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PROJECTIVITY OF LIE TRIPLE ALGEBRAS

Michihiko Kikkawa

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Abstract. The new algebraic concept of *projectivity* is introduced for Lie triple algebras which induces from a given Lie triple algbera $\mathfrak{g}$ some other Lie triple algebra $\tilde{\mathfrak{g}}$ in projective relation with $\mathfrak{g}$, by using some *Lie algebra* $\mathfrak{l}$ *of projectivity* on the same underlying vector space. The projectivity is discussed for Lie algebras and Lie triple systems as special cases.

1. Introduction

The concept of *Lie triple algebra* has been introduced by Yamaguti [20] under the name of *general Lie triple system*, related with the canonical connection of reductive homogeneous space of Nomizu [18]. It is an algebraic system on a vector space equipped with a bilinear multiplication and a trilinear multiplication, which contains both of the concepts of Lie algebra and Lie triple system as special cases.

In 1975, the author has introduced the concept of homogeneous Lie loops (cf. Kikkawa [4], [5]) as a non-associative generalization of the concept of Lie groups, that is, a kind of differentiable algebraic binary systems on manifolds. Then, he defined and investigated the *tangent Lie triple algebra* of the homogeneous (left) Lie loop (cf. Kikkawa [5], [7], [8], [9], [10]) with the bilinear and trilinear multiplications on the tangent space at the unit element. It gives the tangent Akivis algebra of Lie loops for the case of homogeneous Lie loops (cf. Akivis [1], [2] and Hofmann - Strambach [3]).

A non-associative generalization of the theory of Lie groups and Lie algebras has been established by the author to the theory of geodesic homogeneous left Lie loops and tangent Lie triple algebras (cf. [5], [6], [15]). Furthermore, the concept of projectivity of geodesic homogeneous left Lie loops has been introduced and some conditions for projectivity of geodesic homogeneous left Lie loops have been discussed on their tangent Lie triple algebras (cf. Kikkawa [11], [12], [13], [14], [15], [16], [17] and Sanami-Kikkawa [19]).

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Motivated by these works, we will introduce in this paper the concept of projectivity of Lie triple algebras in general. For any given Lie triple algebra $\mathfrak{g}$, we consider a Lie algebra $\mathfrak{l}$ whose inner derivations are derivations of $\mathfrak{g}$ and the inner derivations of $\mathfrak{g}$ are derivations of $\mathfrak{l}$, and we define new binary and ternary operations by means of the bracket operation of $\mathfrak{l}$ and multiplications of $\mathfrak{g}$.

In §2, we show that the binary and ternary operation thus defined form a Lie triple algebra $\tilde{\mathfrak{g}}$ again, which will be said to be *in projective relation* with $\mathfrak{g}$. The Lie algebra $\mathfrak{l}$ is called *Lie algebra of projectivity of* $\mathfrak{g}$.

In §3, we discuss some properties of Lie algebras of projectivity of any given Lie triple algebra. In §4, we consider projectivity of Lie algebras and Lie triple systems as special cases of Lie triple algebras.

2. Projectivity of Lie triple algebras

In this section, we consider Lie triple algebras in general, and introduce the concept of projectivity between Lie triple algebras.

Let $\boldsymbol{V}$ be a vector space over a field $\boldsymbol{K}$ of characteristic 0.

Definition 2.1. A *Lie triple algebra* $\mathfrak{g} = \{\boldsymbol{V}; B, D\}$ is an algebraic system on $\boldsymbol{V}$ defined by a pair (B, D) of a bilinear multiplication $B : \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ and a trilinear multiplication $D : \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ which satisfy the following relations:

$$B(X,Y) = -B(Y,X) \tag{2.1}$$

$$D(X,Y,Z) = -D(Y,X,Z) \tag{2.2}$$

$$\mathfrak{S}_{X,Y,Z}\{D(X,Y,Z) + B(B(X,Y),Z)\} = 0 \tag{2.3}$$

$$\mathfrak{S}_{X,Y,Z}\{D(B(X,Y),Z,W)\} = 0 \tag{2.4}$$

$$D(U,V,B(X,Y)) = B(D(U,V,X),Y) + B(X,D(U,V,Y)) \tag{2.5}$$

$$\begin{aligned} D(U,V,D(X,Y,Z)) &= D(D(U,V,X),Y,Z) \\ &\quad + D(X,D(U,V,Y),Z) + D(X,Y,D(U,V,Z)), \end{aligned} \tag{2.6}$$

for any $X, Y, Z, U, V, W \in \boldsymbol{V}$. Here, we denote by $\mathfrak{S}_{X,Y,Z}$ the cyclic sum with respect to X, Y, Z .

