

SEMIBOUNDED PERTURBATION OF GREEN FUNCTION IN AN NTA DOMAIN

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ABSTRACT. Let $\mathcal{L} = -\sum_{i,j} \partial_i(a_{ij}\partial_j)$ be a Hölder continuous uniformly elliptic operator on an NTA domain and let $\mathcal{L}_V = \mathcal{L} + V$ be a generalized Schrödinger operator for a nonnegative function V . Let G and G_V be the Green function for the Dirichlet problem with respect to \mathcal{L} and \mathcal{L}_V , respectively. In case V is a function of the distance to the boundary, a necessary sufficient condition for G and G_V to have comparable decay near the boundary is given. This is based on an integral identity of \mathcal{L} -harmonic functions and integral inequalities of \mathcal{L} -superharmonic and \mathcal{L} -subharmonic functions, which may be of independent interest.

1. INTRODUCTION

Let $\mathcal{L} = -\sum_{i,j} \partial_i(a_{ij}\partial_j)$ be a smooth uniformly elliptic operator on a domain $D \subset \mathbb{R}^n$, $n \geq 2$, such that $a_{ij}(x) = a_{ji}(x)$ is Hölder continuous on D and

$$\Lambda^{-1}|\xi|^2 \leq A[\xi] \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n,$$

where $A[\xi] = \sum_{i,j} a_{ij}\xi_i\xi_j$ and $\Lambda > 1$ is a constant independent of $\xi \in \mathbb{R}^n$ and $x \in D$. We also consider the Schrödinger operator $\mathcal{L}_V = \mathcal{L} + V$ for a nonnegative function V on D . Let G and G_V be the Green function for the Dirichlet problem in D with respect to \mathcal{L} and \mathcal{L}_V , respectively. This means that for any $y \in D$, $G(\cdot, y)$ and $G_V(\cdot, y)$ vanish on ∂D and $\mathcal{L}G(\cdot, y) = \mathcal{L}_VG_V(\cdot, y) = \delta_y$, where δ_y is the Dirac measure at y . Since \mathcal{L} and \mathcal{L}_V are self adjoint, it follows that G and G_V are symmetric.

We are interested in the relationship between G and G_V . It is easy to see that

$$G_V(x, y) \leq G(x, y) \quad \text{for } x, y \in D.$$

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Let $x_0 \in D$ be fixed. We say that V is semibounded perturbation of \mathcal{L}_V if

$$G(x_0, y) \leq CG_V(x_0, y) \quad \text{for } y \in D$$

with $C > 1$ independent of $y \in D$ (see [13, §3]). The main purpose of this paper is to give a necessary and sufficient condition for V of a certain form to be semibounded perturbation in the case when D is an NTA domain (see [10]). By the symbol C we denote an absolute positive constant whose value is unimportant and may change from line to line. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $C \geq 1$ such that $C^{-1}f_1 \leq f_2 \leq Cf_1$. The main result of this paper is the following.

Theorem 1. *Let D be an NTA domain. Let $v(r)$ be a locally bounded nonnegative function for $0 < r \leq r_0 = \text{diam}(D)$ such that $v(r) \approx v(2r)$ and $r^2v(r)$ is nondecreasing for small $r > 0$. Set $V(x) = v(\delta(x))$ with $\delta(x) = \text{dist}(x, \partial D)$. Then V is semibounded perturbation of \mathcal{L}_V if and only if*

$$(1) \quad \int_0^{r_0} rv(r)dr < \infty.$$

It is known that the Martin boundary of an NTA domain with respect to \mathcal{L} is the Euclidean boundary and every boundary point is minimal ([10, Theorem 5.9 and p.158]). Hence, in view of [14, Theorem 2], we have the following corollary immediately.

Corollary 1. *Let D be an NTA domain and let v and V be as above. If (1) holds, then the Martin boundary of D with respect to \mathcal{L}_V is the Euclidean boundary of D and every boundary point is minimal.*

When $\mathcal{L} = -\Delta$ and D is smooth, Suzuki [14] showed the sufficiency part of the above theorem and we gave an alternative proof as well as the necessity part ([4, Theorems 2 and 3] see also [3]). Both proofs use the smoothness of D and do not apply to an NTA domain. Instead, we shall employ the following integral identity of \mathcal{L} -harmonic functions and integral inequalities of \mathcal{L} -superharmonic and \mathcal{L} -subharmonic functions, which may be of independent interest. These are generalizations of our previous results [1] (see also [2] and [5, pp.171–182]), which is based on the coarea formula.

