 0$
(I5) $x \wedge (x \to y) \le y \vee Ny$
(I6) $x \wedge (x \to y) \le Nx \vee y$
(I7) $x \wedge Nx \wedge (x \to y) \le N(x \to y) \vee y$

Example: As a model of *MI*-algebras we list $\mathbf{M} = \{0, a, b, 1\}$, where $0 < a = Na, b = Nb < 1$ and a, b are not comparable:

$\mathbf{M} = \{0, a, b, 1\}$

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	b	1
b	b	a	1	1
1	0	a	b	1

Let L be any *MI*-algebra. A non-empty subset F of L is called a filter when it satisfies the conditions:
(f1) $x, y \in F$ imply $x \wedge y \in F$;
(f2) $x \in F$ and $x \le y$ imply $y \in F$.

A filter F is called proper when it is a proper subset of L, that is, $0 \notin F$. We define two kinds of filters of L. By a maximal filter M, we mean the proper filter M such that $M \subseteq G$ implies $M = G$ for any proper filter G. A proper filter P is called prime if $x \vee y \in P$ implies $x \in P$ or $y \in P$ for every $x, y \in L$.

It is easy to show the next lemmas, so we omit their proofs.

Lemma 1. *If $x \in L$ and $x \neq 0$, then there is a maximal filter M of L such that $x \in M$.*

Lemma 2. *If M is a maximal filter of L, then it is a prime filter.*

Let M be an arbitrary maximal filter of L. We define the subsets L_j ($j \in \{0, a, b, 1\}$) of L:

$L_0 = \{x \in L | x \notin M, Nx \in M\}$
$L_a = \{x \in L | x \in M, Nx \in M\}$
$L_b = \{x \in L | x \notin M, Nx \notin M\}$
$L_1 = \{x \in L | x \in M, Nx \notin M\}$

Now we define an equivalence relation $\sim$ as follows: For $x, y \in L$,

$$x \sim y \Longleftrightarrow \exists L_j (x, y \in L_j) \ .$$

Since each subset L_j of L is an equivalence class, we denote it simply by $j^\sharp$, that is, $L_j = j^\sharp$. The relation $\sim$ is obtained alternatively.

Lemma 3. *$x \sim y$ if and only if $x \in M \Leftrightarrow y \in M$ and $Nx \in M \Leftrightarrow Ny \in M$.*

Lemma 4. *The relation $\sim$ is a congruence on L.*

Proof. It sufficies to show that
(1) $x \sim y$ and $p \sim q$ imply $(x \wedge p) \sim (y \wedge q)$;
(2) $x \sim y$ implies $Nx \sim Ny$;
(3) $x \sim y$ and $p \sim q$ imply $(x \to p) \sim (y \to q)$;
We only consider the case of (3).
Case (3): Suppose that $x \sim y$ and $p \sim q$. It is sufficient to show that

$$(x \to p) \in M \ \Leftrightarrow \ (y \to q) \in M \text{ and } N(x \to p) \in M \ \Leftrightarrow \ N(y \to q) \in M.$$

We prove that the left-hand statement implies the right-hand one in the first case. Suppose that $(x \to p) \in M$. We need to show that $(y \to q) \in M$. If $p \in M$, then we have $q \in M$ by assumption. Thus it follows from (I1) that $y \to q \in M$.

We suppose that $p \notin M$ (i.e. $q \notin M$). We have two cases (a) $x \notin M$ or (b) $x \in M$.

• Case (a) $x \notin M$ (i.e. $y \notin M$): From (I4) $y \vee Nq \vee (y \to q) = 1 \in M$ and assumption $y \notin M$, we have $Nq \in M$ or $(y \to q) \in M$. It goes well if $(y \to q) \in M$. If $Nq \in M$ (i.e. $Np \in M$), there are two subcases (a1) $Nx \in M$ or (a2) $Nx \notin M$.

Subcase (a1): If $Nx \in M$ (i.e. $Ny \in M$), then we have $Ny \wedge Nq \leq (y \to q) \vee q \in M$ by (I3). Since $q \notin M$, it is the case that $y \to q \in M$.

