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ANNIHILATORS IN BCK-ALGEBRAS II

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Abstract. In this paper we show that for every BCI-algebra X,
(1) the family $CI(X)$ of all closed ideal in X forms a pseudo-complemented distributive lattice if and only if X is the BCK-algebra;
(2) $\forall A \in CI(X) A \subseteq A^{**} \iff X$ is the BCK-algebra ;

1. BCK-algebra

Firstly we define a BCI-algebra and a BCK-algebra. An algebra $(X; *, 0)$ of type (2,0) is called a BCI-algebra when it satisfies the conditions: For every $x, y, z \in X$,

(1) $(x * y) * (x * z) \leq z * y$
(2) $x * (x * y) \leq y$
(3) $x * x = 0$
(4) $x * y = y * x = 0 \Longrightarrow x = y$,

,where the relation "$\leq$" is defined by

$$x \leq y \text{ if and only if } x * y = 0.$$

It is well known that the relation is a partially ordered relation on any BCI-algebra (cf [2]).

In BCI-algebras, the following properties hold:

(a) $x * 0 = x$
(b) $0 * (x * y) = (0 * x) * (0 * y)$
(c) $(x * y) * z = (x * z) * y$
(d) $(x * z) * (y * z) \leq x * y$
(e) $x \leq y \Longrightarrow x * z \leq y * z$
(f) $x \leq y \Longrightarrow z * x \leq z * y$

We say that a BCI-algebra X is a BCK-algebra if

(5) $0 * x = 0$ for $x \in X$.

By an ideal we mean a subset I of X such that it satisfies the conditions:

$(I1)$ $0 \in I$
$(I2)$ $x * y, y \in I \Longrightarrow x \in I$.

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An ideal I is called closed whenever $0 * x \in I$ for any $x \in I$. By $CI(X)$, we denote the set of all closed ideals of X.

For every non-empty subset A of X, we define a subset A^* which is called an annihilator of A :

$$A^* = \{x \in X \mid \forall a \in A \ (a * (a * x) = 0)\}.$$

The set A is called involutory if $A = A^{**}$. By $S(X)$ we mean the set of all involutory ideals of X. It is proved in [1] that if X is a BCK-algebra and A is a non-empty subset of X then A^* is the ideal of X. We can show the same result for BCI-algebras.

Proposition 1. *Let A be a non-empty subset of a BCI-algebra X. Then A^* is a closed ideal of X.*

Proof. For every $a \in A$, since $a * (a * 0) = a * a = 0$, we have $0 \in A^*$. Next we suppose that $x, y * x \in A^*$. We obtain from definition that

(6) $a * (a * x) = 0$ and

(7) $a * (a * (y * x)) = 0$.

It follows from (2) and (6) that $0 * x = 0$ and hence $(a * x) * a = (a * a) * x = 0 * x = 0$. This means that $a = a * x$ by (4). Similarly we have $a = a * (y * x)$. From these, we have in turn

$a = a * (y * x) = (a * x) * (y * x) \leq a * y$ (by (d))

$0 = a * a \leq (a * y) * a = 0 * y$ (by (e))

$0 = 0 * (0 * y) \leq y$, which implies that $0 * y = 0$.

It follows that $(a * y) * a = (a * a) * y = 0 * y = 0$ and hence $a = a * y$. Thus $y \in A^*$ and A^* is the ideal of X_I.

Lastly we show that A^* is closed, that is, $0 * x \in A^*$ whenever $x \in A^*$. For $x \in A^*$, we obtain that $0 = a * (a * x) \leq x$ for every $a \in A$. Thus we have $0 * x = 0 \in A^*$. This means that A^* is closed.

Let X be a *BCI*-algebra. We define a subset G called *BCK*-part of X as

$$G = \{x \in X | 0 * x = 0\}$$

For the subset G it is proved that

Proposition 2. *(1) G is the closed ideal, that is, $G \in CI(X)$.*
(2) $G = \{0\}^*$
(3) $G^* = \{0\}$

Proof. We only prove the case of (1). The other cases are proved similarly (cf. [3]). It is obvious that $0 \in G$. If $x, y * x \in G$, then $0 * x = 0 * (y * x) = 0$. Since $0 * (y * x) = (0 * y) * (0 * x)$, we have $0 * y = 0$. Thus G is the ideal. Moreover, for any $x \in G$, it follows from definition of G that $0 * x = 0 \in G$. This means that G is closed.

Proposition 3. *Let X be a BCI-algebra and A a subset of X. Then we have*
$0 \in A$ *if and only if* $A \cap A^* = \{0\}$

Proof. We show the "only if" part. We suppose that $0 \in A$. First of all we note that $A \cap A^* \neq \emptyset$ because of $0 \in A^*$. For every $x \in A \cap A^*$, we have

$x*(x*x)=0$ by definition of A^*. On the other hand $x*(x*x)=x*0=x$. It follows that $x=0$ and hence $A\cap A^*=\{0\}$.

Corollary 1. $A\cap A^*=\emptyset$ *or* $A\cap A^*=\{0\}$

Hence we have $A\cap A^*=\emptyset$ for every ideal A in any BCI-algebra (cf [1]).