Let $\mathfrak{g} = \{\boldsymbol{V}; B, D\}$ be a Lie triple algebra on $\boldsymbol{V}$. We denote by $D(X,Y)$ the endomorphism of $\boldsymbol{V}$ given by $D(X,Y)Z := D(X,Y,Z)$ called an *inner derivation* of $\mathfrak{g}$.

Now, assume that there exists a Lie algebra $\mathfrak{l} = \{\boldsymbol{V}; L\}$ on $\boldsymbol{V}$ whose bracket operation is given by

$$[X,Y]_{\mathfrak{l}} = L(X,Y) \qquad \text{for } X, Y \in \boldsymbol{V}.$$

Definition 2.2. A Lie algebra $\mathfrak{l} = \{\boldsymbol{V}; L\}$ will be called a *Lie algebra of projectivity of a Lie triple algebra* $\mathfrak{g} = \{\boldsymbol{V}; B, D\}$ if it satisfies the following relations:

$$L(X, B(Y, Z)) = B(L(X, Y), Z) + B(Y, L(X, Z)) \tag{2.7}$$

$$\begin{aligned} L(X, D(Y, Z, W)) = D(L(X, Y), Z, W) \\ + D(Y, L(X, Z), W) + D(Y, Z, L(X, W)) \end{aligned} \tag{2.8}$$

$$D(U, V, L(X, Y)) = L(D(U, V, X), Y) + L(X, D(U, V, Y)) \tag{2.9}$$

Theorem 2.1. *Let* $\mathfrak{g} = \{\boldsymbol{V}; B, D\}$ *be a Lie triple algebra on a vector space* $\boldsymbol{V}$ *and* $\mathfrak{l} = \{\boldsymbol{V}; L\}$ *a Lie algebra of projectivity of* $\mathfrak{g}$. *Then,* $\tilde{\mathfrak{g}} = \left\{\boldsymbol{V}; \tilde{B}, \tilde{D}\right\}$ *given by*

$$\begin{aligned} \tilde{B}(X, Y) &:= B(X, Y) + 2L(X, Y) \\ \tilde{D}(X, Y, Z) &:= D(X, Y, Z) - L(B(X, Y), Z) - L(L(X, Y), Z), \end{aligned}$$

for $X, Y, Z \in \boldsymbol{V}$, *forms a Lie triple algebra on* $\boldsymbol{V}$.

Proof. The relations (2.1) and (2.2) for $\tilde{\mathfrak{g}}$ are clear. We show (2.3) for $\tilde{\mathfrak{g}}$. In fact, we have

$$\begin{aligned} \tilde{D}(X, Y, Z) + \tilde{B}(\tilde{B}(X, Y), Z) &= D(X, Y, Z) - L(B(X, Y), Z) - L(L(X, Y), Z) \\ &\quad + B(B(X, Y), Z) + 2B(L(X, Y), Z) \\ &\quad + 2L(B(X, Y), Z) + 4L(L(X, Y), Z) \\ &= D(X, Y, Z) + B(B(X, Y), Z) + A(X, Y, Z), \end{aligned}$$

where $A(X, Y, Z)$ is given by

$$A(X, Y, Z) = 2B(L(X, Y), Z) - B(L(Z, X), Y) - B(L(Y, Z), X) + 3L(L(X, Y), Z),$$

because the Lie algebra $\mathfrak{l}$ satisfies (2.7). Hence we get

$$\mathfrak{S}_{X,Y,Z} A(X, Y, Z) = 0,$$

and (2.3) for $\mathfrak{g}$ implies the required relation for $\tilde{\mathfrak{g}}$.

