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EXTREMUM PROBLEMS ON A HILBERT NETWORK

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Abstract. As a generalization of a usual infinite network, a Hilbert network is defined as a pair of a graph and a resistance taking values in a Hilbert space. With the sets of nodes and arcs of the graph, we associate variables belonging to a Hilbert space. In this situation, we study several extremum problems related to Hilbert-valued functions on the set of nodes or arcs of the graph and their inverse relations.

1. Introduction with preliminaries

Let $G = \{X, Y, K\}$ be a locally finite infinite graph which is connected and has no self-loof as in [4]. Here X is a countable set of nodes, Y is a countable set of arcs and K is the node-arc incidence matrix.

Let $\mathscr{H}$ be a real Hilbert space with an inner product $((\ \cdot, \cdot))$ and the norm $\|\cdot\|$. Denote by $L(X;\mathscr{H})$ the set of all functions u on X such that $u(x) \in \mathscr{H}$ for each $x \in X$ and by $L_0(X;\mathscr{H})$ the set of all $u \in L(X;\mathscr{H})$ such that the support $\{x \in X; u(x) \neq 0\}$ is a finite set. The meaning of the notation $L(Y;\mathscr{H})$ and $L_0(Y;\mathscr{H})$ is similar. Let $\mathscr{L}(\mathscr{H})$ be the set of all bounded, linear, positive and invertible linear operators from $\mathscr{H}$ to $\mathscr{H}$. Assume that $r \in L(Y;\mathscr{L}(\mathscr{H}))$. This is a generalization of the resistance in the ususal network theory as in [3] and [4]. We call the pair $N = \{G, r\}$ of the graph G and this generalized resistance r a Hilbert network as in [1], [6] and [7].

For each $y \in Y$, there exists $\rho(y) > 0$ by our assumption (cf. [5]) such that

$$((r(y)h, h)) \geq \rho(y)\|h\|^2 \quad \text{for all} \quad h \in \mathscr{H}.$$

Here $r(y)h$ means the image of h under $r(y)$, i.e., $r(y)(h)$. In this paper, we use this convention unless no confusion occurs from the context. Denote by $r(y)^{-1}$ the inverse operator of $r(y)$. Notice that there exists $\rho^*(y) > 0$ such that

$$((r(y)^{-1}h, h)) \geq \rho^*(y)\|h\|^2 \quad \text{for all} \quad h \in \mathscr{H}.$$

For each $y \in Y$, there exists a unique *square root* $r(y)^{1/2} \in \mathscr{L}(\mathscr{H})$ of $r(y)$ by [2] i.e.,

$$[r(y)^{1/2}]^2 = r(y).$$

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Before introducing extremum problems on the Hilbert network N, we need several preparations.

Definition 1.1. *Let e be a fixed element of $\mathscr{H}$ such that $\|e\| = 1$.*

Definition 1.2. *For $u \in L(X;\mathscr{H})$, the potential drop δu of u and the discrete derivative du of u are defined by*

$$\begin{aligned} \delta u(y) &:= \textstyle\sum_{x\in X} K(x,y)u(x), \\ du(y) &:= -r(y)^{-1}(\delta u(y)) = -r(y)^{-1}\delta u(y). \end{aligned}$$

The Dirichlet sum of u is defined by

$$D(u) := \textstyle\sum_{y\in Y}((r(y)du(y), du(y))) = \sum_{y\in Y}((r(y)^{-1}\delta u(y), \delta u(y))).$$

Definition 1.3. *For $w \in L(Y;\mathscr{H})$, the divergence $\partial w(x)$ of w and the energy $H(w)$ of w are defined by*

$$\begin{aligned} \partial w(x) &:= \textstyle\sum_{y\in Y} K(x,y)w(y), \\ H(w) &:= \textstyle\sum_{y\in Y}((r(y)w(y), w(y))). \end{aligned}$$

Notice that $D(u) = H(du)$. Let us put

$$\begin{aligned} D(N;\mathscr{H}) &:= \{u \in L(X;\mathscr{H}); D(u) < \infty\}, \\ L_H(Y;\mathscr{H}) &:= \{w \in L(Y;\mathscr{H}); H(w) < \infty\}. \end{aligned}$$

For every $w_1, w_2 \in L_H(Y;\mathscr{H})$, we define the inner product $H(w_1, w_2)$ by

$$H(w_1, w_2) := \textstyle\sum_{y\in Y}((r(y)w_1(y), w_2(y))).$$

For every $u_1, u_2 \in D(N;\mathscr{H})$, we define the mutual Dirichlet sum $D(u_1, u_2)$ by

$$D(u_1, u_2) := H(du_1, du_2) = \textstyle\sum_{y\in Y}((r(y)^{-1}\delta u_1(y), \delta u_2(y))).$$

Lemma 1.1. *Let $h \in \mathscr{H}$. For every $y \in Y$, the following relations hold:*
(1) $|((r(y)w(y), h))|^2 \le ((r(y)w(y), w(y)))((r(y)h, h))$.
(2) $1 \le ((r(y)^{-1}h, h))((r(y)h, h))$.

Proof. By the Schwarz inequality, we have

$$\begin{aligned} |((r(y)w(y), h))|^2 &= |((r(y)^{1/2}w(y), r(y)^{1/2}h))|^2 \\ &\le \|r(y)^{1/2}w(y)\|^2\|r(y)^{1/2}h\|^2 \\ &= ((r(y)w(y), w(y)))((r(y)h, h)). \end{aligned}$$

(2) follows from (1) by taking $w(y) := r(y)^{-1}h$. $\square$

Lemma 1.2. $|H(w_1, w_2)| \le H(w_1)^{1/2}H(w_2)^{1/2}$.

Proof. From the Schwarz inequality, it follows that

$$\begin{aligned}|H(w_1,w_2)| &\le \textstyle\sum_{y\in Y}|((r(y)w_1(y),w_2(y)))|\\ &= \textstyle\sum_{y\in Y}|((r(y)^{1/2}w_1(y),r(y)^{1/2}w_2(y)))|\\ &\le \textstyle\sum_{y\in Y}\|r(y)^{1/2}w_1(y)\|\|r(y)^{1/2}w_2(y)\|\\ &\le [\textstyle\sum_{y\in Y}\|r(y)^{1/2}w_1(y)\|^2]^{1/2}[\textstyle\sum_{y\in Y}\|r(y)^{1/2}w_2(y)\|^2]^{1/2}\\ &= H(w_1)^{1/2}H(w_2)^{1/2}. \quad \square\end{aligned}$$

Notice that $L_H(Y;\mathscr{H})$ is a Hilbert space with this inner product.

Lemma 1.3. *If $w\in L_0(Y;\mathscr{H})$, then $\sum_{y\in Y}r(y)w(y)\in\mathscr{H}$ and*

$$\textstyle\sum_{y\in Y}((r(y)w(y),h)) = ((\sum_{y\in Y}r(y)w(y),h))$$

for every $h\in\mathscr{H}$.