Subcase (a2): If $Nx \notin M$, then from (I3) we get that $N(x \to p) \wedge Np \leq x \vee p \notin M$. Since $Np \in M$, it follows that $N(x \to p) \notin M$. But in this case we obtain that $x \to p \notin M$ by (I7). This is a contradiction.

• Case (b) $x \in M$: We get from (I5) and (I6) that $Nx \in M$ (i.e. $Ny \in M$) and $Nq \in M$. It follows from (I3) that $Ny \wedge Nq \leq (y \to q) \vee q \in M$ and hence that $y \to q \in M$ by $q \notin M$.

Thus $x \to p \in M$ implies $y \to q \in M$. The other cases are proved similarly.

Hence the relation $\sim$ is the congruent relation on L.

Since $\sim$ is congruent, we can consistently define operations $\wedge$, $\vee$, $\to$, and N in $L^\sharp = \{0^\sharp, a^\sharp, b^\sharp, 1^\sharp\}$: For every $i^\sharp, j^\sharp \in L^\sharp$,

$i^\sharp \wedge j^\sharp = (i \wedge j)^\sharp$

$i^\sharp \vee j^\sharp = (i \vee j)^\sharp$

$i^\sharp \to j^\sharp = (i \to j)^\sharp$

$N(i^\sharp) = (Ni)^\sharp$

The general theory of universal algebras shows the next theorem.

Theorem 1. *Let L be a MI-algebra and M be a maximal filter of L. There exists a quotient MI-algebra $L^\sharp = (L^\sharp; \wedge, \vee, N, \to, 0^\sharp, 1^\sharp)$ which is embeddable to the MI-algebra* $\mathbf{M}$.

Proof. The map $\xi : L^\sharp \longrightarrow \mathbf{M}$ defined by $\xi(x^\sharp) = j$ when $x \in L_j$ gives the desired result.

3. de Morgan logic *ML*

In this section we shall define a de Morgan logic *ML*. As a language we use a countable set of propositional variables $p_1, p_2, ..., p_n, ...$ and logical symbols $\wedge, \vee, \neg$, and $\to$. We denote the set of propositional variables by Π, that is, $\Pi = \{p_1, p_2, ..., p_n, ...\}$. The formulas of *ML* are defined as usual. Let $A, B, C, ...$ be arbitrary formulas of *ML*. In the following we list an axiom system of *ML*.

Axioms;

(A1) $A \to A$

(A2) $A \wedge B \to A$

(A2)' $A \to A \vee B$

(A3) $A \wedge B \to B \wedge A$

(A3)' $A \vee B \to B \vee A$

(A4) $A \to \neg\neg A$

(A5) $\neg\neg A \to A$

(A6) $A \to (B \to A)$

(A7) $\neg A \wedge \neg B \to (A \to B) \vee B$

(A8) $A \vee \neg B \vee (A \to B)$

(A9) $A \wedge (A \to B) \to B \vee \neg B$

(A10) $A \wedge (A \to B) \to \neg A \vee B$

(A11) $A \wedge \neg A \wedge (A \to B) \to \neg(A \to B) \vee B$

(A12) $\neg(A \wedge B) \to \neg A \vee \neg B$

(A13) $\neg A \wedge \neg B \to \neg(A \vee B)$

(A14) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

(A15) $(A \vee B) \wedge (A \vee C) \to A \vee (B \wedge C)$

Rules of inference;

(R1) B is deduced from A and $A \to B$ (modus ponens, MP);
(R2) $A \to C$ is done from $A \to B$ and $B \to C$;
(R3) $A \to B \wedge C$ from $A \to B$ and $A \to C$;
(R4) $B \vee C \to A$ from $B \to A$ and $C \to A$;
(R5) $\neg A \to \neg B$ from $B \to A$;
(R6) $(B \to C) \to (A \to C)$ from $A \to B$;
(R7) $(C \to A) \to (C \to B)$ from $A \to B$ and $B \to A$.