Next, we show the relation (2.4) for $\tilde{\mathfrak{g}}$. We have

$$\begin{aligned} \tilde{D}(\tilde{B}(X, Y), Z, W) = &D(B(X, Y), Z, W) + 2D(L(X, Y), Z, W) \\ &- L(B(B(X, Y), Z), W) - 2L(B(L(X, Y), Z), W) \\ &- L(L(B(X, Y), Z), W) - 2L(L(L(X, Y), Z), W). \end{aligned}$$

From (2.4) for $\mathfrak{g}$ and Jacobi identity for $\mathfrak{l}$, we see that the first term and the last term of the right hand side of the equation above vanish by taking cyclic sum with respect to X, Y, Z. So, we show that the cyclic sum of the remaining two parts vanish. In fact, by using (2.3) for $\mathfrak{g}$, (2.9) and (2.8), we get

$$\begin{aligned}
&\mathfrak{S}_{X,Y,Z}\{2D(L(X,Y),Z,W) - L(B(B(X,Y),Z),W)\} \\
&\quad = \mathfrak{S}_{X,Y,Z}\{2D(L(X,Y),Z,W) + L(D(X,Y,Z),W)\} \\
&\quad = \mathfrak{S}_{X,Y,Z}\{2D(L(X,Y),Z,W) + D(X,Y,L(Z,W)) - L(Z,D(X,Y,W)\} \\
&\quad = \mathfrak{S}_{X,Y,Z}\{2D(L(X,Y),Z,W) - D(L(Z,X),Y,W) - D(X,L(Z,Y),W)\} \\
&\quad = 0.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
&\mathfrak{S}_{X,Y,Z}\{2B(L(X,Y),Z + L(B(X,Y),Z)\} \\
&\quad = \mathfrak{S}_{X,Y,Z}\{2B(L(X,Y),Z) + B(L(X,Z),Y) + B(X,L(Y,Z))\} \\
&\quad = \mathfrak{S}_{X,Y,Z}\{2B(L(X,Y),Z) - B(L(Z,X),Y) - B(L(Y,Z),X)\} \\
&\quad = 0.
\end{aligned}$$

Thus the relation (2.4) is shown for $\tilde{\mathfrak{g}}$.

The relation (2.5) for $\tilde{\mathfrak{g}}$ is shown as follows: Since any inner derivation $D(X,Y)$ of the Lie triple algebra $\mathfrak{g}$ and any inner derivation of the Lie algebra $\mathfrak{l}$ are derivations of the operations B, D and L, we get

$$\begin{aligned}
D(U,V,B(X,Y)) &= B(D(U,V,X),Y) + B(X,D(U,V,Y)) \\
D(U,V,2L(X,Y)) &= 2L(D(U,V,X),Y) + 2L(X,D(U,V,Y)) \\
-L(B(U,V),B(X,Y)) &= -B(L(B(U,V),X),Y) - B(X,L(B(U,V),Y)) \\
-L(B(U,V),2L(X,Y)) &= -2L(L(B(U,V),X),Y) - 2L(X,L(B(U,V),Y)) \\
-L(L(U,V),B(X,Y)) &= -B(L(L(U,V),X),Y) - B(X,L(L(U,V),Y)) \\
-L(L(U,V),2L(X,Y)) &= -2L(L(L(U,V),X),Y) - 2L(X,L(L(U,V),Y)).
\end{aligned}$$

By summing up each side of these equations, we get

$$\begin{aligned}
\tilde{D}(U,V,\tilde{B}(X,Y)) &= (B+2L)(D(U,V,X),Y) + (B+2L)(X,D(U,V,Y)) \\
&\quad - (B+2L)(L(B(U,V),X),Y) - (B+2L)(X,L(B(U,V),Y)) \\
&\quad - (B+2L)(L(L(U,V),X),Y) - (B+2L)(X,L(L(U,V),Y)) \\
&= \tilde{B}(D(U,V,X) - L(B(U,V),X) - L(L(U,V),X),Y) \\
&\quad + \tilde{B}(X,D(U,V,Y) - L(B(U,V),Y) - L(L(U,V),Y)) \\
&= \tilde{B}(\tilde{D}(U,V,X),Y) + \tilde{B}(X,\tilde{D}(U,V,Y)),
\end{aligned}$$

which shows that $\tilde{\mathfrak{g}}$ satisfies (2.5).