Proof. Since $r(y)w(y)\in\mathscr{H}$ for every $y\in Y$ and $w\in L_0(Y;\mathscr{H})$, our assertion is clear. $\square$

For $a\in X$, let us put

$$D(N;\mathscr{H};a) := \{u\in D(N;\mathscr{H}); u(a)=0\}.$$

Lemma 1.4. *For any $x\in X$, there exists a constant M_x which such that*

$$\|u(x)\|\le M_x D(u)^{1/2}$$

for all $u\in D(N;\mathscr{H};a)$.

Proof. We may assume that $x\ne a$. There exists a path P from a to x. Let $C_X(P)$ and $C_Y(P)$ be the sets of nodes and arcs on P respectively (cf. [4]), i.e.,

$$C_X(P) := \{x_0,x_1,\cdots,x_n\}\ (x_0=a, x_n=x),$$

$$C_Y(P) := \{y_1,y_2,\cdots,y_n\},$$

$$\{x\in X; K(x,y_i)\ne 0\} = \{x_{i-1},x_i\}\ (i=1,2,\cdots,n).$$

Let $u\in D(N;\mathscr{H};a)$. Then we have

$$\begin{aligned}D(u) &\ge \textstyle\sum_{y\in C_Y(P)}((r(y)^{-1}\delta u(y),\delta u(y)))\\ &= \sum_{i=1}^n((r(y_i)^{-1}\delta u(y_i),\delta u(y_i)))\\ &\ge \sum_{i=1}^n\rho^*(y_i)\|u(x_i)-u(x_{i-1})\|^2\\ &\ge \sum_{i=1}^n\rho^*(y_i)[\|u(x_i)\|-\|u(x_{i-1})\|]^2,\end{aligned}$$

so that

$$\|u(x_i)\|-\|u(x_{i-1})\|\le D(u)^{1/2}[\rho^*(y_i)]^{-1/2}$$

, for $i = 1, 2, \cdots$. Since $u(a) = 0$, we have

$$\|u(x)\| = \sum_{i=1}^{n}[\|u(x_i)\| - \|u(x_{i-1})\|] \leq M_x D(u)^{1/2}$$

with

$$M_x := \sum_{i=1}^{n}[\rho^*(y_i)]^{-1/2}.$$

This completes the proof. □

We see that $D(u)^{1/2}$ is a norm on $D(N;\mathscr{H};a)$.

Proposition 1.1. *$D(N;\mathscr{H};a)$ is a Hilbert space with respect to the inner product $D(u_1, u_2)$.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $D(N;\mathscr{H};a)$, i.e., $D(u_n - u_m) \to 0$ as $n, m \to \infty$. Then $\{D(u_n)\}$ is bounded. It follows from Lemma 1.4 that $\{u_n(x)\}$ is a Cauchy sequence in $\mathscr{H}$ for each $x \in X$. Therefore there exists $\tilde{u}(x) \in \mathscr{H}$ such that $\|u_n(x) - \tilde{u}(x)\| \to 0$ as $n \to \infty$ for each $x \in X$. Thus $\tilde{u}(a) = 0$ and $\|du_n(y) - d\tilde{u}(y)\| \to 0$ as $n \to \infty$ for each $y \in Y$. Since $\{D(u_n)\}$ is bounded, we see that $\tilde{u} \in D(N;\mathscr{H})$ by Fatou's lemma. For any $\epsilon > 0$, there exists n_0 such that $D(u_n - u_m) < \epsilon^2$ for all $n, m \geq n_0$. For any finite subset Y' of Y,

$$\sum\nolimits_{y \in Y'}((r(y)d(u_n - u_m)(y), d(u_n - u_m)(y))) \leq D(u_n - u_m).$$

Letting $m \to \infty$, we have

$$\sum\nolimits_{y \in Y'}((r(y)d(u_n - \tilde{u})(y), d(u_n - \tilde{u})(y))) \leq \epsilon^2$$

for all $n \geq n_0$, so that $D(u_n - \tilde{u}) \leq \epsilon^2$. Hence, $D(u_n - \tilde{u}) \to 0$ as $n \to \infty$. □

Denote by $D_0(N;\mathscr{H};a)$ the closure of the set

$$L_0(X;\mathscr{H};a) := \{u \in L_0(X;\mathscr{H}); u(a) = 0\}$$

in the Hilbert space $D(N;\mathscr{H};a)$.

2. $\mathscr{H}$-FLOWS

Definition 2.1. *Let a and b be distinct two nodes. We say that $w \in L(Y;\mathscr{H})$ is an $\mathscr{H}$-flow from a to b if the following conditions are fulfilled:*
(F.1) $\partial w(x) = 0$ *for all* $x \in X \setminus \{a, b\}$;
(F.2) $\partial w(a) + \partial w(b) = 0$.

Denote by $F(a, b;\mathscr{H})$ the set of all $\mathscr{H}$-flows from a to b.

Definition 2.2. *For each $w \in F(a, b;\mathscr{H})$, we introduce the following two quantities:*

$$\begin{aligned} I_e(w) &:= ((\partial w(b), e)) = -((\partial w(a), e)), \\ I(w) &:= \|\partial w(a)\| = \|\partial w(b)\|. \end{aligned}$$

Let us put $F_0(a, b;\mathscr{H}) := F(a, b;\mathscr{H}) \cap L_0(Y;\mathscr{H})$ and denote by $F_H(a, b;\mathscr{H})$ the closure of $F_0(a, b;\mathscr{H})$ in $L_H(Y;\mathscr{H})$.

Lemma 2.1. *Assume that N is a finite network. If $w \in L(Y;\mathscr{H})$ satisfies (F.1), then it does also (F.2).*

Proof. Since N is a finite network and

$$\sum_{x\in X} K(x,y) = 0$$

for each $y \in Y$, we have by changing the order of summation

$$\partial\tilde{w}(a) + \partial\tilde{w}(b) = \sum_{x\in X} \partial\tilde{w}(x) = \sum_{y\in Y}[\sum_{x\in X} K(x,y)]\tilde{w}(y) = 0. \quad \square$$

Similarly we have

Lemma 2.2. *If $w \in L_0(Y;\mathscr{H})$ satisfies (F.1), then it does (F.2).*

Corollary 2.1. *(F.1) implies (F.2) for every $w \in F_H(a,b;\mathscr{H})$.*

Lemma 2.3. *Let $u \in L(X;\mathscr{H})$ and $w \in L_0(Y;\mathscr{H})$. Then*

$$\sum_{y\in Y}((w(y), \delta u(y))) \leq H(w)^{1/2} D(u)^{1/2}.$$

Proof. We have by Lemma 1.2

$$\sum_{y\in Y}((w(y), \delta u(y))) = H(w, du) \leq H(w)^{1/2} H(du)^{1/2} \leq H(w)^{1/2} D(u)^{1/2}.$$