Let A be a formula of *ML*. By $\vdash_{ML} A$ we mean that there is a sequence of formulas $A_1, A_2, ..., A_n$ of *ML* such that
(1) $A = A_n$
(2) For every A_i, it is an axiom or it is deduced from A_j and A_k $(j, k < i)$ by rules of inference.

We say that A is provable in *ML* when $\vdash_{ML} A$. If no confusion arises, we denote it simply by $\vdash A$.

It is easy to show the next lemmas, so we omit their proofs.

Lemma 5. *For every formula A and B of ML, we have that* $\vdash (A \to A) \to (B \to B)$.

We denote $p_1 \to p_1$ by t. Thus it is clear that $\vdash t \to (A \to A)$ for every formula A.

A function $\tau : \Pi \longrightarrow \mathbf{M}$ is called a valuation function. The domain of the valuation function can be extended uniquely to the set Φ of all formulas of *ML* as follows:
$\tau(A \wedge B) = \tau(A) \wedge \tau(B)$
$\tau(A \vee B) = \tau(A) \vee \tau(B)$
$\tau(\neg A) = N\tau(A)$
$\tau(A \to B) = \tau(A) \to \tau(B)$

Henceforth we use the same symbol τ for the extended valuation function.

We can show that the de Morgan logic *ML* is sound for the *MI*-algebra $\mathbf{M}$, that is, if $\vdash_{ML} A$ then $\tau(A) = 1$ for any valuation function τ.

Theorem 2. *Let A be an arbitrary formula of ML. If $\vdash_{ML} A$ then $\tau(A) = 1$ for every valuation function τ.*

Proof. By induction on the construction of a proof. It sufficies to show that $\tau(X) = 1$ for every axiom X and that the rules of inference preserve the validity.

As corollaries to the theorem we have the following.

Corollary 1. *ML is consistent.*

Proof. Since $\tau(\neg t) = 0$, the formula $\neg t$ is not provable in *ML*. Thus the de Morgan logic is consistent.

Corollary 2. *The de Morgan logic ML is different from the classical propositional logic (CPL), the intuitionistic propositional logic (IPL), and Kleene logic (KL).*

Proof. If we think of a valuation function τ such that $\tau(p_1) = a$ and $\tau(p_n) = b$ for $n = 2, 3, ...$, then we have that $\tau(p_1 \wedge \neg p_1) = a$ and $\tau(p_2 \vee \neg p_2) = b \neq 1$ and hence that the formulas $p_2 \vee \neg p_2$ and $(p_1 \wedge \neg p_1) \to (p_2 \vee \neg p_2)$ are not provable in *ML*. Thus *ML* is different from *CPL* and from *KL*. Next, the formula $\neg\neg A \to A$ is not provable in *IPL* in general, but it is provable in *ML*.

4. Completeness Theorem

In this section we shall establish the completeness theorem of the de Morgan logic *ML*, and it is the main theorem of this paper. The completeness theorem of *ML* means that a formula A is provable in *ML* if $\tau(A) = 1$ for any valuation function τ. As a method to show the theorem, we consider the Lindenbaum algebra of *ML* and investigate the property of that algebra.

We introduce the relation $\equiv$ on Φ as follows. For $A, B \in \Phi$,

$$A \equiv B \ \text{ iff } \ \vdash_{ML} A \to B \text{ and } \vdash_{ML} B \to A.$$

Lemma 6. *The relation $\equiv$ is a congruent relation on Φ.*

Proof. We only show that the relation $\equiv$ satisfies the next conditions: If $A \equiv X$ and $B \equiv Y$, then

(a) $(A \wedge B) \equiv (X \wedge Y)$
(b) $(A \vee B) \equiv (X \vee Y)$
(c) $\neg A \equiv \neg X$
(d) $(A \to B) \equiv (X \to Y)$

It is evident that the conditions (a), (b), and (c) hold from axioms (A2),(A2)',(A3),(A3)', and the rules of inference (R2),(R5),(R6). We prove that the condition (d) holds.