Finally, by using the same method as above, we get

$$\begin{aligned}
\tilde{D}(U,V,\tilde{D}(X,Y,Z)) = \tilde{D}(\tilde{D}(U,V,X),Y,Z) &+ \tilde{D}(X,\tilde{D}(U,V,Y),Z) \\
&+ \tilde{D}(X,Y,\tilde{D}(U,V,Z)).
\end{aligned}$$

□

Remark . Let $\mathfrak{g} = \{\boldsymbol{V}; B, D\}$, $\mathfrak{l} = \{\boldsymbol{V}; L\}$ and $\tilde{\mathfrak{g}} = \left\{\boldsymbol{V}; \tilde{B}, \tilde{D}\right\}$ be as in Theorem 2.1 above. Then, the Lie algebra $-\mathfrak{l} = \{\boldsymbol{V}; -L\}$ is a Lie algebra of projectivity of the Lie triple algebra $\tilde{\mathfrak{g}}$ which induces the Lie triple algebra $\mathfrak{g}$ by

$$B(X,Y) = \tilde{B}(X,Y) - 2L(X,Y)$$
$$D(X,Y,Z) = \tilde{D}(X,Y,Z) + L(\tilde{B}(X,Y),Z) - L(L(X,Y),Z).$$

In fact, we get

$$L(\tilde{B}(X,Y),Z) - L(L(X,Y),Z) = L(B(X,Y),Z) + L(L(X,Y),Z),$$

and the second equation above is obtained by the definition of $\tilde{D}(X,Y,Z)$.

By this fact, we can define the projectivity of Lie triple algebras as follows.

Definition 2.3. Two Lie triple algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ on the same vector space $\boldsymbol{V}$ will be said to be *in projective relation* if there exists a Lie algebra $\mathfrak{l} = \{\boldsymbol{V}; L\}$ of projectivity of $\mathfrak{g}$ such that $\tilde{\mathfrak{g}}$ is induced from $\mathfrak{g}$ by Theorem 2.1.

Remark . The theory of geodesic homogeneous left Lie loops has been established by the author as an exact non-associative generalization of the theory of Lie groups (cf. Kikkawa [5], [6], [15]) . For instance, it has been shown that any connected and simply connected geodesic homogeneous left Lie loop (G, μ) is characterized by its tangent Lie triple algebra $\mathfrak{g}$. By introducing in [11] a differential geometric concept of *affine homogeneous structures* on manifolds with linear connections, the author defined the concept of *projectivity for geodesic homogeneous left Lie loops* in [12], and invetigated various properties of projectivity of them in Kikkawa [12], [14], [15], [16], etc. Especially, projectivity of geodesic homogeneous left Lie loops has been investigated in terms of their tangent Lie triple algebras.

Putting together these results on projectivity of geodesic homogeneous left Lie loops and the projectivity of Lie triple algebras introduced in this section, we get the following theorem which is based on Theorem 5 in [11] .

Theorem 2.2. *Two geodesic homogeneous left Lie loops (G, μ) and $(G, \tilde{\mu})$ on the same connected and simply connected analytic manifold G are in projective relation if and only if their tangent Lie triple algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are in projective relation.*

3. Lie algebras of projectivity of Lie triple algebras

In this section, we investigate sum and scalar product of Lie algebras of projectivity of a given Lie triple algebra. To do this, we will give a lemma for Lie algebras on the same underlying vector space.

