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AN INTRODUCTION OF KURAMOCHI BOUNDARY OF AN INFINITE NETWORK

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Abstract. We develop a theory of Kuramochi boundary on an infinite network. The discrete Laplacian and the Dirichlet sum in the discrete potential theory on an infinite network play an important role in our study.

1. Introduction and preliminaries

The theory of Kuramochi boundary on Riemann surfaces due to [10] has been developed analogous as in the case of Martin boundary (cf. [5]). We study a discrete analogy of the theory of Kuramochi boundary on Riemann surfaces, regarding an infinite network as a Riemann surface. We take the same line as in the discrete potential theory in [11], [12] and [13] in the sense that the discrete Laplacian and the Dirichlet sum play the role of the Laplacian and the Dirichlet integral. In order to emphasize the analogy to the continuous case, the contents of this paper are arranged parallel to those of the paper [10] of M. Ohtsuka. For notation and terminology concerning the infinite network, we mainly follow [7] and [12].

Let $N = \{X, Y, K, r\}$ be an infinite network which is connected, locally finite and has no self-loop. Here X is the countable set of nodes, Y is the countable set of arcs, K is the node-arc incidence function (matrix) and r is a positive real valued function on Y. For each $y \in Y$, denote by $e(y)$ the extremities of y, i.e.,

$$e(y) := \{x \in X;\ K(x, y) \neq 0\}.$$

For each $a \in X$, let $Y(a)$ be the set of arcs which are incident to node a:

$$Y(a) := \{y \in Y;\ K(a, y) \neq 0\}.$$

The geodesic distance $\rho(a, b)$ between two nodes a and b is the number of arcs in the shortest path joining a and b. Let us put

$$X(a) := \{x \in X; \rho(a, x) \leq 1\}, \quad W(a) := \{x \in X; \rho(a, x) = 1\}.$$

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It is clear that

$$X(a) = \cup\{e(y);\ y \in Y(a)\} \text{ and } W(a) = X(a) \setminus \{a\}.$$

An exhaustion $\{N_n\}(N_n = < X_n, Y_n >)$ of N is the sequence of finite subnetworks N_n such that $X(x) \subset X_{n+1}$ for all $x \in X_n$, $Y_n \subset Y_{n+1}$ and

$$X = \cup_{n=1}^{\infty} X_n, \quad Y = \cup_{n=1}^{\infty} Y_n.$$

We say that a subset A of X which contains more than or equal to two nodes is connected if, for every distinct nodes $a, b \in A$, there exists a path P from a to b such that the set of nodes on P is a subset of A. For a subset A of X, the components of $X \setminus A$ are defined as the maximal connected subsets of $X \setminus A$, in case $X \setminus A$ is not connected.

Denote by $L(X)$ the set of all real valued functions on X and by $L^+(X)$ the set of all non-negative real valued functions on X. We use the notation $L(Y)$ similarly. The support S_f of a function $f \in L(X)$ is defined by

$$S_f := \{x \in X; f(x) \neq 0\}.$$

Denote by $L_0(X)$ the set of all $f \in L(X)$ with finite support.

For a subset A of X, let ε_A be the characteristic function of A. In case $A = \{a\}$, we simply set $\varepsilon_a := \varepsilon_{\{a\}}$.We often identify ε_a with the unit point measure $\tilde{\varepsilon}_a$ at a. In this way, we identify a function $\mu \in L^+(X)$ with the measure

$$\tilde{\mu} = \sum_{a \in X} \mu(a)\tilde{\varepsilon}_a$$

and call the quantity

$$\mu(X) := \sum_{x \in X} \mu(x)$$

the total mass of μ. Conversely, a measure on X is identified with a function which belongs to $L^+(X)$.

The discrete derivative $du \in L(Y)$ of $u \in L(X)$ and the Dirichlet mutual sum $D(u, v)$ of $u, v \in L(X)$ are defined by

$$du(y) := -r(y)^{-1} \sum_{x \in X} K(x, y)u(x).$$

$$D(u, v) := \sum_{y \in Y} r(y)[du(y)][dv(y)].$$

We set $D(u) := D(u, u)$ and call it the Dirichlet sum of u. Denote by $D(N)$ the set of all $u \in L(X)$ with finite Dirichlet sum:

$$D(N) := \{u \in L(X); D(u) < \infty\}.$$

The discrete Laplacian $\Delta u \in L(X)$ of $u \in L(X)$ is defined by

$$\Delta u(x) := \sum_{y \in Y} K(x, y)[du(y)].$$

We have

Lemma 1.1. *Let $u, f \in L(X)$. Then the discrete Dirichlet integral formula:*

$$D(u, f) = -\sum_{x\in X}[\Delta u(x)]f(x)$$

holds if any one of the following conditions holds:
(1) $f \in L_0(X)$ *or* $u \in L_0(X)$;
(2) $du \in L_0(Y)$

Proof. For $w \in L(Y)$, put

$$\partial w(x) := \sum_{y\in Y} K(x, y)w(y).$$

Then $\Delta u = \partial[du]$. Observe that the relation

$$\sum_{x\in X} v(x)\partial w(x) = -\sum_{y\in Y} r(y)[dv(y)]w(y)$$

holds if $v \in L_0(X)$ or $w \in L_0(Y)$. In fact, in these cases, the order of summation can be interchanged, since N is locally finite. □

Corollary 1.1. $D(u, \varepsilon_a) = -\Delta u(a)$ *for every* $a \in X$.

To rewrite Δu in a more familiar form, we introduce $c(x, z)$ for $x, z \in X, x \neq z$ and $c(x)$ for $x \in X$ as follows:

$$\begin{aligned} c(x, z) &:= \sum_{y\in Y} r(y)^{-1}|K(x, y)K(z, y)|, \\ c(x) &:= \sum_{y\in Y} r(y)^{-1}|K(x, y)|. \end{aligned}$$

For simplicity, we set $c(x, x) := 0$. Clearly, $c(x, z) = c(z, x)$ and $c(x, z) = 0$ for all $z \in X \setminus W(x)$. Recall that $c(x)$ is the total conductance at x, i.e.,

$$c(x) = \sum_{z\in X} c(x, z) = \sum_{z\in W(x)} c(x, z).$$

We have

$$\Delta u(x) = -c(x)u(x) + \sum_{z\in X} c(x, z)u(z).$$

Remark 1.1. *Let $u_1, u_2 \in L(X)$ and $u_1 \leq u_2$ on X. If $u_1(a_0) = u_2(a_0)$, then the inequality $\Delta u_1(a_0) \leq \Delta u_2(a_0)$ holds.*