Corollary 2.2. *Let $u \in D(N;\mathscr{H})$ and $w \in F_H(a,b;\mathscr{H})$. Then*

$$\sum_{y\in Y}((w(y), \delta u(y))) \leq H(w)^{1/2} D(u)^{1/2}.$$

Proof. There exists a sequence $\{w_n\}$ in $F_0(a,b;\mathscr{H})$ such that $H(w_n - w) \to 0$ as $n \to \infty$. We have by Lemma 2.3 $H(w_n, du)) \leq H(w_n)^{1/2} D(u)^{1/2}$. Since $du \in L_H(Y;\mathscr{H})$, we see that $H(w_n, du) \to H(w, du)$ and $H(w_n) \to H(w)$ as $n \to \infty$. $\square$

Lemma 2.4. *Let $u \in D(N;\mathscr{H})$ and $w \in F_H(a,b;\mathscr{H})$. Then*

$$\sum_{x\in X}((u(x), \partial w(x))) = \sum_{y\in Y}((\delta u(y), w(y))).$$

Proof. There exists a sequence $\{w_n\}$ in $F_0(a,b;\mathscr{H})$ such that $H(w_n - w) \to 0$ as $n \to \infty$. Since the support of w_n is a finite set, we have

$$\begin{aligned} ((u(a), \partial w_n(a))) + ((u(b), \partial w_n(b))) &= \sum_{x\in X}((u(x), \partial w_n(x))) \\ &= \sum_{y\in Y}((\delta u(y), w_n(y))) = H(du, w_n). \end{aligned}$$

By letting $n \to \infty$, we obtain the desired inequality, since $du \in L_H(Y;\mathscr{H})$ and $\partial w(x) = 0$ for $x \in X \setminus \{a,b\}$. $\square$

Denote by $C_0(N)$ the set of all finite cycles on N, i.e.,

$$C_0(N) := \{\omega \in L_0(Y;\mathscr{H}); \partial\omega(x) = 0 \text{ on } X\}.$$

Lemma 2.5. *Let $\tilde{w} \in F(a,b;\mathscr{H})$ such that $H(\tilde{w}) < \infty$. Suppose that $H(\tilde{w},\omega) =$ 0 for every $\omega \in C_0(N)$. Then there exists $\tilde{u} \in D(N;\mathscr{H};a)$ such that $d\tilde{u} = -\tilde{w}$.*

Proof. Let p_1, p_2 be path indices of paths from a to x (cf. [4]). First we shall prove

$$\sum_{y\in Y} p_1(y)r(y)\tilde{w}(y) = \sum_{y\in Y} p_2(y)r(y)\tilde{w}(y).$$

In fact, for any $h \in \mathscr{H}$, $\omega(y) := (p_1(y) - p_2(y))h$ belongs to $C_0(N)$, so that we have by our assumption

$$0 = H(\tilde{w}, (p_1 - p_2)h) = \sum_{y\in Y}((r(y)[(p_1(y) - p_2(y))\tilde{w}(y)], h)).$$

Since $(p_1 - p_2)\tilde{w} \in L_0(Y; \mathscr{H})$, we see by Lemma 1.3.

$$((\sum_{y\in Y} r(y)[(p_1(y) - p_2(y))\tilde{w}(y)], h)) = 0.$$

Since $h \in \mathscr{H}$ is arbitrary, our assertion follows. Define $\tilde{u} \in L(X; \mathscr{H})$ by $\tilde{u}(a) = 0$ and

$$\tilde{u}(x) := \sum_{y\in Y} p_x(y)\tilde{w}(y) \text{ for } x \neq a,$$

where p_x is the path index of a path from a to x. This function is well-defined by the above observation. Let $y' \in Y$ and $\{x \in X; K(x, y') \neq 0\} = \{x_1, x_2\}$. Let p_{x_2} be the path index of a path P_{x_2} from a to x_2 which passes the arc y' after the node x_1. Namely P_{x_2} consists of a path P_{x_1} from a to x_1 and the single arc y'. We have

$$\begin{aligned}
\tilde{u}(x_2) &= \sum_{y\in Y} p_{x_2}(y)\tilde{w}(y) \\
&= \sum_{y\in Y} p_{x_1}(y)\tilde{w} + r(y')K(x_1, y')\tilde{w}(y') \\
&= \tilde{u}(x_1) + r(y')K(x_1, y')\tilde{w}(y'),
\end{aligned}$$

so that $\tilde{u}(x_2) = \tilde{u}(x_1) + r(y')K(x_1, y')\tilde{w}(y')$, or $\delta\tilde{u}(y') = -r(y')\tilde{w}(y')$. □

3. Inverse relation I

Now let us consider the following pair of extremum problems on the Hilbert network N which are related to $\mathscr{H}$-valued functions on X or Y:

$$\begin{aligned}
d_e(a, b; \mathscr{H}) &:= \inf\{D(u); u \in L(X; \mathscr{H}), ((u(a), e)) = 0,\ ((u(b), e)) = 1\}, \\
d^*(a, b; \mathscr{H}; e) &:= \inf\{H(w); w \in F_H(a, b; \mathscr{H}),\ \partial w(b) = e\}
\end{aligned}$$

First we have

Theorem 3.1. $1 \leq d_e(a, b; \mathscr{H})d^*(a, b; \mathscr{H}, e)$.

Proof. Let u be a feasible solution for $d_e(a, b; \mathscr{H})$ and let w be a feasible solution for $d^*(a, b; \mathscr{H}; e)$. It suffices to show that $1 \leq H(w)^{1/2}D(u)^{1/2}$. There exists a sequence $\{w_n\}$ in $F_0(a, b; \mathscr{H})$ such that $H(w - w_n) \to 0$ as $n \to \infty$. We have by Lemma 2.3

$$\begin{aligned}
1 = ((u(b), e)) &= ((u(b), \partial w(b))) = \lim_{n\to\infty}((u(b), \partial w_n(b))) \\
&= \lim_{n\to\infty}\sum_{x\in X}((u(x), \partial w_n(x))) = \lim_{n\to\infty}\sum_{y\in Y}((\delta u(y), w_n(y))) \\
&\leq \lim_{n\to\infty} H(w_n)^{1/2}D(u)^{1/2} = H(w)^{1/2}D(u)^{1/2}. \quad \square
\end{aligned}$$

To prove the converse inequality, we prepare

Lemma 3.1. *There exists a unique optimal solution for* $d^*(a, b; \mathscr{H}; e)$.