Lemma 3.1. *Let $\mathfrak{l}_1 = \{\boldsymbol{V}; L_1\}$ and $\mathfrak{l}_2 = \{\boldsymbol{V}; L_2\}$ be two Lie algebras on the same vector space $\boldsymbol{V}$. Assume that they are related with each other under the relations: The inner derivations of $\mathfrak{l}_i$ are derivations of $\mathfrak{l}_j$ for $i, j = 1, 2$. Then, the sum $L = L_1 + L_2$ forms a Lie algebra on $\boldsymbol{V}$, which will be denoted by $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$.*

Proof. The proof is easy. Since it is clear that L is anti-symmetric, we show the Jacobi identity for $L = L_1 + L_2$. Denote $L_X(Y) := L(X,Y)$. By assumption, we see that each L_X is a derivation of both of $\mathfrak{l}_1$ and $\mathfrak{l}_2$, and so it is a derivation of the bilinear multiplication $L = L_1 + L_2$. □

Theorem 3.2. *Let* $\mathfrak{g} = \{\boldsymbol{V}; B, D\}$ *be a Lie triple algebra and* $\mathfrak{l} = \{\boldsymbol{V}; L\}$, $\tilde{\mathfrak{l}} = \left\{\boldsymbol{V}; \tilde{L}\right\}$ *be two Lie algebras of projectivity of* $\mathfrak{g}$. *Assume that any inner derivation of each of* $\mathfrak{l}$, $\tilde{\mathfrak{l}}$ *is a derivation of the other Lie algebra. Then,* $\mathfrak{l} + \tilde{\mathfrak{l}} = \left\{\boldsymbol{V}; L + \tilde{L}\right\}$ *forms a Lie algebra of projectivity of* $\mathfrak{g}$ *again.*

Proof. By Lemma 3.1 above, $\mathfrak{l} + \tilde{\mathfrak{l}}$ forms a Lie algebra. On the other hand, it is easy to show that $\mathfrak{l} + \tilde{\mathfrak{l}}$ satisfies the relations (2.7), (2.8) and (2.9) by means of the corresponding relations for $\mathfrak{l}$ and $\tilde{\mathfrak{l}}$, respectively. □

Now, let $\mathfrak{l} + \tilde{\mathfrak{l}}$ be the Lie algebra of projectivity for the Lie triple algebra $\mathfrak{g}$ obtained in Theorem 3.2 above. Then, the induced Lie triple algebra in projective relation with $\mathfrak{g}$ is give by

$$\mathfrak{g}^{\mathfrak{l}+\tilde{\mathfrak{l}}} = \left\{\boldsymbol{V}; B^{L+\tilde{L}}, D^{L+\tilde{L}}\right\},$$

where

$$\begin{aligned} B^{L+\tilde{L}}(X,Y) &= B(X,Y) + 2L(X,Y) + 2\tilde{L}(X,Y) \\ D^{L+\tilde{L}}(X,Y,Z) &= D(X,Y,Z) - L(B(X,Y),Z) - \tilde{L}(B(X,Y),Z) \\ &\qquad - (L+\tilde{L})((L+\tilde{L})(X,Y),Z). \end{aligned}$$

Especially, if $\mathfrak{l}$ is a Lie algebra of projectivity of the Lie triple algebra $\mathfrak{g}$, the Lie algebra $\mathfrak{l}^k = k\mathfrak{l} = \{\boldsymbol{V}; kL\}$, for any non-zero element $k \in \boldsymbol{K}$, is a Lie algebra of projectivity of $\mathfrak{g}$. It is evident that $\mathfrak{l}^k + \mathfrak{l}^h = \mathfrak{l}^{k+h}$ holds for any $k, h \in \boldsymbol{K}$, where $\mathfrak{l}^0$ is considered to be an Abelian Lie algebra.

Thus, we obtain the following corollary:

Corollary 3.3. *For any Lie algebra* $\mathfrak{l}$ *of projectivity of a Lie triple algebra* $\mathfrak{g}$, *there exists a one-parameter family* $\left\{\mathfrak{l}^k; k \in \boldsymbol{K}\right\}$ *of Lie algebras of projectivity of* $\mathfrak{g}$. *The induced Lie triple algebras* $\mathfrak{g}^k$ *in projective relation with* $\mathfrak{g}$ *is given by*

$$\begin{aligned} B^k(X,Y) &= B(X,Y) + 2kL(X,Y) \\ D^k(X,Y,Z) &= D(X,Y,Z) - kL(B(X,Y),Z) - k^2L(L(X,Y),Z), \end{aligned}$$

for $X, Y, Z \in \boldsymbol{V}$. *Any two Lie triple algebras* $\mathfrak{g}^k$ *and* $\mathfrak{g}^h$ *among them are in projective relation under the Lie algebra* $\mathfrak{l}^{h-k}$ *of projectivity.*