Suppose that $A \equiv X$ and $B \equiv Y$. Since $\vdash X \to A$, we have $\vdash (A \to B) \to (X \to B)$ by (R6). On the other hand, since $\vdash B \to Y$ and $\vdash Y \to B$, it follows that $(X \to B) \to (X \to Y)$ by (R7) and hence that $\vdash (A \to B) \to (X \to Y)$. Similarly it follows that $\vdash (X \to Y) \to (A \to B)$.

Hence the relation $\equiv$ is the congruent relation.

We think of the property of the quotient set $\Phi/_{\equiv}$ by the congruent relation $\equiv$. We set $\Phi/_{\equiv} = \{[A] | \ A \in \Phi\}$, where $[A] = \{X \in \Phi | \ A \equiv X\}$. We introduce an order relation $\sqsubseteq$ on $\Phi/_{\equiv}$ as follows: For any $[A]$, $[B] \in \Phi/_{\equiv}$,

$$[A] \sqsubseteq [B] \ \text{ iff } \ \vdash_{ML} A \to B.$$

Since the relation $\equiv$ is congruent, it is clear that the definition of $\sqsubseteq$ is well-defined and that the relation $\sqsubseteq$ is a partial order. Concerning to this order we have

Lemma 7. *inf*$\{[A],[B]\}=[A \wedge B]$, *sup*$\{[A],[B]\}=[A \vee B]$

Proof. We shall show the first case for the sake of simplicity. The second case can be proved analogously. Since $\vdash (A \wedge B) \to A$ and $\vdash (A \wedge B) \to B$, we obtain $[A \wedge B] \sqsubseteq [A]$, $[B]$. For any $[C]$ such that $[C] \sqsubseteq [A]$, $[B]$, since $\vdash C \to A$ and $\vdash C \to B$, it follows that $\vdash C \to (A \wedge B)$ by (R3). Thus we have *inf*$\{[A],[B]\} = [A \wedge B]$.

By the lemma we can define the operations $\sqcap$ and $\sqcup$ by

$$[A] \sqcap [B] = \inf\{[A],[B]\} = [A \wedge B]$$
$$[A] \sqcup [B] = \sup\{[A],[B]\} = [A \vee B].$$

It is easy to show that the structure $(\Phi/_{\equiv}; \sqcap, \sqcup)$ is a lattice. Moreover, if we put $[t] = \mathbf{1}$, $\mathbf{N}[A] = [\neg A]$, and $[A] \Rightarrow [B] = [A \to B]$, then the axioms of *ML* assures that the structure $(\Phi/_{\equiv}; \sqcap, \sqcup, \mathbf{N}, \Rightarrow, \mathbf{0}, \mathbf{1})$ is a *MI*-algebra. The structure is called a Lindenbaum algebra of *ML*. Hence we have the theorem.

Theorem 3. *The Lindenbaum algebra $\Phi/_{\equiv}$ of the de Morgan logic ML is the MI-algebra.*

As to that algebra $\Phi/_{\equiv}$, we have an important lemma.

Lemma 8. *For every formula A, $\vdash_{ML} A$ iff $[A] = \mathbf{1}$ in $\Phi/_{\equiv}$.*

Proof. Suppose that $\vdash A$. Since $A \to (t \to A)$ is provable in *ML*, we get that $\vdash t \to A$, that is, $[A] = \mathbf{1}$. Conversely if we assume that $[A] = \mathbf{1}$ then it follows $\vdash t \to A$ by definition. Thus we have $\vdash A$ by $\vdash t$.