2. Structure of $I(X)$

In this section we consider the set $I(X)$ of all ideals of a BCK-algebra X. In case of the BCK-algebra X, every ideal of X is identical with the closed ideal, that is, $I(X)=CI(X)$. In [4], it is proved that if X is a BCK-algebra then the set $I(X)$ of all ideals of X forms the pseudo-complemented distributive lattice. In this section we show the converse, that is, for every BCI-algebra X, if $CI(X)$ is the pseudo-complemented distributive lattice then X is the BCK-algebra.

Proposition 4. $A\subseteq B \implies B^*\subseteq A^*$

Proof. [3]

We consider the following condition (BCK) which is proved to be equivalent to that X is a BCK-algebra ([4]):

(BCK) For every ideal A, if $x\in X$ and $a\in A$ then $a*(a*x)\in A$.

Proposition 5. *Let X be any BCI-algebra. Then we have*
X *: BCK-algebra* $\iff$ *the condition (BCK) holds.*

Lemma 1. *Let X be a BCK-algebra and $A,B\in I(X)$.*
$A\cap B=\{0\}\iff A\subseteq B^*$.

Proof. See [4]

Lemma 2. *If X is a BCK-algebra, then $A\subseteq A^{**}$ for every $A\in I(X)$.*

Proof. Let $a\in A$. For any $x\in A^*$, since A^* is the ideal, it follows from (BCK) that $x*(x*a)\in A^*$. On the other hand, since $x*(x*a)\le a\in A$, we have $x*(x*a)\in A\cap A^*=\{0\}$. This implies that $x*(x*a)=0$ for every $x\in A^*$. Hence we have $A\subseteq A^{**}$.

Remark: In [1] the results above are proved for every commutative BCK-algebra. But we see that the commutativity is unnecessary.

Lemma 3. *Let X be a BCI-algebra. Then*
$\{0\}^*=X \iff X: BCK-algebra.$

Proof. For any $x\in X=\{0\}^*$, we have $0*(0*x)=0$ by definition of annihilator. This means that $0*x=0$, that is, X is the BCK-algebra.

The converse is trivial.

Theorem 1. *Let X be a BCI-algebra. Then*

$$\forall A\in CI(X)\ (A\subseteq A^{**}) \iff X: BCK-algebra$$

Proof. Suppose that $A \subseteq A^{**}$ for every $A \in CI(X)$. Clearly X is a closed ideal of X, we have $X \subseteq X^{**}$ and hence $X = X^{**} = \{0\}^*$. It follows from lemma 3 that X is the BCK-algebra.

In [6], it is proved that the set $I(X)$ of all ideals of BCK-algebra X is the distributive lattice, where

$A \wedge B = A \cap B$

$A \vee B = \{x \in X \mid \exists a \in A, \exists b \in B; (x * a) * b = 0\}$.

It is clear from the argument above that the set $I(X) = CI(X)$ of all ideals of BCK-algebra X forms the pseudo-complemented distributive lattice (cf [4]). In the following we show that the converse holds, that is, if $CI(X)$ is a pseudo-complemented distributive lattice then X is the BCK-algebra. We also give an example which indicates that the pseudo-complementedness of $CI(X)$ is essential.

Theorem 2. *Let X be a BCI-algebra. If $(CI(X); \wedge, \vee, \phi, X)$ is the pseudo-complemented distributive lattice, then X is the BCK-algebra.*

Proof. We assume that $CI(X)$ is the pseudo-complemented distributive lattice. For every closed ideal $A \in CI(X)$, we have

$$\begin{aligned} A \wedge (A \vee A^{**}) &= (A^* \wedge A) \vee (A^* \wedge A^{**}) \\ &= \{0\} \vee \{0\} \\ &= \{0\}. \end{aligned}$$

By pseudo-complementedeness, it follows that $A \vee A^{**} \subseteq A^{**}$ and hence $A \vee A^{**} = A^{**}$. This implies that $A \subseteq A^{**}$ for every $A \in CI(X)$. Thus X is the BCK-algebra.

By the example below we know that the pseudo-complementedness is essential, that is, X is not need the BCK-algebra even if $CI(X)$ is distributive.

Example

*	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

It is easy to see that $CI(X) = \{\{0\}, \{0,1\}, \{0,3\}, X\}$ and hence that $CI(X)$ is a distributive lattice. The algebra is not a BCK-algebra, because we have $\{0,1\} \cap \{0,3\} = \{0\}$ but $\{0,3\}$ is not a subset of $\{0,1\}^* = \{0\}$.

It is proved in [4] that the set $S(X) = (S(X); \wedge, \sqcup, *, \phi, X)$ of all involutory ideals of a BCK-algebra X forms a Boolean algebra, where $A \sqcup B$ is defined as $(A^* \cap B^*)^*$ for every $A, B \in S(X)$. If we consider the map $\xi : I(X) \to S(X)$ defined by $\xi(A) = A^*$, then we have $ker\xi$ is a congruence over $I(X)$. From the general theory of universal algebras, we can conclude that

Theorem 3. *$I(X)/ker\xi \cong S(X)$. Hence I(X)/ker ξ is the Boollean algebra.*

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