Proof. Let $\{w_n\}$ be a minimizing sequence for $d^*(a, b; \mathscr{H}; e)$, i.e., $\{w_n\} \subset F_H(a, b; \mathscr{H})$, $\partial w_n(b) = e$ and $H(w_n) \to d^*(a, b; \mathscr{H}; e)$ as $n \to \infty$. Since $(w_n + w_m)/2$ is a feasible solution for $d^*(a, b; \mathscr{H}; e)$, we have

$$\begin{aligned} d^*(a, b; \mathscr{H}; e) &\leq H((w_n + w_m)/2) \\ &\leq H((w_n + w_m)/2) + H((w_n - w_m)/2) \\ &= [H(w_n) + H(w_m)]/2 \to d^*(a, b; \mathscr{H}; e) \end{aligned}$$

as $m, n \to \infty$. Therefore $H(w_n - w_m) \to 0$ as $n, m \to \infty$. It follows that $\{w_n\}$ is a Cauchy sequence in the Hilbert space $L_H(Y; \mathscr{H})$. There exists $\tilde{w} \in L_H(Y; \mathscr{H})$ such that $H(w_n - \tilde{w}) \to 0$ as $n \to \infty$. Then $\tilde{w} \in F_H(a, b; \mathscr{H})$, $\partial\tilde{w}(b) = e$ and $d^*(a, b; \mathscr{H}; e) = H(\tilde{w})$. Namely $\tilde{w}$ is an optimal solution for $d^*(a, b; \mathscr{H}; e)$. Since $H(w)$ is a strictly convex function of $w \in L_H(Y; \mathscr{H})$, the uniqueness of the optimal solution follows. □

Lemma 3.2. *Let* $\tilde{w}$ *be the optimal solution for* $d^*(a, b; \mathscr{H}; e)$. *Then* $H(\tilde{w}, \omega) = 0$ *for every* $\omega \in C_0(N)$.

Proof. For any $\omega \in C_0(N)$ and $t \in \mathbf{R}$, $\tilde{w} + t\omega$ is a feasible solution for $d^*(a, b; \mathscr{H}; e)$. Thus

$$H(\tilde{w}) \leq H(\tilde{w} + t\omega) = H(\tilde{w}) + 2tH(\tilde{w}, \omega) + t^2 H(\omega).$$

By the standard variational argument, we obtain $H(\tilde{w}, \omega) = 0$. □

Lemma 3.3. *Let* $\tilde{w}(y)$ *be the same as above. There exists* $\tilde{u} \in D(N; \mathscr{H})$ *such that* $\tilde{u}(a) = 0$, $((\tilde{u}(b), e)) = d^*(a, b; \mathscr{H}; e)$ *and* $\delta\tilde{u} = -\tilde{w}$.

Proof. Let $\tilde{u}$ be the function defined by $\tilde{w}$ in Lemma 3.2. Then $\tilde{u}(a) = 0$ and $d\tilde{u} = -\tilde{w}$. There exists $\{w_n\} \subset F_0(a, b; \mathscr{H})$ such that $H(w_n - \tilde{w}) \to 0$ as $n \to \infty$. Let p_b a path index of a path from a to b. Since $w_n - p_b\partial w_n(b) \in C_0(N)$, we have $H(\tilde{w}, w_n - p_b\partial w_n(b)) = 0$. From $\partial w_n(b) \to \partial w(b) = e$, it follows that $H(\tilde{w}, \tilde{w} - p_b e) = 0$, so that

$$d^*(a, b; \mathscr{H}; e) = H(\tilde{w}) = H(\tilde{w}, p_b e) = ((\tilde{u}(b), e)). \quad \square$$

Theorem 3.2. $d_e(a, b; \mathscr{H})d^*(a, b; \mathscr{H}; e) = 1$.

Proof. Let $\tilde{w}$ be the optimal solution for $d^*(a, b; \mathscr{H}; e)$ and let $\tilde{u}$ be the function defined in Lemma 3.3. Then $v := \tilde{u}/d^*(a, b; \mathscr{H}; e)$ is a feasible solution for $d_e(a, b; \mathscr{H})$ and

$$\begin{aligned} d_e(a, b; \mathscr{H}) &\leq D(v) = D(\tilde{u})/d^*(a, b; \mathscr{H}; e)^2 \\ &= H(\tilde{w})/(d^*(a, b; \mathscr{H}; e)^2 = 1/d^*(a, b; \mathscr{H}; e), \end{aligned}$$

so that $d_e(a, b; \mathscr{H})d^*(a, b; \mathscr{H}; e) \leq 1$. Thus the equality holds by Theorem 3.1. □

4. Inverse relation II

Let us consider further extremum problems on the Hilbert network N:

$$\begin{aligned} d(a,b;\mathscr{H};e) &:= \inf\{D(u); u \in L(X;\mathscr{H}), u(a)=0,\ u(b)=e\}, \\ d(a,b;\mathscr{H}) &:= \inf\{D(u); u \in L(X;\mathscr{H}), u(a)=0,\ \|u(b)\|=1\}, \\ d_e^*(a,b;\mathscr{H}) &:= \inf\{H(w); w \in F_H(a,b;\mathscr{H}),\ I_e(w)=1\}, \\ d^*(a,b;\mathscr{H}) &:= \inf\{H(w); w \in F_H(a,b;\mathscr{H}),\ I(w)=1\}. \end{aligned}$$

Clearly

$$d_e(a,b;\mathscr{H}) \le d(a,b;\mathscr{H};e), \quad d(a,b;\mathscr{H}) \le d(a,b;\mathscr{H};e),$$

$$d_e^*(a,b;\mathscr{H}) \le d^*(a,b;\mathscr{H};e), \quad d^*(a,b;\mathscr{H}) \le d^*(a,b;\mathscr{H};e).$$

We have

Theorem 4.1. $1 \le d(a,b;\mathscr{H};e)d_e^*(a,b;\mathscr{H})$.

Proof. It suffices to show that $1 \le H(w)^{1/2}D(u)^{1/2}$ holds for any feasible solution u for $d(a,b;\mathscr{H};e)$ and any feasible solution w for $d_e^*(a,b;\mathscr{H})$. By the corollary of Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} 1 = I_e(w) &= ((\partial w(b),e)) = \sum\nolimits_{x\in X}((\partial w(x),u(x))) \\ &= \sum\nolimits_{y\in Y}((w(y),\delta u(y))) \\ &\le H(w)^{1/2}D(u)^{1/2}. \quad \square \end{aligned}$$

To prove the converse inequality, we prepare

Lemma 4.1. *There exists a unique optimal solution for* $d(a,b;\mathscr{H};e)$.