Now we shall prove the completeness theorem of *ML*. In order to show that, it sufficies to indicate the existence of a valuation function τ such that $\tau(A) \neq \mathbf{1}$ if A is not provable in *ML*. Suppose that a formula A is not provable in *ML*. In the Lindenbaum algebra $\Phi/_{\equiv}$ of *ML*, we have $[A] \neq \mathbf{1}$ by the lemma above. It means that $\mathbf{N}[A] \neq \mathbf{0}$. By lemma 1, there is a maximal filter M in $\Phi/_{\equiv}$ such that $\mathbf{N}[A] \in M$. By theorem 1, there exists a quotient *MI*-algebra $(\Phi/_{\equiv})^{\sharp}$ which is embeddable to $\mathbf{M}$ by $\xi : (\Phi/_{\equiv})^{\sharp} \to \mathbf{M}$, where $\xi([X]^{\sharp}) = j$ if $[X] \in (\Phi/_{\equiv})_j$. For any propositional variable p, we put $\tau(p) = j$ if $[p] \in (\Phi/_{\equiv})_j$, that is,

$$\tau(p) = \begin{cases} 1 & \text{if } [p] \in M,\ \mathbf{N}[p] \notin M \\ a & \text{if } [p] \in M,\ \mathbf{N}[p] \in M \\ b & \text{if } [p] \notin M,\ \mathbf{N}[p] \notin M \\ 0 & \text{if } [p] \notin M,\ \mathbf{N}[p] \in M. \end{cases}$$

As to that function τ, we can show the next lemma.

Lemma 9. *For any formula $X \in \Phi$,*

$$\tau(X) = \begin{cases} 1 & \textit{if } [X] \in M,\ \mathbf{N}[X] \notin M \\ a & \textit{if } [X] \in M,\ \mathbf{N}[X] \in M \\ b & \textit{if } [X] \notin M,\ \mathbf{N}[X] \notin M \\ 0 & \textit{if } [X] \notin M,\ \mathbf{N}[X] \in M. \end{cases}$$

Proof. It sufficies to show that τ satisfies the followings: For arbitrary formulas X and Y,

(a) $\tau(X \wedge Y) = \tau(X) \wedge \tau(Y)$
(b) $\tau(X \vee Y) = \tau(X) \vee \tau(Y)$
(c) $\tau(\neg X) = N\tau(X)$
(d) $\tau(X \to Y) = \tau(X) \to \tau(Y)$

We only show the case (d). Let $x = [X]$ and $y = [Y]$.

Case (d):

• $\tau(X) = a, \tau(Y) = 0$: It is suffifient to prove that $x \Rightarrow y \in L_a$, that is, $x \Rightarrow y$ and $\mathbf{N}(x \Rightarrow y)$ are in M. By assumption we have $x \in M$, $\mathbf{N}x \in M, y \notin M, \mathbf{N}y \in M$. From (I3) $\mathbf{N}x \sqcap \mathbf{N}y \leq (x \Rightarrow y) \sqcup y$, since $y \notin M$, we have $x \Rightarrow y \in M$. Next, it follows from (I7) that $\mathbf{N}(x \Rightarrow y) \sqcup y \in M$ and hence that $\mathbf{N}(x \Rightarrow y) \in M$ because of $y \notin M$. These imply that $x \Rightarrow y \in L_a$.

• $\tau(X) = a, \tau(Y) = b$: In this case we have $x, \mathbf{N}x \in M$ and $y, \mathbf{N}y \notin M$ by definition. We can conclude that $x \Rightarrow y \notin M$, because (I5) $x \sqcap (x \Rightarrow y) \leq y \sqcup \mathbf{N}y \notin M$ and $x \in M$. That $\mathbf{N}y \notin M$ yields $\mathbf{N}(x \Rightarrow y) \notin M$ by (I1).

The other cases are proved similarly.

Well, since $\mathbf{N}[A] \in M$, it follows that $\tau(A) \neq \mathbf{1}$ by that lemma. Hence we have the completeness theorem of *ML*.

Theorem 4. *For any formula A, $\vdash_{ML} A$ iff $\tau(A) = 1$ for every valuation function τ.*

It turns out from the theorem that it is sufficient to calculate the value of $\tau(A)$ for every valuation τ, in order to show whether a formula A is provable or not in *ML*. Since any formula has at most finite numbers of propositional variables, say n, the possible values of the n-tuple of the propositional variables of that formula are finite (at most 4^n). Thus we can establish that

Theorem 5. *The de Morgan logic ML is decidable.*

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