Definition 1.1. *We say that a function $u \in L(X)$ is harmonic (resp. superharmonic) on a subset A of X if $\Delta u(x) = 0$ (resp. $\Delta u(x) \leq 0$) for all $x \in A$.*

Now we give the framework of this paper. We always fix a node x_0 and set $X_0 = X \setminus \{a_0\}$. The operation φ_A defined in §2 with a finite subset A of X_0 and a $\varphi \in L(X)$ plays a fundamental role in our study. The discrete Kuramochi function $\tilde{g}_a$ is defined in §3 as in [7] with the aid of $1_{\{a\}}$. Taking the Kuramochi function as a potential kernel, we define the Kuramochi potential $\tilde{G}\mu$ of a non-negative function μ on X_0. In §4, we introduce an SHS function (or full superharmonic function) as a non-negative real valued function v on X which satisfies $v_A \leq v$ for every finite subset A of X_0. We say that an SHS function is an HS function if it is harmonic on X_0. The main result in S 4 is a Riesz decomposition theorem which assures that every SHS function is represented as the sum of a Kuramochi potential and an HS function. In §5, the operation v_A^* (the reduced function of v onto $X \setminus A$) is introduced for an SHS function and a subset A of X_0 as a generalization of v_A. It will be shown that v_A^* has properties analogous to v_A. Thus except this section we use the notation v_A instead of v_A^*. The Kuramochi boundary of an infinite network will be introduced in §6 more in detail than in [7]. Extending the Kuramochi kernel to the Kuramochi boundary ∂N of N continuously, we study in §7 the representation of HS_0 function by means of Kuramochi potential of a measure on the Kuramochi boundary. By means of a reduced function v_F of an SHS function v onto a closed subset F of ∂N, the points of ∂N are classified into minimal points and non-minimal points in §8. The relation between a minimal function and a minimal point will be given in §10. Finally, the uniqueness of the canonical representation of HS_0 is proved in §11.

2. Dirichlet principle

Denote by R the set of all real numbers and let $a_0 \in X$ be a fixed node in this paper. We introduce the following notation:

$$\begin{aligned} X_0 : \ &= \ X \setminus \{a_0\} \\ D(N; a_0) \ &:= \ \{u \in D(N); u(a_0) = 0\}. \end{aligned}$$

Notice that $D(N; a_0)$ is a Hilbert space with respect to the inner product $(u, v) := D(u, v)$ and that $\|u\| := [D(u)]^{1/2}$ is a norm on $D(N; a_0)$. Observe that the norm convergence on $D(N; a_0)$ implies the pointwise convergence (cf. [12]). Namely, if $u_n, u \in D(N; a_0)$ and if $D(u_n - u) \to 0$ as $n \to \infty$, then $u_n(x) \to u(x)$ as $n \to \infty$ for every $x \in X$.

Hereafter, we always assume that an exhaustion $\{N_n\}(N_n = < X_n, Y_n >)$ of N satisfies the condition $a_0 \in X_1$.

For a subset A of X_0 and $\varphi \in L(X)$, set

$$\mathcal{D}_A(\varphi) := \{u \in D(N; a_0); u = \varphi \text{ on } A\}.$$

The following Dirichlet principle in N is well-known (cf. [7] and [12]):

Theorem 2.1. *Assume that $\mathcal{D}_A(\varphi) \neq \emptyset$. Then there exists a unique $h \in \mathcal{D}_A(\varphi)$ which has the minimum Dirichlet sum among the functions in $\mathcal{D}_A(\varphi)$. The function h is harmonic in $X_0 \setminus A$ and is characterized by $h \in \mathcal{D}_A(\varphi)$ and*

$$(u - h, h) = 0 \text{ for all } u \in \mathcal{D}_A(\varphi). \tag{2.1}$$

Definition 2.1. *Denote by φ_A the unique function h in the above theorem, i.e., $\varphi_A \in \mathcal{D}_A(\varphi)$ and*

$$D(\varphi_A) = \min\{D(u); u \in \mathcal{D}_A(\varphi)\}.$$

Corollary 2.1. *$(\varphi_A, f) = 0$ for every $f \in D(N; a_0)$ with $f = 0$ on A.*

Proof. Let $f \in D(N; a_0)$ with $f = 0$ on A. Then $u = f + \varphi_A \in \mathcal{D}_A(\varphi)$, so that we have $(f, \varphi_A) = 0$ by (2.1). □

Proposition 2.1. *The condition $\mathcal{D}_A(\varphi) \neq \emptyset$ holds if any one of the following conditions is satisfied:*
(1) A is a finite subset of X_0 and $\varphi \in L(X)$.
(2) A is a subset of X_0 and $D(\varphi) < \infty$.

We recall some properties of φ_A studied in [7]:

Theorem 2.2. *Let A and B be subsets of X_0 such that $A \subset B$ and let $\varphi \in L(X)$. If $\mathcal{D}_B(\varphi) \neq \emptyset$, then $(\varphi_A)_B = \varphi_A$.*

Proof. By the relation $\mathcal{D}_A(\varphi) \supset \mathcal{D}_B(\varphi) \neq \emptyset$, $\varphi_A \in D(N; a_0)$ exists by Theorem 2.1, so that $(\varphi_A)_B$ exists by Proposition 2.1. Since $(\varphi_A)_B \in \mathcal{D}_A(\varphi)$, $D(\varphi_A) \leq D((\varphi_A)_B)$. Since $\varphi_A \in \mathcal{D}_B(\varphi_A)$, $D((\varphi_A)_B) \leq D(\varphi_A)$. Hence $D((\varphi_A)_B) = D(\varphi_A)$ and $\varphi_A = (\varphi_A)_B$ by the uniqueness of φ_A. □

For a subset A of X_0, put

$$L_A(X) := \{\varphi \in L(X); \mathcal{D}_A(\varphi) \neq \emptyset\}.$$

Theorem 2.3. *φ_A is a positive linear mapping from $L_A(X)$ into $D(N; a_0)$, i.e.,*
(1) $\varphi = 0$ on A if and only if $\varphi_A = 0$.
(2) The condition: $\varphi \geq 0$ on A implies that $\varphi_A \geq 0$ on X.
(3) $(\varphi + \psi)_A = \varphi_A + \psi_A$ for $\varphi, \psi \in L_A(X)$
(4) $(c\varphi)_A = c\varphi_A$ for $\varphi \in L_A(X)$, $c \in R$.