Proof. Let $\{u_n\}$ be a minimizing sequence for $d(a,b;\mathscr{H};e)$, i.e., $\{u_n\} \subset D(N;\mathscr{H};a)$, $u_n(b)=e$ and $D(u_n) \to d(a,b;\mathscr{H};e)$ as $n \to \infty$. Since $(u_n+u_m)/2$ is a feasible solution for $d(a,b;\mathscr{H};e)$, we have

$$\begin{aligned} d(a,b;\mathscr{H};e) &\le D((u_n+u_m)/2) \\ &\le D((u_n+u_m)/2) + D((u_n-u_m)/2) \\ &= [D(u_n)+D(u_m)]/2 \to d(a,b;\mathscr{H};e) \end{aligned}$$

as $n \to \infty$. Therefore $D(u_n-u_m) \to 0$ as $n,m \to \infty$. It follows from Proposition 1.1 that there exists $\tilde{u} \in D(N;\mathscr{H};a)$ such that$D(u_n-\tilde{u}) \to 0$ as $n \to \infty$. Clearly $\tilde{u}(b)=e$ and $\alpha = D(\tilde{u})$. Namely $\tilde{u}$ is an optimal solution. The uniqueness of the optimal solution follows from the fact that $D(u)$ is strict convex on $D(N;\mathscr{H};a)$. $\square$

Lemma 4.2. *Assume that N is a finite network. Let $\tilde{u}$ be the optimal solution for $d(a,b;\mathscr{H};e)$ and put $\tilde{w}(y) := d\tilde{u}(y)$. Then $\tilde{w} \in F(a,b;\mathscr{H})$ and $I_e(\tilde{w}) = D(\tilde{u})$.*

Proof. Let $f \in D(N;\mathscr{H})$ satisfy $f(a)=f(b)=0$. Then, for any $t \in \mathbf{R}$, $\tilde{u}+tf$ is a feasible solution for $d(a,b;\mathscr{H};e)$, so that

$$D(\tilde{u}) \le D(\tilde{u}+tf) = D(\tilde{u}) + 2tD(\tilde{u},f) + t^2D(f).$$

By the standard variational argument, we have $D(\tilde{u}, f) = 0$. On the other hand, we have

$$\begin{aligned} D(\tilde{u}, f) &= \sum_{y\in Y}((\tilde{w}(y), \sum_{z\in X} K(z,y)f(z))) \\ &= \sum_{z\in X}\sum_{y\in Y}((K(z,y)\tilde{w}(y), f(z))) \\ &= \sum_{z\in X}((\partial\tilde{w}(z), f(z))). \end{aligned}$$

Denote by ε_x the characteristic function of $\{x\}$, i.e., $\varepsilon_x(x) = 1$ and $\varepsilon_x(z) = 0$ for $z \neq x$. Let $x \neq a, b$. For any $h \in \mathscr{H}$, we may take $\varepsilon_x h$ for f, which leads to

$$((\partial\tilde{w}(x), h)) = 0.$$

Therefore $\partial\tilde{w}(x) = 0$ for $x \neq a, b$. Namely $\tilde{w}$ satisfies (F.1). Since N is a finite network, we have $\tilde{w} \in F(a, b; \mathscr{H})$ by Lemma 2.1. By taking $\tilde{u} - \varepsilon_b e$ for f, we obtain $D(\tilde{u}, \tilde{u} - \varepsilon_b e) = 0$, so that

$$D(\tilde{u}) = D(\tilde{u}, \varepsilon_b e) = ((\partial\tilde{w}(b), e)).$$

Therefore $I_e(\tilde{w}) = D(\tilde{u})$. □

Theorem 4.2. *Assume that N is a finite network. Then the inverse relation $d(a, b; \mathscr{H}; e)d_e^*(a, b; \mathscr{H}) = 1$ holds.*

Proof. Let $\tilde{u}$ be the optimal solution for $d(a, b; \mathscr{H}; e)$ and let $\tilde{w} = d\tilde{u}$. We see by Lemma 4.2 that $\tilde{w}(y)/D(\tilde{u})$ is a feasible solution for $d_e^*(a, b; \mathscr{H})$, so that

$$\begin{aligned} d_e^*(a, b; \mathscr{H}) &\leq H(\tilde{w}(y)/D(\tilde{u})) \\ &= D(\tilde{u})/D(\tilde{u})^2 \\ &= 1/D(\tilde{u}) = 1/d(a, b; \mathscr{H}; e). \end{aligned}$$

Thus $d(a, b; \mathscr{H}; e)d_e^*(a, b; \mathscr{H}) \leq 1$. □

In order to establish the equality in Theorem 4.2 in the case where N is an infinite network, we consider an exhaustion $\{G_n\}(G_n := < X_n, y_n >)$ of G (cf. [4]) with $a, b \in X_1$. A Hilbert subnetwork N_n of N is defined as the pair of the pair of G_n and the restriction of r onto Y_n.

On each finite subnetwork N_n, we define the Dirichlet mutual sum of $u_1, u_2 \in L(X_n; \mathscr{H})$ by

$$D_n(u_1, u_2) := \sum_{y\in Y_n}((r(y)du_1(y), du_2(y)))$$

and put $D_n(u) = D_n(u, u)$. For $w \in L(Y_n; \mathscr{H})$, we define $H_n(w)$ and $\partial_n w$ by

$$\begin{aligned} H_n(w) &:= \sum_{y\in Y_n}((r(y)w(y), w(y))), \\ \partial_n w(x) &:= \sum_{y\in Y_n} K(x,y)w(y). \end{aligned}$$

For large n, we have $\partial_n w(a) = \partial w(a)$ and $\partial_n w(b) = \partial w(b)$. Let us consider the following extremum problems on N_n:

$$\begin{aligned} d_n := d(a, b; N_n; \mathscr{H}; e) &:= \inf\{D_n(u); u \in L(X_n; \mathscr{H}), u(a) = 0,\ u(a) = e\}, \\ d_n^* := d_e^*(a, b; N_n; \mathscr{H}) &:= \inf\{H_n(w); w \in F_n(a, b; \mathscr{H}), ((\partial_n w(b), e)) = 1\}, \end{aligned}$$

where $F_n(a, b; \mathscr{H}) := \{w \in L(Y_n; \mathscr{H}); \partial_n w(x) = 0 \text{ on } X_n \setminus \{a, b\}\}$.

Lemma 4.3. *$\{d(a,b;N_n;\mathscr{H};e)\}$ converges to $d(a,b;\mathscr{H};e)$ as $n\to\infty$.*

Proof. Let $\tilde{u}$ and u_n be the optimal solutions of $d(a,b;\mathscr{H};e)$ and d_n respectively. Then for every $f\in L(X_n;\mathscr{H})$ satisfying $f(a)=f(b)=0$, we have $D_n(u_n,f)=0$ as in the proof of Lemma 4.2. For $n<m$, we have

$$D_n(\tilde{u}-u_n,u_n)=0 \quad \text{and} \quad D_n(u_m-u_n,u_n)=0.$$

Furthermore

$$D_n(u_n)\le D_n(\tilde{u})\le D(\tilde{u})<\infty.$$

By the relation

$$0\le D_n(u_m-u_n)=D_n(u_m)-D_n(u_n)\le D_m(u_m)-D_n(u_n),$$

we see that the limit of $\{D_n(u_n)\}$ exists, and hence

$$\lim_{n\to\infty} D_n(u_m-u_n)=0.$$

For $k<n<m$, we have

$$D_k(u_m-u_n)\le D_n(u_m-u_n)\to 0 \quad (n\to\infty).$$

Thus $\{u_n\}$ is a Cauchy sequence with respect to D_k, and the limit of $\{u_n(x)\}$ exists for all $x\in X_k$ both in the sense of D_k and in the sense of norm convergence in $\mathscr{H}$. Let v be the limit of $\{u_n\}$. Then $v(a)=0$ and $v(b)=e$, so that $D(\tilde{u})\le D(v)$. Since $D_k(u_n)\le D_n(u_n)$ if $k\le n$, we have

$$D_k(v)=\lim_{n\to\infty} D_k(u_n)\le \lim_{n\to\infty} D_n(u_n)\le D(\tilde{u}).$$

Letting $k\to\infty$, we obtain $D(v)\le D(\tilde{u})$, and hence $D(v)=D(\tilde{u})$. By the uniqueness of the optimal solution, we have $v=\tilde{u}$ and

$$\lim_{n\to\infty} D_n(u_n)=D(\tilde{u}). \quad \square$$

Theorem 4.3. $d(a,b;\mathscr{H};e)d_e^*(a,b;\mathscr{H})=1$.