Theorem 2. *Let D be a regular domain. Let $\varphi(t)$ be a nonnegative function for $t > 0$. Let $x_0 \in D$ and $g(x) = G(x_0, x)$. Suppose $0 \leq a < b \leq \infty$. Then the following statements hold:*

(i) *If h is \mathcal{L} -harmonic on D , then*

$$\int_{\{x \in D: a < g(x) < b\}} h(x)\varphi(g(x))A[\nabla g(x)]dx = h(x_0) \int_a^b \varphi(t)dt.$$

(ii) If u is \mathcal{L} -superharmonic on D , then

$$\int_{\{x \in D: a < g(x) < b\}} u(x) \varphi(g(x)) A[\nabla g(x)] dx \leq u(x_0) \int_a^b \varphi(t) dt.$$

(iii) If s is \mathcal{L} -subharmonic on D , then

$$\int_{\{x \in D: a < g(x) < b\}} s(x) \varphi(g(x)) A[\nabla g(x)] dx \geq s(x_0) \int_a^b \varphi(t) dt.$$

We shall estimate $|\nabla g(x)|$ to obtain some corollaries of Theorem 2, which will be needed for the proof of Theorem 1. The upper estimate is easy. The lower estimate is more difficult. If D is an NTA domain, then $|\nabla g(x)|$ can be estimated by $g(x)/\delta(x)$ in a certain sense with the help of the boundary Harnack principle ([10, Theorem 5.1 and p.138]). For details see Section 2.

For the proof of Theorem 1 we also need the decay estimate of $g(x)$ near the boundary. We find $0 < \beta \leq 1 \leq \alpha < \infty$ such that

$$(2) \quad C^{-1} \delta(x)^\alpha \leq g(x) \leq C \delta(x)^\beta \quad \text{for } x \in D \text{ close to } \partial D.$$

The above inequalities are rather easy. The first inequality actually holds for a John domain (see e.g. [9, p.185 and (2.6)]) and the second does for a domain satisfying the capacity density condition (see e.g. [7]). In general, $\beta < 1 < \alpha$. Only in the case when D is smooth, we have $\alpha = \beta = 1$, which was essential in [4, Theorems 2 and 3]. Surprisingly, the weak estimate (2) is sufficient for the proof of Theorem 1.

Recently, Ancona [8] studied Green functions for elliptic operators on manifolds or domains. By using a completely different method, he proved a better result than the sufficiency part of Theorem 1. He proved that under the condition (1), V becomes bounded perturbation of \mathcal{L}_V , i.e., $G(x, y) \approx G_V(x, y)$ for all $x, y \in D$.

2. PROOF OF THEOREM 2 AND COROLLARIES

Proof of Theorem 2. The proof is essentially the same as in [1]. We shall prove only (i), since the remaining can be proved in the same fashion. Let h be \mathcal{L} -harmonic on D . Let $D_t = \{x \in D : g(x) > t\}$ for $t > 0$. Observe that D_t is a relatively compact subset of D . We see that the Green function G^t for D_t with respect to \mathcal{L} satisfies that $G^t(x_0, x) = g(x) - t$. By the Sard theorem (see e.g. [11, Corollary on p.35]), ∂D_t is a smooth surface for a.e. $t > 0$. For such t we see that the outward normal \mathbf{n} of ∂D_t is given by $-\nabla g/|\nabla g|$. On the other hand the Poisson integral formula for D_t with respect to \mathcal{L} shows that

$$h(x_0) = - \int_{\partial D_t} h(x) (\nabla g(x), A(x) \mathbf{n}(x)) d\sigma(x),$$

where $A(x)\mathbf{n}(x) = (\sum_j a_{ij}n_j)$ is the conormal vector (see [12, (10.4) on p.21]). Hence

$$h(x_0) = \int_{\partial D_t} h(x) \frac{A[\nabla g(x)]}{|\nabla g(x)|} d\sigma(x).$$

By the coarea formula (see e.g. [11, pp.37–39])

$$\begin{aligned} h(x_0) \int_a^b \varphi(t) dt &= \int_a^b dt \int_{\partial D_t} h(x) \varphi(g(x)) \frac{A[\nabla g(x)]}{|\nabla g(x)|} d\sigma(x) \\ &= \int_{\{x \in D : a < g(x) < b\}} h(x) \varphi(g(x)) A[\nabla g(x)] dx. \end{aligned}$$

Thus the required identity holds.

Let us estimate $|\nabla g(x)|$ and give some corollaries of Theorem 2, which will be needed for the proof of Theorem 1. Since $g(x)$ is positive and \mathcal{L} -harmonic near ∂D , it follows that $|\nabla g(x)| \leq Cg(x)/\delta(x)$. Since $A = (a_{ij})$ is uniformly elliptic, we have the following corollary from the above proof.

Corollary 2. *Let $0 \leq a < b \leq \infty$. Suppose that s is a positive \mathcal{L} -subharmonic function on $\{x \in D : g(x) > a\}$. Then*

$$s(x_0) \int_a^b \varphi(t) dt \leq C \int_{\{x \in D : a < g(x) < b\}} s(x) \varphi(g(x)) \frac{g(x)^2}{\delta(x)^2} dx,$$

where C is independent of s , a and b .