Proof. (1) Since $\varphi_A = \varphi$ on A, the "if" part is clear. If $\varphi = 0$ on A, then $0 \in \mathcal{D}_A(\varphi)$ and $D(\varphi_A) = 0$. Since $\varphi_A \in D(N; a_0)$, we have $\varphi_A = 0$.

(2) Assume that $\varphi \geq 0$ on A. Then $\varphi_A^+ = \max(\varphi_A, 0) \in \mathcal{D}_A(\varphi)$, so that $D(\varphi_A) \leq D(\varphi_A^+)$. By Corollary 2 of Lemma 2 in [12], we have $D(\varphi_A^+) \leq D(\varphi_A)$. Hence $D(\varphi_A^+) = D(\varphi_A)$ and $\varphi_A = \varphi_A^+ \in L^+(X)$ by the uniqueness of φ_A.

(3) Let φ and ψ satisfy $\mathcal{D}_A(\varphi) \neq \emptyset$ and $\mathcal{D}_A(\psi) \neq \emptyset$ and put $f = \varphi + \psi$ and $h = \varphi_A + \psi_A$. Then $h \in \mathcal{D}_A(f)$ and $(h - f_A, f_A) = 0$ by (2.1). Since $h - f_A = 0$ on A, we have by Corollary 2.1

$$\begin{aligned} D(h - f_A) &= D(h) + D(f_A) - 2(h, f_A) \\ &= D(h) - (h, f_A) = (h, h - f_A) \\ &= (\varphi_A, h - f_A) + (\psi_A, h - f_A) = 0 \end{aligned}$$

Thus $h = f_A$.

(4) Let $\varphi \in L_A(X)$, and let c be a number and put $h = (c\varphi)_A$. We have $c\varphi_A \in \mathcal{D}_A(c\varphi)$ and $c\varphi_A - h = 0$ on A, so that

$$D(c\varphi_A - h) = (c\varphi_A - h, c\varphi_A) - (c\varphi_A - h, h) = 0$$

by (2.1) and Corollary 2.1. Thus $h = c\varphi_A$. □

Corollary 2.2. *Let $\varphi, \psi \in L_A(X)$. If $\varphi \geq \psi$ on A, then $\varphi_A \geq \psi_A$ on X.*

Theorem 2.4. *Let A be a finite subset of X_0. Then for each $x \in X_0$, there exists a unique function $\omega_A^x \in L^+(X)$ such that $\omega_A^x(z) = 0$ on $X_0 \setminus A$ and*

$$\varphi_A(x) = \sum\nolimits_{z \in A} \varphi(z)\omega_A^x(z)$$

for every $\varphi \in L(X)$.

Proof. The uniqueness of ω_A^x is clear. To prove the existence of ω_A^x, let $x \in X_0$. Since A is a finite subset of X_0. Then $L_A(X) = L(X)$ from Proposition 2.1. Hence for each $x \in X_0$ we may consider that $\varphi_A(x)$ is a linear functional on $D(N; a_0)$ by Theorem 2.3. By Lemma 1 in [12], there exists a constant M_x depending only on $\{x\}$ such that

$$|u(x)| \leq M_x[D(u)]^{1/2} \quad \text{for all} \quad u \in D(N; a_0). \tag{2.2}$$

For $\varphi \in D(N; a_0)$, we have by (2.1) and (2.2)

$$|\varphi_A(x)| \leq M_x[D\varphi_A)]^{1/2} \leq M_x[D(\varphi)]^{1/2},$$

which implies the continuity of φ_A on $D(N; a_0)$. By the Riesz representation theorem, there exists $\beta_A^x \in D(N; a_0)$ such that $\varphi_A(x) = (\varphi, \beta_A^x)$ for all $\varphi \in D(N; a_0)$. Put $\omega_A^x = -\Delta\beta_A^x$. Then by Lemma 1.1

$$\varphi_A(x) = (\varphi, \beta_A^x) = \sum\nolimits_{z \in X} \varphi(z)\omega_A^x(z) \tag{2.3}$$

for every $\varphi \in L_0(X)$. For $b \in X_0$, we have by (2.3)

$$0 \leq (\varepsilon_b)_A(x) = \sum\nolimits_{z \in X} \varepsilon_b(z)\omega_A^x(z) = \omega_A^x(b), \tag{2.4}$$

i.e., $\omega_A^x \in L^+(X)$. In case $b \in X_0 \setminus A$, we have $\omega_A^x(b) = 0$ by (2.4) and Theorem 2.2(1), namely, $S_{\omega_A^x} \subset A$. Therefore, it follows that

$$\varphi_A(x) = \sum\nolimits_{z \in A} \varphi(z)\omega_A^x(z) \tag{2.5}$$

for all $\varphi \in L_0(X)$. For $\varphi \in L(X)$, consider the function $\psi \in L_0(X)$ defined by $\psi = \varphi$ on A and $\psi = 0$ on $X \setminus A$. Then $\psi_A = \varphi_A$ by Theorem 2.2(1),(2). It follows that (2.5) holds for all $\varphi \in L(X)$. □

Remark 2.1. *Notice that $\omega_A^x = \varepsilon_x$ if A is a finite subset of X_0 and $x \in A$.*

Proposition 2.2. *Let $x \in X_0$ and take $A := W(x)$. Then $\omega_A^x(z) = c(x,z)/c(x)$ for $z \in W(x)$ and $\omega_A^x(z) = 0$ for $z \in X \setminus W(x)$.*

Proof. Since φ_A is harmonic on $X_0 \setminus A$ and $\varphi_A(z) = \varphi(z)$ on A, we have

$$\begin{aligned} 0 = \Delta\varphi_A(x) &= -c(x)\varphi_A(x) + \sum\nolimits_{z\in X} c(x,z)\varphi_A(z) \\ &= -c(x)\varphi_A(x) + \sum\nolimits_{z\in A} c(x,z)\varphi(z), \end{aligned}$$

so that

$$\varphi_A(x) = \sum\nolimits_{z\in A} \frac{c(x,z)}{c(x)}\varphi(z).$$

Therefore

$$\sum\nolimits_{z\in A} \frac{c(x,z)}{c(x)}\varphi(z) = \sum\nolimits_{z\in A} \varphi(z)\omega_A^x(z)$$

for all $\varphi \in L(X)$. By taking $\varphi = \varepsilon_a$ for $a \in A = W(x)$, we obtain $\omega_A^x(a) = c(x,a)/c(x)$. □

Theorem 2.5. *Let $\varphi \in L_A(X)$. Then*

$$\min(0, \inf\{\varphi(z); z \in A\}) \le \varphi_A \le \max(0, \sup\{\varphi(z); z \in A\}). \tag{2.2}$$