Proof. It is easily seen that for large n we have

$$d_n^*=\inf\{H(w);w\in F(a,b;\mathscr{H}),I_e(w)=1,w_n=0 \text{ on } Y\setminus Y_n\}.$$

Therefore we obtain $d_n^*\ge d_{n+1}^*\ge d_e^*(a,b;\mathscr{H})$, so that

$$d_e^*(a,b;\mathscr{H})\le \lim_{n\to\infty} d_n^*.$$

Since $d_n\cdot d_n^*=1$ by Theorem 4.2, we have by Lemma 4.3

$$d(a,b;\mathscr{H};e)d_e^*(a,b;\mathscr{H})\le \lim_{n\to\infty} d_n\cdot d_n^*=1.$$

Our equality follows from Theorem 4.1. $\square$

Corollary 4.1. *$\{d_e^*(a,b;N_n;\mathscr{H})\}$ converges to $d_e^*(a,b;\mathscr{H})$ as $n\to\infty$.*

5. Extremal length

Let a and b be two distinct nodes and let $\mathbf{P}_a, b$ be the set of all paths from a to b. For a path P and a function w on Y, we set for simplicity

$$\sum_P w(y) := \sum_{y \in C_Y(P)} w(y)$$

The extremal length $EL(a, b; \mathscr{H})$ of N between a and b is defined by the inverse of the value of the extremum problem:

$$EL(a, b; \mathscr{H})^{-1} := \inf\{H(w); w \in EL(\mathbf{P}_{a,b}; \mathscr{H})\},$$

where $EL(\mathbf{P}_{a,b}; \mathscr{H})$ is the set of all $w \in L(Y; \mathscr{H})$ satisfying

$$\sum_P \|r(y)w(y)\| \geq 1 \quad \text{for all} \quad P \in \mathbf{P}_{a,b}.$$

The extremal length $EL_e(a, b; \mathscr{H})$ of N between a and b is defined by the inverse of the value of the extremum problem:

$$EL_e(a, b; \mathscr{H})^{-1} := \inf\{H(w); w \in EL_e(\mathbf{P}_{a,b}; \mathscr{H})\},$$

where $EL_e(\mathbf{P}_{a,b}; \mathscr{H})$ is the set of all $w \in L(Y; \mathscr{H})$ satisfying

$$\sum_P |((r(y)w(y), e))| \geq 1 \quad \text{for all} \quad P \in \mathbf{P}_{a,b}.$$

We have

$$EL(a, b; \mathscr{H}) \geq EL_e(a, b; \mathscr{H}),$$

since $|((r(y)w(y), e))| \leq \|r(y)w(y)\|\|e\| = \|r(y)w(y)\|$.

Lemma 5.1. $EL_e(a, b; \mathscr{H})^{-1} \leq d_e(a, b; \mathscr{H})$.

Proof. Let u be any feasible solution for $d_e(a, b; \mathscr{H})$ and put $w(y) := du(y)$. Then $w(y) \in \mathscr{H}$ for each $y \in Y$. Let $P \in \mathbf{P}_{a,b}$ with $C_X(P) := \{x_0, x_1, \cdots, x_n\}$ ($x_0 = a, x_n = b$), $C_Y(P) := \{y_1, y_2, \cdots, y_n\}$ and $\{x \in X; K(x, y_i) \neq 0\} = \{x_{i-1}, x_i\}$ for $(i = 1, 2, \cdots, n)$ as in the proof of Lemma 1.4. Then we have

$$\begin{aligned}
\sum_P |((r(y)w(y), e))| &= \sum_{i=1}^{n} |((\delta u(y_i), e))| \\
&\geq \sum_{i=1}^{n} |((u(x_i) - u(x_{i-1}), e))| \\
&\geq ((u(b), e)) - ((u(a), e)) = 1.
\end{aligned}$$

Therefore

$$EL_e(a, b; \mathscr{H})^{-1} \leq H(w) = D(u),$$

and hence $EL_e(a, b; \mathscr{H})^{-1} \leq d_e(a, b; \mathscr{H})$. □

Lemma 5.2. *Let w be a feasible solution for $EL_e(a, b; \mathscr{H})$. Then*

$$d_e(a, b; \mathscr{H}) \leq \sum_{y \in Y} ((r(y)w(y), w(y)))((r(y)e, e))((r(y)^{-1}e, e)).$$

Proof. Put $V(y) := |((r(y)w(y), e))|$. Then

$$\sum_P V(y) \geq 1 \quad \text{for all} \quad P \in \mathbf{P}_{a,b}.$$

By the duality between the max-potential problem and the min-work problem (cf. [4]), we can find $\beta \in L(X; \mathbf{R})$ such that $\beta(a) = 0,\ \beta(b) = 1$ and $|\delta\beta(y)| \leq V(y)$ on Y. Let $u(x) := \beta(x)e$. Then $u \in L(X; \mathscr{H})$, $u(a) = 0$ and $u(b) = e$, so that by Lemma 1.1

$$\begin{aligned} d_e(a, b; \mathscr{H}) &\leq D(u) = \sum_{y\in Y}(r(y)^{-1}\delta u(y), \delta u(y)) \\ &= \sum_{y\in Y}(\delta\beta(y))^2((r(y)^{-1}e, e)) \\ &\leq \sum_{y\in Y} V(y)^2((r(y)^{-1}e, e)) \\ &\leq \sum_{y\in Y}((r(y)w(y), w(y)))((r(y)e, e))((r(y)^{-1}e, e)) \quad \square \end{aligned}$$

Theorem 5.1. *Let* $M(r) := \sup\{((r(y)e, e))((r(y)^{-1}e, e)); y \in Y\}$. *Then*

$$EL_e(a, b; \mathscr{H})^{-1} \leq d_e(a, b; \mathscr{H}) \leq M(r)EL_e(a, b; \mathscr{H})^{-1}.$$

Corollary 5.1. *Assume that* $((r(y)e, e))((r(y)^{-1}e, e)) = 1$ *for all* $y \in Y$. *Then* $d_e(a, b; \mathscr{H}) = EL_e(a, b; \mathscr{H})^{-1}$.