Proof. The previous half of Corollary is clear, and we can show that the Lie triple algebra $\mathfrak{g}^h$ is induced from $\mathfrak{g}^h$ by the Lie algebra $\mathfrak{l}^{h-k}$. In fact, the multiplications $\tilde{B}$ and $\tilde{D}$ of the Lie triple algebra $\tilde{\mathfrak{g}}$ induced from $\mathfrak{g}^k$ by the Lie algebra $\mathfrak{l}^{h-k}$ are given by:

$$\begin{aligned}\tilde{B}(X,Y) =& B^k(X,Y) + 2(h-k)L(X,Y)\\ =& B(X,Y) + 2kL(X,Y) + 2(h-k)L(X,Y)\\ =& B^h(X,Y),\\ \tilde{D}(X,Y,Z) =& D^k(X,Y,Z) - (h-k)L(B^k(X,Y),Z) - (h-k)^2L(L(X,Y),Z)\\ =& D(X,Y,Z) - kL(B(X,Y),Z) - k^2L(L(X,Y),Z)\\ &+ (h-k)L(B(X,Y) + 2kL(X,Y),Z) - (h-k)^2L(L(X,Y),Z)\\ =& D(X,Y,Z) - hL(B(X,Y),Z) - h^2L(L(X,Y),Z)\\ =& D^h(X,Y,Z).\end{aligned}$$

□

4. Projectivity of Lie algebras and Lie triple systems

In this section, we consider some special cases of projectivity of Lie triple algebras.

It is evident that any Lie algebra $\mathfrak{g} = \{\boldsymbol{V}; B\}$ can be regarded as a Lie triple algebra with the trilinear multiplication D is trivially zero. In this case, any Lie algebra $\mathfrak{l} = \{\boldsymbol{V}; L\}$ of projectivity of $\mathfrak{g}$ is given by the following relations (cf. Definition 2.2):

$$L(X, B(Y,Z)) = B(L(X,Y),Z) + B(Y, L(X,Z)), \tag{4.1}$$

for $X, Y, Z \in \boldsymbol{V}$. That is, any inner derivation of $\mathfrak{l}$ is a derivation of $\mathfrak{g}$. If $\mathfrak{l}$ satisfies the relation above, it induces a Lie triple algebra $\tilde{\mathfrak{g}} = \left\{\boldsymbol{V}; \tilde{B}, \tilde{D}\right\}$ by Theorem 2.1 from $\mathfrak{g}$, with the multiplications;

$$\tilde{B}(X,Y) = B(X,Y) + 2L(X,Y)$$
$$\tilde{D}(X,Y,Z) = -L(B(X,Y),Z) - L(L(X,Y),Z)$$

for $X, Y, Z \in \boldsymbol{V}$.

Any Lie algebra $\mathfrak{g} = \{\boldsymbol{V}; B\}$ has a Lie algebra of projectivity since $\mathfrak{g}$ itself satisfies the relation above. So, by Corollary 3.3, we get a 1-parameter family of Lie algebras $k\mathfrak{g} = \{\boldsymbol{V}; kB\}$ of projectivity of $\mathfrak{g}$ which induces a 1-parameter family of Lie triple algebras $\mathfrak{g}^k = \left\{\boldsymbol{V}; B^k, D^k\right\}, k \in \boldsymbol{K}$, whose multiplications are given by

$$B^k(X,Y) = (2k+1)B(X,Y)$$
$$D^k(X,Y,Z) = -k(1+k)B(B(X,Y),Z), \text{ for } X,Y,Z \in \boldsymbol{V},$$

which is in projective relation with $\mathfrak{g}^0 = \mathfrak{g}$.

Remark . In [19] the Lie algebra $\mathfrak{l}$ satisfying (4.1) is called a *projective double Lie algebra* on $\mathfrak{g}$, and it is shown that, if $\mathfrak{g} = \{\boldsymbol{V}; B\}$ is a real simple Lie algebra on an odd-dimensional real vector space $\boldsymbol{V}$, all the Lie triple algebras which are in projective relation with $\mathfrak{g}$ are given by those 1-parameter family $\mathfrak{g}^k, k \in \boldsymbol{R}$ above.