Proof. Put

$$c_1 := \min(0, \inf\{\varphi(z); z \in A\}) \le 0,\ c_2 := \max(0, \sup\{\varphi(z); z \in A\}) \ge 0.$$

We may assume that both c_1 and c_2 are finite. Define f_1, f_2 by

$$f_1(x) := \max(\varphi_A, c_1),\ f_2(x) := \min(\varphi_A, c_2).$$

Then $f_1, f_2 \in \mathcal{D}_A(\varphi)$, so that $D(\varphi_A) \le D(f_k)$ for $k = 1, 2$. Noting that $\max(t, c_1)$ and $\min(t, c_2)$ are contractions on R, we see that $D(f_k) \le D(\varphi_A)$ for $k = 1, 2$ by Lemma 2 in [12]. Therefore $D(f_1) = D(f_2) = D(\varphi_A)$. By the uniqueness of φ_A, we have $f_1 = f_2 = \varphi_A$, and hence $c_1 \le \varphi_A \le c_2$. □

Corollary 2.3. *If A is a finite subset of X_0, then $\omega_A^x(X) \le 1$ for $x \in X_0 \setminus A$.*

Proof. By Theorem 2.5, $1_A(x) \le 1$ on X. Our assertion follows immediately from Theorem 2.4.

Lemma 2.1. *Let A be a finite subset of X_0, and let $\{\varphi_k\}$ be a sequence of functions of $L(X)$ which converges pointwise to $\varphi \in L(X)$. Then $\{(\varphi_k)_A\}$ converges uniformly to φ_A as $k \to \infty$.*

Proof. For any $\epsilon > 0$, there exists k_0 such that

$$\sup\{|\varphi_k(a) - \varphi(a)|; a \in A\} < \epsilon$$

for all $k \ge k_0$, since A is a finite set. By Theorems 2.4 and 2.5, we have

$$\begin{aligned} |(\varphi_k)_A(x) - \varphi_A(x)| &= |\sum\nolimits_{a\in A} [\varphi_k(a) - \varphi(a)]\omega_A^x(a)| \\ &\le \sum\nolimits_{a\in A} |\varphi_k(a) - \varphi(a)|\omega_A^x(a) \\ &\le \epsilon\, \omega_A^x(A) \le \epsilon \end{aligned}$$

for all $k \ge k_0$ and all $x \in X$. This completes the proof. □

Lemma 2.2. *Let $\varphi \in L^+(X)$. Then $\varphi_{A\cup B} \le \varphi_A + \varphi_B$ holds for any finite subsets A, B of X_0.*

Proof. Put $h := \varphi_A + \varphi_B - \varphi_{A\cup B}$. Then $h_{A\cup B} = h$ by Theorems 2.2 and 2.3. Since $h \ge 0$ on $A \cup B$, we have by Theorem 2.5

$$h(x) = h_{A\cup B}(x) \ge \min\{h(z); z \in A \cup B\} \ge 0$$

on X. □

3. Kuramochi function

Now we introduce a discrete analogue of the Kuramochi function in the theory of Riemann surface and study its fundamental properties as in [7] and [10].

Definition 3.1. *Let $a \in X_0$ and denote by $1_{\{a\}}$ the function φ_A when we take A as $\{a\}$ and φ as 1. We set*

$$\tilde{g}_a(x) = \tilde{g}(x, a) := 1_{\{a\}}/D(1_{\{a\}})$$

and call it the Kuramochi function of N with pole at $a \in X_0$.

Needless to say, $1_{\{a\}} \in D(N; a_0)$ and $D(1_{\{a\}}) \neq 0$.
The following properties are well-known (cf. [7] and [8]):

Theorem 3.1. *The Kuramochi function $\tilde{g}_a$ has the following properties:*
(1) $\Delta\tilde{g}_a(x) = -\varepsilon_a(x)$ *on* X_0.
(2) $0 < \tilde{g}_a(x) \le \tilde{g}_a(a) < \infty$ *on the component of X_0 which contains a and $\tilde{g}_a(x) = 0$ on every component of X_0 which does not contain a.*
(3) $\tilde{g}_a$ *is a reproducing kernel of $D(N; a_0)$, i.e.,*

$$(\tilde{g}_a, u) = u(a) \quad \text{for every} \quad u \in D(N; a_0). \tag{3.1}$$

(4) *If $f \in L_0(X)$ and $f(a_0) = 0$, then*

$$f(a) = (\tilde{g}_a, f) = -\sum\nolimits_{x\in X} [\Delta f(x)]\tilde{g}_a(x). \tag{3.2}$$

(5) $\tilde{g}_a(b) = \tilde{g}_b(a)$ *for every* $a, b \in X_0$.
(6) $\Delta\tilde{g}_a(a_0) = 1$.

We give the proof of the following theorem (Theorem 3.2 in [7]), since this is a key result in this paper.

Theorem 3.2. *Let A be a nonempty subset X_0. Then:*
(1) $(\tilde{g}_a)_A = g_a$ *on X if $a \in A$.*
(2) $(\tilde{g}_a)_A \le \tilde{g}_a$ *on X if A is a finite set and $a \notin A$.*

Proof. For simplicity we put $\tilde{g} = \tilde{g}_a$, $h = 1_{\{a\}}$, $\alpha = D(1_{\{a\}})$ and $\tilde{g}_A = (\tilde{g}_a)_A$. Then $\tilde{g} = h/\alpha$ and $\tilde{g}_A = h_A/\alpha$ by Theorem 2.3.

(1) Assume that $a \in A$. Then $h_A = (1_{\{a\}})_A = 1_{\{a\}} = h$ by Theorem 2.2, and hence $(\tilde{g}_a)_A = g_a$ on X.