Remark 1. Let I be the identity map of $\mathscr{H}$ and let $\gamma \in L(Y; \mathbf{R})$ be positive. Then $r(y) = \gamma(y)I$ is positive and invertible. Clearly, we have $((r(y)e, e)) = \gamma(y)$ and $((r(y)^{-1}e, e)) = 1/\gamma(y)$, so that the condition in the above theorem holds in this case.

We shall prove

Theorem 5.2. *Assume that the graph* $G = \{X, Y, K\}$ *is a tree. Then*

$$d_e(a, b; \mathscr{H}) = EL_e(a, b; \mathscr{H})^{-1} = H(pe)^{-1} = \sum_P((r(y)e, e)),$$

where p *is the path index of the path* P *from* a *to* b.

Proof. Since the graph is a tree, there exists a unique path P from a to b. Let p be the path index of P. Then

$$F_H(a, b; \mathscr{H}) = \{tph; h \in \mathscr{H},\ t \in \mathbf{R}\}.$$

If w is a feasible solution for $d^*(a, b; \mathscr{H}; e)$, then $w = pe$ and

$$\begin{aligned} d^*(a, b; \mathscr{H}; e) &= H(pe) = \sum_{y\in Y}|p(y)|((r(y)e, e)) \\ &= \sum_P((r(y)e, e)). \end{aligned}$$

Let w be a feasible solution for $EL_e(a, b; \mathscr{H})^{-1}$. Then we have by Lemma 1.2

$$\begin{aligned} 1 &\leq \sum_P|((r(y)w(y), e))| = \sum_{y\in Y}|((r(y)w(y), p(y)e))| \\ &\leq H(w)^{1/2}H(pe)^{1/2}, \end{aligned}$$

so that $H(pe)^{-1} \leq H(w)$. Therefore by Theorem 3.2

$$d_e(a, b; \mathscr{H}) = H(pe)^{-1} \leq EL_e(a, b; \mathscr{H})^{-1}.$$

Our equality follows from Lemma 5.1. □

We show by an example that the equality $d_e(a,b;\mathscr{H}) = EL_e(a,b;\mathscr{H})^{-1}$ does not hold in general.

Example. Let $X = \{x_0, x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$ and define K by

$$\begin{aligned} K(x_0,y_1) &= K(x_0,y_2) = K(x_1,y_3) = -1, \\ K(x_1,y_2) &= K(x_2,y_1) = K(x_2,y_3) = 1 \end{aligned}$$

and $K(x,y) = 0$ for any other pair. Then $G = \{X, Y, K\}$ is a finite graph. Take $\mathscr{H}$ as $\mathbf{R}^2$ with the usual inner product and define $r(y)$ by

$$r(y_i) := \begin{pmatrix} 1 & 0 \\ 0 & t_i \end{pmatrix}$$

with $t_i > 0$ for $i = 1, 2, 3$. Then

$$r(y_i)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/t_i \end{pmatrix}.$$

Let $a = x_0,\ b = x_2$ in the above setting and let $e = (e_1, e_2)^T \in \mathbf{R}^2$. For $w \in L(Y; \mathbf{R}^2)$, set $w(y_i) = (\xi_i, \eta_i)^T$ for $i = 1, 2, 3$. Then

$$H(w) = \sum_{i=1}^{3} (\xi_i^2 + t_i \eta_i^2).$$

Let w be a feasible solution for $d^*(a,b;\mathbf{R}^2;e)$. Then $w(y_2) = w(y_3)$ or $\xi_2 = \xi_3,\ \eta_2 = \eta_3$ and

$$\xi_1 + \xi_2 = e_1, \quad \eta_1 + \eta_2 = e_2.$$

Minimizing $H(w)$ subject to this constraints, we obtain

$$d^*(a,b;\mathbf{R}^2;e) = \frac{2}{3}e_1^2 + \frac{t_1(t_2+t_3)}{t_1+t_2+t_3}e_2^2,$$

so that by Theorem 3.2

$$d_e(a,b;\mathbf{R}^2) = \frac{3(t_1+t_2+t_3)}{2(t_1+t_2+t_3)e_1^2 + 3t_1(t_2+t_3)e_2^2}.$$

On the other hand, the feasibility of $w \in L(Y;\mathbf{R}^2)$ for $EL_e(a,b;\mathbf{R}^2)$ implies

$$\begin{aligned} \xi_1 e_1 + t_1 \eta_1 e_2 &\geq 1, \\ (\xi_2 + \xi_3)e_1 + (t_2\eta_2 + t_3\eta_3)e_2 &\geq 1. \end{aligned}$$

Minimizing $H(w)$ subject to this constraints, we obtain

$$EL_e(a,b;\mathbf{R}^2)^{-1} = \frac{3e_1^2 + (t_1+t_2+t_3)e_2^2}{(e_1^2 + t_1e_2^2)[2e_1^2 + (t_2+t_3)e_2^2]}.$$

We have

$$d_e(a,b;\mathbf{R}^2) - EL_e(a,b;\mathbf{R}^2)^{-1} = \frac{(t_2+t_3-2t_1)^2 e_1^2 e_2^2}{\alpha} \geq 0,$$

where

$$\alpha = (e_1^2 + t_1 e_2^2)[2e_1^2 + (t_2 + t_3)e_2^2][2(t_1 + t_2 + t_3)e_1^2 + 3t_1(t_2 + t_3)e_2^2].$$

The equality holds in case $e_1 = 0$, or $e_2 = 0$ or $t_2 + t_3 = 2t_1$.

6. Extremal width

Let a and b be distinct two nodes and let $\mathbf{Q}_{a,b}$ be the set of all cuts between a and b (cf. [4]).

The extremal width $EW(a, b; \mathscr{H})$ of N between a and b is defined by the inverse of the value of the extremum problem:

$$EW(a, b; \mathscr{H})^{-1} := \inf\{H(w); w \in EW(\mathbf{Q}_{a,b}; \mathscr{H})\},$$

where $EW(\mathbf{Q}_{a,b}; \mathscr{H})$ is the set of all $w \in L(Y; \mathscr{H})$ satisfying

$$\sum\nolimits_{y \in Q} \|w(y)\| \geq 1 \quad \text{for all} \quad Q \in \mathbf{Q}_{a,b}.$$

The extremal width $EW_e(a, b; \mathscr{H})$ of N between a and b is defined by the inverse of the value of the extremum problem:

$$EW_e(a, b; \mathscr{H})^{-1} := \inf\{H(w); w \in EW_e(\mathbf{Q}_{a,b}; \mathscr{H})\},$$

where $EW_e(\mathbf{Q}_{a,b}; \mathscr{H})$ is the set of all $w \in L(Y; \mathscr{H})$ satisfying

$$\sum\nolimits_{y \in Q} |((w(y), e))| \geq 1 \quad \text{for all} \quad Q \in \mathbf{Q}_{a,b}.$$

We have

$$EW(a, b; \mathscr{H}) \geq EW_e(a, b; \mathscr{H}),$$

since $|((w(y), e))| \leq \|w(y)\| \|e\| = \|w(y)\|$.