Definition 4.1. A *Lie triple system* $\mathfrak{s} = \{\boldsymbol{V}; D\}$ is a vector space $\boldsymbol{V}$ equipped with a trilinear multiplication $D : \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ satisfying the following relations:

$$D(X,Y,Z) = -D(Y,X,Z) \tag{4.2}$$

$$\mathfrak{S}_{X,Y,Z} D(X,Y,Z) = 0 \tag{4.3}$$

$$\begin{aligned} D(U,V,D(X,Y,Z)) &= D(D(U,V,X),Y,Z) \\ &\quad + D(X,D(U,V,Y),Z) + D(X,Y,D(U,V,W)), \end{aligned} \tag{4.4}$$

for $X,Y,Z,U,V,W \in \boldsymbol{V}$. Tha is, $\mathfrak{g}$ is a Lie triple algebra with the trivial bilinear multiplication $B = 0$.

Example 1. It is well-known that any Lie algebra $\mathfrak{l} = \{\boldsymbol{V}; L\}$ forms a Lie triple system $\mathfrak{s}_{\mathfrak{l}} = \{\boldsymbol{V}; D_L\}$ with the trilinear multiplication

$$D_L(X,Y,Z) = L(L(X,Y),Z), \text{ for } X,Y,Z \in \boldsymbol{V}.$$

Let $\mathfrak{s} = \{\boldsymbol{V}\}$ be a Lie triple system. Then, projectivity of $\mathfrak{s}$ can be considered as follows: Let $\mathfrak{l} = \{\boldsymbol{V}; L\}$ be a Lie algebra on $\boldsymbol{V}$. Then, by Definition 2.2, $\mathfrak{l}$ is a Lie algebra of projectivity of $\mathfrak{g}$ if and only if it satisfies the following relations:

$$\begin{aligned} L(X,D(Y,Z,W)) &= D(L(X,Y),Z,W) + D(Y,L(X,Z),W) \\ &\quad + D(Y,Z,L(X,W)) \end{aligned} \tag{4.5}$$

$$D(U,V,L(X,Y)) = L(D(U,V,X),Y) + L(X,D(U,V,Y)). \tag{4.6}$$

Assume that $\mathfrak{l} = \{\boldsymbol{V}; L\}$ is a Lie algebra of projectivity of the Lie triple system $\mathfrak{s} = \{\boldsymbol{V}; D\}$. Then, by Theorem 2.1, a Lie triple algebra $\mathfrak{g} = \left\{\boldsymbol{V}; \tilde{B}, \tilde{D}\right\}$ is induced from $\mathfrak{g}$ by $\mathfrak{l}$ as follows:

$$\tilde{B}(X,Y) = 2L(X,Y) \tag{4.7}$$

$$\tilde{D}(X,Y,Z) = D(X,Y,Z) - L(L(X,Y),Z). \tag{4.8}$$

So we get a Lie algebra $\tilde{\mathfrak{l}} = \left\{\boldsymbol{V}; \tilde{B}\right\}$ and a Lie triple system $\tilde{\mathfrak{s}} = \left\{\boldsymbol{V}; \tilde{D}\right\} = \mathfrak{s} - \mathfrak{s}_{\mathfrak{l}}$, where $\mathfrak{s}_{\mathfrak{l}}$ is the Lie triple system formed by the Lie algebra $\mathfrak{l}$ in Example 1 . Moreover, we can see that the following relation holds:

$$\begin{aligned} \tilde{B}(X,\tilde{D}(Y,Z,W)) =& \tilde{D}(\tilde{B}(X,Y),Z,W) + \tilde{D}(Y,\tilde{B}(X,Z),W) \\ & \tilde{D}(Y,Z,\tilde{B}(X,W)). \end{aligned} \tag{4.9}$$

Definition 4.2. (cf. [17]) A Lie triple algebra $\mathfrak{g} = \{\boldsymbol{V}; B, D\}$ is said to be *feigned* if $\mathfrak{l} = \{\boldsymbol{V}; B\}$ forms a Lie algebra, $\mathfrak{s} = \{\boldsymbol{V}; D\}$ forms a Lie triple system and any inner derivation of $\mathfrak{l}$ is a derivation of $\mathfrak{s}$, that is, the relation (4.9) holds for B and D .