(2) Assume that A is a finite set and $a \notin A$. Put $u := \tilde{g} - \tilde{g}_A$ and $B := A \cup \{a\}$. Then by Theorems 2.2 and 2.3

$$u_B = \tilde{g}_B - (\tilde{g}_A)_B = \tilde{g} - \tilde{g}_A = u,$$

since $a \in B$ and $A \subset B$. Since $u = 0$ on A, Theorem 2.5 implies that $u \geq \min(0, u(a))$. Since u is superharmonic on $X \setminus A$, u cannot attain its minimum at $x = a$ (cf. Lemma 2.1 in [13]). Hence $u \geq 0$ on X, i.e., $(\tilde{g}_a)_A \leq \tilde{g}_a$ on X. $\square$

Theorem 3.3. *Let A be a nonempty finite subset of X_0. Then $(\tilde{g}_a)_A(b) = (\tilde{g}_b)_A(a)$ for every $a, b \in X_0$.*

Proof. By Theorem 2.4 and Theorem 3.2(1), we have

$$\begin{aligned}
(\tilde{g}_a)_A(b) &= \textstyle\sum_{x\in A} \tilde{g}_a(x)\omega_A^b(x) = \sum_{x\in A} \tilde{g}_x(a)\omega_A^b(x) \\
&= \textstyle\sum_{x\in A}[(\tilde{g}_x)_A(a)]\omega_A^b(x) = \sum_{x\in A}[\sum_{z\in A} \tilde{g}_x(z)\omega_A^a(z)]\omega_A^b(x) \\
&= \textstyle\sum_{z\in A}[\sum_{x\in A} \tilde{g}_z(x)\omega_A^b(x)]\omega_A^a(z) \\
&= \textstyle\sum_{z\in A}[(\tilde{g}_z)_A(b)]\omega_A^a(z) = \sum_{z\in A} \tilde{g}_z(b)\omega_A^a(z) \\
&= \textstyle\sum_{z\in A} \tilde{g}_b(z)\omega_A^a(z) = (\tilde{g}_b)_A(a) \quad \square
\end{aligned}$$

For $\mu \in L^+(X)$, we define the $\tilde{g}$-potential $\tilde{G}\mu$ of μ by

$$\tilde{G}\mu(x) = \textstyle\sum_{a\in X} \tilde{g}_a(x)\mu(a).$$

We call $\tilde{G}\mu$ the Kuramochi potential of μ if $\tilde{G}\mu \in L^+(X)$ and put

$$M(\tilde{G}) := \{\mu \in L^+(X); \mu(a_0) = 0, \tilde{G}\mu \in L^+(X)\}.$$

Furthermore, denote by $\tilde{G}_A\mu$ the $(\tilde{g}_a)_A$-potential of μ, i.e.,

$$(\tilde{G}_A\mu)(x) := \textstyle\sum_{a\in X} (\tilde{g}_a)_A(x)\mu(a).$$

Lemma 3.1. *Let A be a finite subset of X_0. Then $\tilde{G}_A\mu = (\tilde{G}\mu)_A$ for every $\mu \in M(\tilde{G})$.*

Proof. By Theorem 3.2, $\tilde{G}_A\mu \in L(X)$ for every $\mu \in M(\tilde{G})$. We have by Theorem 2.4

$$(\tilde{g}_a)_A(x) = \textstyle\sum_{z\in A} \tilde{g}_a(z)\omega_A^x(z)$$

for every $x \in X_0$. Therefore, we have

$$\begin{aligned}(\tilde{G}_A\mu)(x) &= \sum_{a\in X}(\tilde{g}_a)_A(x)\mu(x) \\ &= \sum_{a\in X}[\sum_{z\in A}\tilde{g}_a(z)\omega_A^x(z)]\mu(a) \\ &= \sum_{z\in A}[\sum_{a\in X}\tilde{g}_a(z)\mu(a)]\omega_A^x(z) \\ &= \sum_{z\in A}\tilde{G}\mu(z)\omega_A^x(z) \\ &= (\tilde{G}\mu)_A(x). \quad \square\end{aligned}$$

By Theorem 3.1(1), we have

Lemma 3.2. *If $\mu \in M(\tilde{G})$, then $\Delta\tilde{G}\mu = -\mu$ on X_0. Consequently, $\tilde{G}\mu$ is harmonic on $X_0 \setminus S_\mu$ and superharmonic on X_0.*

Lemma 3.3. *Let $\mu \in M(\tilde{G})$. Then $\mu(X) = \Delta\tilde{G}\mu(a_0)$.*

Proof. Since $\Delta\tilde{g}_a(a_0) = 1$ for all $a \in X_0$ by Theorem 3.1(6) and $\mu(a_0) = 0$, we have

$$\Delta\tilde{G}\mu(a_0) = \sum_{x\in X}\Delta\tilde{g}_a(a_0)\mu(x) = \mu(X). \quad \square$$

Corollary 3.1. *$\mu(X) < \infty$ for every $\mu \in M(\tilde{G})$.*

Notice that another useful characterization of the Kuramochi function $\tilde{g}$ was given in [7] by using the concept of flows from a_0 to $a \in X_0$. Some examples of Kuramochi functions were given there. We recall a simple one of them. Let Z^+ be the set of all non-negative integers.

Example 3.1. Let $X = \{x_k; k \in Z^+\}, Y = \{y_{k+1}; k \in Z^+\}$,

$$K(x_{k-1}, y_k) = -1,\ K(x_k, y_k) = 1 \text{ for } k \in Z^+, k \geq 1,$$

and $K(x, y) = 0$ for any other pair of (x, y). Take $a_0 = x_0$ and set

$$R_k := \sum_{j=1}^{k} r(y_j).$$

For $n \geq 1$, we have $\tilde{g}_{x_n}(x_0) = 0,\ \tilde{g}_{x_n}(x_k) = R_k$ for $1 \leq k \leq n$ and $\tilde{g}_{x_n}(x_k) = R_n$ for $k \geq n$.

4. HS functions and SHS functions

We begin with

Definition 4.1. *Let $v \in L^+(X)$ which is not equal to 0 identically. We say that v is an SHS (or full superharmonic) function if $v_A(x) \le v(x)$ on X for every finite subset A of X_0. We say that v is an HS function if it is an SHS function which is harmonic on X_0. We say that v is an SHS_0 (resp. HS_0) function if it is an SHS (resp. HS) which satisfies $v(a_0) = 0$.*

Notice that for any SHS function v, there exists an SHS_0 function which takes same values as v on X_0. In fact, for an SHS function v, define u by $u = v$ on X_0 and $u(a_0) = 0$. Since $u_A = v_A$ for any finite subset A of X_0 by Theorem 2.3(1), u is an SHS_0 function.