The inequality $|\nabla g(x)| \geq Cg(x)/\delta(x)$ does not hold in a pointwise sense. However, it holds in a certain weak sense for an NTA domain. In [1] this was observed for $\mathcal{L} = -\Delta$. The key ingredient was the boundary Harnack principle. For a general uniformly elliptic operator \mathcal{L} the boundary Harnack principle is available also (see [10, Theorem 5.1 and p.138]) and hence the same proof as in [1] works for the present case. We have the following corollary.

Corollary 3. *Let D be an NTA domain. Let $\varphi(t) \approx \varphi(2t)$ for $t > 0$ and let $0 \leq a < b < \infty$. Suppose that u is a positive \mathcal{L} -superharmonic function on D . Then*

$$\int_{\{x \in D : a < g(x) < b\}} u(x) \varphi(g(x)) \frac{g(x)^2}{\delta(x)^2} dx \leq Cu(x_0) \int_a^b \varphi(t) dt,$$

where C is independent of u , a and b .

Remark. Note that the simplified proofs in [2] and [5, pp.171–182] use the analyticity of harmonic functions ([2, Lemma 1] and [5, Lemma 9.6.2 on p.180]). In general, \mathcal{L} -harmonic functions need not be analytic. Instead of the analyticity, we can use the unique continuation property of those functions for the present situation.

3. PROOF OF THEOREM 1

Proof of Theorem 1. Sufficiency: Let us recall the resolvent equation

$$(3) \quad G(x, y) = G_V(x, y) + \int_D G_V(x, z)V(z)G(z, y)dz \quad \text{for } x, y \in D.$$

In view of [14, §4] and [13, Theorem 1.5], it is sufficient to show that for any $\varepsilon > 0$ there is a compact subset K of D such that

$$(4) \quad \int_{D \setminus K} g(z)V(z)G(z, y)dz \leq \varepsilon g(y) \quad \text{for } y \in D,$$

where we recall $g(y) = G(x_0, y)$.

Suppose (1) holds. Let $\varphi(t) = t^{2/\alpha-1}v(t^{1/\alpha})$, in other words, $v(r) = r^{\alpha-2}\varphi(r^\alpha)$ with $\alpha > 1$ in (2). By assumption $t\varphi(t)$ is nondecreasing. Let $b > 0$ be sufficiently small such that

$$V(z) = v(\delta(z)) = \delta(z)^{\alpha-2}\varphi(\delta(z)^\alpha) \leq Cg(z)\varphi(g(z))\delta(z)^{-2} \quad \text{for } g(z) < b,$$

where (2) and the monotonicity of $t\varphi(t)$ are used. Hence, Corollary 3 with $u = G(\cdot, y)$ yields

$$\begin{aligned} \int_{\{z \in D: g(z) < b\}} g(z)V(z)G(z, y)dz &\leq C \int_{\{z \in D: g(z) < b\}} G(z, y)\varphi(g(z))\frac{g(z)^2}{\delta(z)^2}dz \\ &\leq Cg(y) \int_0^b \varphi(t)dt. \end{aligned}$$

By an elementary calculation

$$\int_0^b \varphi(t)dt = \alpha \int_0^{b^{1/\alpha}} rv(r)dr < \infty.$$

Hence (4) holds with $K = \{z \in D : g(z) \geq b\}$ for sufficiently small $b > 0$.

Necessity: Suppose V is semibounded perturbation of \mathcal{L}_V . Then by the resolvent equation (3)

$$(5) \quad \int_D g(z)V(z)G(z, y)dz \leq Cg(y) \quad \text{for } y \in D.$$

Let $\psi(t) = t^{2/\beta-1}v(t^{1/\beta})$, in other words, $v(r) = r^{\beta-2}\psi(r^\beta)$ with $0 < \beta < 1$ in (2). By assumption $t\psi(t)$ is nondecreasing. We choose $b > 0$ sufficiently small such that

$$V(z) = v(\delta(z)) = \delta(z)^{\beta-2}\psi(\delta(z)^\beta) \geq Cg(z)\psi(g(z))\delta(z)^{-2} \quad \text{for } g(z) < b,$$

where (2) and the monotonicity of $t\psi(t)$ are used. Let $0 < a < b$ and take $y \in D$ close to the boundary such that $g(y) < a$. Then $G(\cdot, y)$ is \mathcal{L} -harmonic on $\{z \in D : g(z) > a\}$. Hence, Corollary 2 with $G(\cdot, y)$ and $\psi(t)$ in place of s and $\varphi(t)$ yields

$$\begin{aligned} \int_{\{z \in D : a < g(z) < b\}} g(z)V(z)G(z, y)dz &\geq C \int_{\{z \in D : a < g(z) < b\}} G(z, y)\psi(g(z))\frac{g(z)^2}{\delta(z)^2}dz \\ &\geq Cg(y) \int_a^b \psi(t)dt. \end{aligned}$$

In view of (5) we have

$$\int_a^b \psi(t)dt \leq C,$$

where C is independent of a and hence

$$\infty > \int_0^b \psi(t)dt = \beta \int_0^{b^{1/\beta}} rv(r)dr.$$

Thus (1) holds. The proof is complete.

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