Theorem 4.1. *A Lie triple algebra $\tilde{\mathfrak{g}} = \left\{\boldsymbol{V}; \tilde{B}, \tilde{D}\right\}$ is in projective relation with a Lie triple system $\mathfrak{s} = \{\boldsymbol{V}; D\}$ if and only if $\tilde{\mathfrak{g}}$ is a feigned Lie triple algebra.*

Proof. We have seen above that any Lie triple algebra $\tilde{\mathfrak{g}} = \left\{\boldsymbol{V}; \tilde{B}, \tilde{D}\right\}$ which is in projective relation with a Lie triple system $\mathfrak{s}$ by a Lie algebra $\mathfrak{l} = \{\boldsymbol{V}; L\}$ is a feigned Lie triple algebra with the Lie algebra $\tilde{\mathfrak{l}} = \left\{\boldsymbol{V}; \tilde{B}\right\}$ and the Lie triple system $\tilde{\mathfrak{s}} = \left\{\boldsymbol{V}; \tilde{D}\right\} = \mathfrak{s} - \mathfrak{s}_{\mathfrak{l}}$.

Conversely, assume that a Lie triple algebra $\tilde{\mathfrak{g}} = \left\{\boldsymbol{V}; \tilde{B}, \tilde{D}\right\}$ is feigned so that $\tilde{\mathfrak{l}} = \left\{\boldsymbol{V}; \tilde{B}\right\}$ forms a Lie algebra, $\tilde{\mathfrak{s}} = \left\{\boldsymbol{V}; \tilde{D}\right\}$ forms a Lie triple system and they are related by the relation (4.9). Then, the Lie algebra $\mathfrak{l} = -\frac{1}{2}\tilde{\mathfrak{l}}$ forms a Lie algebra of projectivity of $\tilde{\mathfrak{g}}$. In fact, the relation (2.7) in Definition 2.2 is clear since $\tilde{\mathfrak{l}}$ forms a Lie algebra, and (2.8) is assured by the relation (4.9). The relation (2.9) in Definition 2.2 is obtained by the relation (2.5) in Definition 2.1 for the Lie triple algebra $\tilde{\mathfrak{g}}$. Then, the induced Lie triple algebra $\mathfrak{g} = \{\boldsymbol{V}; B, D\}$ from the Lie algebra $\mathfrak{l}$ of projectivity is given by:

$$\begin{aligned} B(X,Y) &= \tilde{B}(X,Y) + 2(\frac{-1}{2})\tilde{B}(X,Y) = 0 \\ D(X,Y,Z) &= \tilde{D}(X,Y,Z) + \frac{1}{2}\tilde{B}(\tilde{B}(X,Y),Z) - \frac{1}{4}\tilde{B}(\tilde{B}(X,Y),Z) \\ &= \tilde{D}(X,Y,Z) + \frac{1}{4}\tilde{B}(\tilde{B}(X,Y),Z). \end{aligned}$$

Thus, we have a Lie triple system $\mathfrak{s} = \{\boldsymbol{V}; D\} = \tilde{\mathfrak{s}} + \frac{1}{4}\mathfrak{s}_{\tilde{\mathfrak{l}}}$ which is in projective relation with the given feigned Lie triple algebra $\tilde{\mathfrak{g}}$ □

Remark . Theorem 4.1 above has been proved in Kikkawa [17] in the case of tangent Lie teriple algebra of geodesic homogeneous left Lie loops. There has been shown that the tangent Lie triple algebra is in projective relation with a Lie triple system if and only if the geodesic homogeneous left Lie loop is in projective relation with a symmetric Lie loop.

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Department of Mathematics, Shimane Univarsity, Matsue 690-8504 JAPAN
E-mail address: kikkawa@@riko.shimane-u.ac.jp