Proposition 4.1. *If v is an SHS function, then it is superharmonic on X_0.*

Proof. Let $x \in X_0$ and let take A as $W(x)$. From Proposition 2.2 it follows that

$$v(x) \ge v_A(x) = \sum\nolimits_{z \in A} v(z)\omega_A^x(z) = \sum_{z \in W(x)} [c(x,z)/c(x)]v(z),$$

so that $\Delta v(x) \le 0$. $\square$

By Lemma 2.1 we have

Theorem 4.1. *Let $\{v_n\}$ be a sequence of SHS (resp. SHS_0, HS and HS_0) functions. If $\{v_n\}$ converges pointwise to $v \in L(X)$, then v is an SHS (resp. SHS_0, HS and HS_0 function.*

Theorem 4.2. *Every Kuramochi potential $\tilde{G}\mu$ is an SHS_0 function.*

Proof. For a finite subset A of X_0, we have by Lemma 3.1 and Theorem 3.2

$$\begin{aligned} (\tilde{G}\mu)_A(x) &= \sum\nolimits_{a \in X} (\tilde{g}_a)_A(x)\mu(a) \\ &\le \sum\nolimits_{a \in X} \tilde{g}_a(x)\mu(a) = \tilde{G}\mu(x), \end{aligned}$$

so that $\tilde{G}\mu$ is an SHS function. Since $\tilde{g}_a(a_0) = 0$, we have

$$\tilde{G}\mu(a_0) = \sum\nolimits_{a \in X} \tilde{g}_a(a_0)\mu(a) = 0. \quad \square$$

Theorem 4.3. *Let v be an SHS function and let A be a finite subset of X_0. Then there exists a unique $\nu \in L^+(X)$ such that $v_A = \tilde{G}\nu$ and $S_\nu \subset A$.*

Proof. By definition, v_A is harmonic on $X_0 \setminus A$. To prove that v_A is superharmonic on A, let $a \in A$. Then we have $v_A(a) = v(a)$ and $v_A(x) \leq v(x)$ on X, so that

$$\begin{aligned} \Delta v_A(a) &= -c(a)v_A(a) + \sum\nolimits_{x\in X} c(a,x)v_A(x) \\ &\leq -c(a)v(a) + \sum\nolimits_{x\in X} c(a,x)v(x) = \Delta v(a) \leq 0, \end{aligned}$$

since v is superharmonic on X. Thus v_A is superharmonic on A. Let $\nu(x) := -\Delta v_A(x)$ on X_0 and $\nu(a_0) := 0$. Then $\nu \in L^+(X)$ and $\nu(x) = 0$ on $X \setminus A$. Notice that $\tilde{G}\nu \in D(N; a_0)$. To prove $v_A = \tilde{G}\nu$, put $h := v_A - \tilde{G}\nu$. Then h is harmonic on X_0 by Lemma 3.2. Since $S_\nu \subset A$ and $(\tilde{g}_a)_A(x) = \tilde{g}_a(x)$ for every $a \in A$ by Theorem 3.2(1), we have $(\tilde{G}\nu)_A = \tilde{G}\nu$ and

$$h_A = (v_A)_A - (\tilde{G}\nu)_A = v_A - \tilde{G}\nu = h$$

on X_0. Since $D(h) < \infty$, we have by Theorem 2.5,

$$(4.1) \qquad \min(0, \min\{h(x); z \in A\}) \leq h \leq \max(0, \max\{h(x); z \in A\})$$

on X, so that h attains its minimum on A. By the minimum principle, h is equal to 0 on the connected component X^* of $X \setminus \{a_0\}$ which contains the node $x_0 \in A$ such that $h(x_0) = \min\{h(x); z \in A\}$. Thus $\min\{h(x); z \in A\} = 0$. By applying the same reasoning to $-h$, we obtain $\max\{h(x); z \in A\} = 0$. By (4.1), we have $h = 0$, and hence $v_A = \tilde{G}\nu$. The uniqueness of ν follows from Lemma 3.2. □

Corollary 4.1. *If v is an SHS function and if A is a finite subset of X_0, then v_A is an SHS_0 function.*

Now we shall prove the following Riesz decomposition theorem which assures that every SHS function is equal to the sum of a Kuramochi potential and an HS function. More precisely

Theorem 4.4. *For an SHS function v, there exist $\mu \in M(\tilde{G})$ and a non-negative HS function q such that $v = \tilde{G}\mu + q$ on X_0.*

Proof. Let v be an SHS function and let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N. For $B_n := X_n \setminus \{a_0\}$, there exists $\nu_n \in L^+(X)$ such that $v_{B_n} = \tilde{G}\nu_n$ and $S_{\nu_n} \subset B_n$ by Theorem 4.3. For simplicity, put $v_n := v_{B_n}$. Put $\nu_n(x) := -\Delta v_n(x)$ for $x \in X_0$ and $\nu_n(a_0) := 0$ and define μ_n by $\mu_n = \nu_n$ on X_{n-1} and $\mu_n = 0$ on $X \setminus X_{n-1}$. Since $v_n = v$ on X_n, we have $\mu_n(x) = -\Delta v(x)$ on B_{n-1} and $\mu_n(x) = 0$ on $X \setminus B_{n-1}$. We see easily that $\{\mu_n\}$ is an increasing sequence and converges pointwise to $\mu \in L^+(X)$ defined by $\mu(x) = -\Delta v(x)$ for $x \in X_0$ and $\mu(a_0) = 0$. Since $\mu_n \leq \mu_{n+1} \leq \nu_{n+1}$, we have

$$\tilde{G}\mu_n \leq \tilde{G}\mu_{n+1} \leq \tilde{G}\nu_{n+1} = v_{n+1} \leq v,$$

so that $p = \lim_{n\to\infty} \tilde{G}\mu_n$ exists and $p \in L+(X)$. By Fatou's lemma, we have

$$\tilde{G}\mu(x) = \sum\nolimits_{a\in X} \tilde{g}_a(x)\mu(a) \leq \liminf_{n\to\infty} \sum\nolimits_{a\in X} \tilde{g}_a(x)\mu_n(a) = \lim_{n\to\infty} \tilde{G}\mu_n(x) = p(x) \leq v(x).$$

Thus $\mu \in M(\tilde{G})$. Since $\mu_n \leq \mu$, we have $p \leq \tilde{G}\mu$, and hence $p = \tilde{G}\mu$. Now put $\lambda_n := \nu_n - \mu_n$ and $q_n := \tilde{G}\lambda_n$. Then $v_n = \tilde{G}\nu_n = \tilde{G}\mu_n + q_n$. Notice that $q_n \in L^+(X)$, since $\lambda_n \in L^+(X)$. For $m < n$, we have $v_n = v_m = v$ on X_m and

$$q_m - q_n = \tilde{G}\mu_n - \tilde{G}\mu_m \geq 0$$

on X_m. Namely $\{q_n\}$ converges decreasingly to some $q \in L^+(X)$. Since $v = \tilde{G}\mu_n + q_n$ on B_n, we have $v = \tilde{G}\mu + q$ on X_0. Since q_n is an SHS function by Theorem 4.2, q is an SHS function by Theorem 4.1. Since q_n is harmonic on B_{n-1}, q is harmonic on X_0. Therefore q is an HS function. □

Finally we give a positive superharmonic function which is not an SHS function.