Lemma 6.1. *$EW_e(a, b; \mathscr{H})^{-1} \leq d_e^*(a, b; \mathscr{H})$.*

Proof. Let $Q \in \mathbf{Q}_{a,b}$. Then there exist two disjoint subsets $Q(a)$ and $Q(b)$ of X such that

$$a \in Q(a),\ b \in Q(b),\ X = Q(a) \cup Q(b) \quad \text{and} \quad Q = Q(a) \ominus Q(b).$$

For a subset A of X, denote by $\varepsilon_A \in L(X; \mathbf{R})$ the characteristic function of A. Then $|\delta\varepsilon_{Q(b)}(y)| = 1$ for $y \in Q$ and $|\delta\varepsilon_{Q(b)}(y)| = 0$ for $y \notin Q$. Let w be a feasible solution for $d_e^*(a, b; \mathscr{H})$. There exists a sequence $\{w_n\} \subset F_0(a, b; \mathscr{H})$ such that $H(w - w_n) \to 0$ as $n \to \infty$. We have

$$\begin{aligned} I_e(w_n) &= ((\partial w_n(b), e)) = \sum\nolimits_{x \in X} ((\partial w_n(x), \varepsilon_{Q(b)}(x)e)) \\ &= \sum\nolimits_{y \in Y} ((w_n(y), \delta\varepsilon_Q(y)e)) \\ &\leq \sum\nolimits_{y \in Q} |((w_n(y), e))|. \end{aligned}$$

Namely $w_n / I_e(w_n)$ is a feasible solution for $EW_e(a, b; \mathscr{H})$, so that

$$EW_e(a, b; \mathscr{H})^{-1} \leq H(w_n / I_e(w_n)) = H(w_n)/(I_e(w_n))^2.$$

Letting $n \to \infty$, we obtain $EW_e(a, b; \mathscr{H})^{-1} \leq H(w)$, so that $EW_e(a, b; \mathscr{H})^{-1} \leq d_e^*(a, b; \mathscr{H})$. $\square$

Lemma 6.2. *Let w be a feasible solution for $EW_e(a,b;\mathscr{H})$. Then*

$$d_e^*(a,b;\mathscr{H}) \le \sum\nolimits_{y\in Y}((r(y)w(y),w(y)))((r(y)e,e)(r(y)^{-1}e,e)).$$

Proof. Put $V(y) := |((w(y),e))|$. Then

$$\sum\nolimits_{y\in Q} V(y) \ge 1 \quad \text{for all} \quad Q \in \mathbf{Q}_{a,b}.$$

By the duality between the max-flow problem and the min-cut problem (cf. [4]), we can find $\varphi \in L(Y;\mathbf{R})$ such that $|\varphi(y)| \le V(y)$ on Y,

$$\partial\varphi(x) = 0 \quad \text{for} \quad x \in X \setminus \{a,b\} \quad \text{and} \quad -\partial\varphi(a) = \partial\varphi(b) = 1.$$

Let $w(y) := \varphi(y)e$. Then $w \in F(a,b;\mathscr{H})$ and $I_e(w) = 1$. Thus we have

$$\begin{aligned} d_e^*(a,b;\mathscr{H}) &\le H(w) = \sum\nolimits_{y\in Y}((r(y)\varphi(y)e,\varphi(y)e)) \\ &= \sum\nolimits_{y\in Y}[\varphi(y)]^2((r(y)e,e)) \\ &\le \sum\nolimits_{y\in Y}|((w(y),e))|^2((r(y)e,e)) \\ &\le \sum\nolimits_{y\in Y}((r(y)w(y),w(y)))((r(y)^{-1}e,e))((r(y)e,e)). \quad \square \end{aligned}$$

Theorem 6.1. *Let $M(r) := \sup\{((r(y)e,e))((r(y)^{-1}e,e)); y \in Y\}$. Then*

$$EW_e(a,b;\mathscr{H})^{-1} \le d_e^*(a,b;\mathscr{H}) \le M(r)EW_e(a,b;\mathscr{H})^{-1}.$$

Corollary 6.1. *Assume that $((r(y)e,e))((r(y)^{-1}e,e)) = 1$ for all $y \in Y$. Then $d_e^*(a,b;\mathscr{H}) = EW_e(a,b;\mathscr{H})^{-1}$.*

We show by an example that the equality $d_e^*(a,b;\mathscr{H}) = EW_e(a,b;\mathscr{H})^{-1}$ does not hold in general.

Example. Let $X = \{x_0,x_1,x_2\}$ and $Y = \{y_1,y_2\}$ and define K by

$$K(x_i,y_i) = 1,\ K(x_{i-1},y_i) = -1 \quad (i = 1,2)$$

and $K(x,y) = 0$ for any other pair. Then $G = \{X,Y,K\}$ is a finite graph. Notice that G is a tree. Take $\mathscr{H}$ as $\mathbf{R}^2$ and define $r(y)$ by

$$r(y_i) := \begin{pmatrix} 1 & 0 \\ 0 & t_i \end{pmatrix}$$

where $t_i > 0$ for $i = 1,2$. Then

$$r(y_i)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/t_i \end{pmatrix}.$$

Let $a = x_0,\ b = x_2$ in the above setting and let $e = (e_1,e_2)^T \in \mathbf{R}^2$. For $w(y_i) = (\xi_i,\eta_i) \in L(Y;\mathbf{R}^2)$, we have

$$H(w) = \sum_{i=1}^{2}(\xi_i^2 + t_i\eta_i^2).$$

If w is a feasible solution for $d_e^*(a,b;\mathbf{R}^2)$, then $\xi_1=\xi_2, \eta_1=\eta_2$ and $I_e(w)=1$ implies $\xi_1 e_1+\eta_1 e_2=1$. Minimizing $H(w)$ subject to this constraints, we obtain

$$d_e^*(a,b;\mathbf{R}^2)=\frac{1}{e_1^2/2+e_2^2/(t_1+t_2)}.$$

On the other hand, if w is feasible for $EW_e(a,b;\mathbf{R}^2)^{-1}$, then we have

$$\xi_1 e_1+\eta_1 e_1\geq 1, \quad \xi_2 e_1+\eta_2 e_2\geq 1.$$

Minimizing $H(w)$ subject to this constraints, we obtain

$$EW_e(a,b;\mathbf{R}^2)^{-1}=\frac{t_1}{t_1 e_1^2+e_2^2}+\frac{t_2}{t_2 e_1^2+e_2^2}.$$

Therefore

$$d_e^*(a,b;\mathbf{R}^2)-EW_e(a,b;\mathbf{R}^2)^{-1}=\frac{(t_1-t_2)^2 e_1^2 e_2^2}{[(t_1+t_2)e_1^2+2e_2^2](t_1e_1^2+e_2^2)(t_2e_1^2+e_2^2)}\geq 0$$

and the equality holds if $t_1=t_2$ or $e_1=0$ or $e_2=0$.

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