Example 4.1. Let N, a_0 and R_n be the same as in Example 3.1. Assume that $R := \sum_{y \in Y} r(y) < \infty$. Fix $n \geq 1$ and consider the function u defined by $u(a_0) := 0$, $u(x_k) := R_k/R_n$ for $1 \leq k \leq n$ and $u(x_k) := 1 - (R_k - R_n)/R$ for $k \geq n+1$. It is easy to show that $u(x)$ is a positive superharmonic function in X_0. Take $A := \{x_n\}$. We see that $u_A(x_k) = u(x_k)$ for $1 \leq k \leq n$ and $u_A(x_k) = 1$ for $k \geq n+1$, since u_A is harmonic on $\{x_k; k \neq 0, n\}$, $u_A = u$ on A and u_A has the minimum Dirichlet sum among $\mathcal{D}_A(u)$. Namely, $u_A(x_k) > u(x_k)$ for $k > n+1$ and u is not an SHS function.

5. Reduced function of an SHS function

The function v_A is always defined for a finite subset A of X_0 and $v \in L(X)$. Now we introduce a function v_A^* similar to v_A for any SHS function v and for any subset A of X_0. We begin with

Lemma 5.1. *Let v be an SHS function and let A be a subset of X_0. For an exhaustion $\{N_n\}(N_n = < X_n, Y_n >)$ of N, put $A_n := A \cap X_n$. Then $\{v_{A_n}\}$ converges increasingly to a function v_A^*. This v_A^* does not depend on the choice of an exhaustion of N and has the following properties:*
(1) $v_A^* = v$ on A.
(2) $v_A^* \leq v$ on X.
(3) v_A^* is an SHS_0 function.

Proof. For simplicity, put $v_n := v_{A_n}$. Then $v_n \leq v$. For $m < n$, we have by Theorems 2.2 and 2.3

$$0 \leq (v - v_m)_{A_n} = v_n - (v_m)_{A_n} = v_n - v_m,$$

so that $0 \leq v_m \leq v_n \leq v$. Thus $\{v_n\}$ converges to a function $f \in L^+(X)$.

Let $\{N'_n\}(N'_n =< X'_n, Y'_n >)$ be another exhaustion of N and put $A'_n = A \cap X'_n$. Let f' be the limit of $\{v_{A'_n}\}$. For a fixed A_n, there exists A'_k such that $A_n \subset A'_k$, so that $v_{A_n} \leq v_{A'_k} \leq f'$. Thus $f \leq f'$. By the symmetry of our discussion, we conclude that $f = f'$. Denote by v_A^* the limit of $\{v_n\}$. (1) and (2) follow from the fact that $v_n = v$ on A_n and $v_n \leq v$ on X.

(3) Since v_n is an SHS$_0$ function, v_A^* is also an SHS$_0$ function by Theorem 4.1. □

For an SHS function v and a subset A of X_0, we call the function v_A^* defined in Lemma 5.1 the *reduced function* of v onto $X \setminus A$. Furthermore, we call the sequence $\{v_n\}$ used in Lemma 5.1 a *determining sequence* of v_A^*.

By our definition, it is clear that $v_A^* = v_A$ for every finite subset A of X_0.

Theorem 5.1. *Let u and v be SHS functions and let A and B be subsets of X_0. Then*
(1) $(u+v)_A^* = u_A^* + v_A^*$.
(2) $v_{A\cup B}^* \le v_A^* + v_B^*$.
(3) If $A \subset B$, then $v_A^* \le v_B^*$.
(4) If $A \subset B$, then $(v_A^*)_B^* = v_A^*$.

Proof. (1) Let $\{u_n\}$ and $\{v_n\}$ be determining sequences of u_A^* and v_A^* respectively. Then $(u+v)_{A_n} = u_n + v_v$ by Theorem 2.3. Letting $n \to \infty$, we obtain (1).

(2) Let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N and put $A_n = A \cap X_n$ and $B_n = B \cap X_n$. Then $v_{A_n \cup B_n} \le v_{A_n} + v_{B_n}$ by Lemma 2.2. We let $n \to \infty$ and obtain (2).

(3) It suffices to show (3) in case both A and B are finite sets. We have by Theorems 2.2 and 2.3

$$0 \le (v - v_A)_B = v_A - (v_A)_B = v_A - v_B.$$

(4) By Lemma 5.1(3), $(v_A^*)_B^* \le v_A^*$. To prove the converse inequality, we consider an exhaustion $\{N_n\}(N_n = < X_n, Y_n >)$ of N and put $A_n := A \cap X_n$ and $B_n := B \cap X_n$. For $m < n$, we have $A_m \subset B_n$, so that $(v_{A_m})_{B_n} = v_{A_m}$ by Theorem 2.2. Since $v_{A_n} \le v_A^*$ by definition, we have $v_{A_m} \le (v_A^*)_{B_n}$. Letting $n \to \infty$ first and then $m \to \infty$, we obtain $v_A^* \le (v_A^*)_B^*$. □

Theorem 5.2. *Let A be a subset of X_0, let v be an SHS function such that $\mathcal{D}_A(v) \ne \emptyset$ and let $\{v_n\}$ be a determining sequence of v_A^*. Then $v_A^* \in D(N; a_0)$ and $D(v_n - v_A^*) \to 0$ as $n \to \infty$.*

Proof. Let $u \in \mathcal{D}_A(v)$, let v_n be a determining sequence of v_A^* and let A_n be the same as in Lemma 5.1. Then $u \in \mathcal{D}_{A_n}(v)$ and $D(v_n) \le D(u) < \infty$. For $m < n$, we have $(v_n - v_m, v_m) = 0$ by Theorem 2.1, so that

$$0 \le D(v_n - v_m) = D(v_n) - D(v_m).$$

Thus $\{D(v_n)\}$ is an increasing sequence which is bounded from above. It follows that $\{v_n\}$ is a Cauchy sequence in $D(N; a_0)$. Since $\{v_n\}$ converges pointwise to v_A^*, we conclude that $D(v_n - v_A^*) \to 0$ as $n \to \infty$ and $v_A^* \in D(N; a_0)$. □

Definition 5.1. *We say that v is a DSHS (resp. DHS) if it is an SHS (resp. HS) function and $D(v) < \infty$. We define a DSHS$_0$ (resp. DHS$_0$) function similarly.*

If v is a DSHS function, then $\mathcal{D}_A(v) \neq \emptyset$ by Proposition 2.1. Thus we have

Corollary 5.1. *Let A be a subset of X_0.*
(1) If v is a DSHS function, then $v_A^* \in D(N; a_0)$.
(2) Let $\{v_n\}$ be a determining sequence of v_A^*. If v is a DSHS function, then $D(v_n - v_A^*) \to 0$ as $n \to \infty$.

Theorem 5.3. *If v is a DSHS function, then $v_A^* = v_A$ for any subset A of X_0.*

Proof. Let A_n be the same as in Lemma 5.1 and put $v_n := v_{A_n}$. Since $\mathcal{D}_A(v) \subset \mathcal{D}_{A_n}(v)$, we have $(u - v_n, v_n) = 0$ for all $u \in \mathcal{D}_A(v)$ and for all n by Theorem 2.1. Since $D(v_n - v_A^*) \to 0$ as $n \to \infty$ by Corollary 5.1, we have $(u - v_A^*, v_A^*) = 0$ for all $u \in \mathcal{D}_A(v)$, and hence $v_A^* = v_A$. □

Corollary 5.2. *If v is a DSHS$_0$ function, then $D(v_A^*) \leq D(v)$.*

Proof. It suffices to note that $v \in \mathcal{D}_A(v)$ □.

Corollary 5.3. *Let v be a DSHS function and let A and B be subsets of X_0. If $A \subset B$, then $D(v_B^* - v_A^*) = D(v_B^*) - D(v_A^*)$.*

Proof. Since $v_B = v_B^* \in \mathcal{D}_B(v) \subset \mathcal{D}_A(v)$ by Theorem 5.3 and Corollary 5.1, we have $(v_B - v_A, v_A) = 0$ by Theorem 2.1, and hence $D(v_B - v_A) = D(v_B) - D(v_A)$. □

6. Definition of Kuramochi boundary of N

In this section we sometimes use the notation $\tilde{g}(x, a)$ for the Kuramochi function $\tilde{g}_a(x)$.

Definition 6.1. *We say that a sequence $\{x_j\}$ of nodes in X_0 tends to the infinity of N if, for any finite subset A of X_0, there exists j_0 such that $x_j \in X_0 \setminus A$ for all $j \geq j_0$.*

Lemma 6.1. *Assume that a sequence $\{x_j\}$ of nodes in X_0 tends to the infinity of N. Then $\{\tilde{g}(x, x_j)\}$ is a bounded sequence for each $x \in X_0$.*

Proof. By Theorem 3.1(2), we have

$$\tilde{g}(x, x_j) = \tilde{g}_{x_j}(x) = \tilde{g}_x(x_j) \leq \tilde{g}_x(x) < \infty$$

for all j and $x \in X_0$. □

Corollary 6.1. *If $\{x_j\}$ is a sequence of nodes in X_0 which tends to the infinity of N, then $\{\tilde{g}(\cdot, x_j)\}$ contains a convergent subsequence.*

We say that $\{x_j\}$ is a *fundamental sequence* if it is a sequence of nodes in X_0 which tends to the infinity of N such that $\{\tilde{g}(\cdot, x_j)\}$ converges. Two fundamental sequences $\{x_j\}$ and $\{x'_j\}$ are said to be *equivalent* if the limit functions of $\{\tilde{g}(\cdot, x_j)\}$ and $\{\tilde{g}(\cdot, x'_j)\}$ are equal to each other.

Definition 6.2. *A Kuramochi boundary point of N is defined as an equivalence class of fundamental sequences. The Kuramochi boundary (denoted by ∂N) of N is the set of Kuramochi boundary points.*

For $\xi \in \partial N$, there exists a fundamental sequence $\{x_j\}$ in X_0 such that $\lim_{j\to\infty} \tilde{g}(x, x_j)$ exists for every $x \in X$. In this case, we set

$$\tilde{g}(x, \xi) = \tilde{g}_\xi(x) := \lim_{j\to\infty} \tilde{g}(x, x_j)$$

and call the sequence $\{x_j\}$ a *determining sequence* of ξ. The definition of $\tilde{g}_\xi(x)$ does not depend on the choice of a determining sequence of ξ.

Let us put $\widetilde{X} := X_0 \cup \partial N$ and introduce a metric d on $\widetilde{X}$ by

$$d(x_1, x_2) := \sum\nolimits_{x\in X} \alpha(x) \frac{|\tilde{g}(x, x_1) - \tilde{g}(x, x_2)|}{1 + |\tilde{g}(x, x_1) - \tilde{g}(x, x_2)|} \tag{6.1}$$

for $x_1, x_2 \in \widetilde{X}$, where $\alpha \in L^+(X)$ is positive on X_0 and $\alpha(a_0) = 0$ and $\alpha(X) < \infty$.

By our definition, a sequence $\{x_j\} \subset \widetilde{X}$ converges to $x \in \widetilde{X}$ in the sense of d if and only if $\tilde{g}(\cdot, x_j) \to \tilde{g}(\cdot, x)$ as $j \to \infty$ in the sense of the pointwise convergence. The topology induced by this metric d on X_0 coincides with the original discrete topology.

Proposition 6.1. *$\widetilde{X} = X_0 \cup \partial N$ is a compact metric space with the metric d.*

Proof. It suffices to notice that $\{\tilde{g}(\cdot, x_j)\}$ contains a convergent subsequence for any sequence $\{x_j\}$ tending to ∂N by Corollary 6.1. □

Proposition 6.2. *Let $\xi \in \partial N$. Then:*

(1) $\tilde{g}_\xi = g(\cdot, \xi)$ is harmonic on X_0, $\tilde{g}_\xi(a_0) = 0$ and $\Delta\tilde{g}_\xi(a_0) = 1$.

(2) $\tilde{g}_\xi$ is an HS_0 function.

Proof. Let $\{x_j\}$ be a determining sequence of $\xi \in \partial N$. Then we have

$$\Delta\tilde{g}_{x_j} = -\varepsilon_{x_j} + \varepsilon_{a_0}.$$

By letting $j \to \infty$, we see that $\Delta\tilde{g}_\xi = \varepsilon_{a_0}$. This implies (1).

(2) Since $\tilde{g}_{x_j}$ is an SHS_0 function, our assertion follows from Theorem 4.1. □

As a direct consequence of the topology induced by the metric d, we have

Proposition 6.3. *$\tilde{g}(x, \xi)$ is a continuous function of $\xi \in X$ for each $x \in X_0$.*

We prepare

Lemma 6.2. *Let a_0 and a_0' be distinct two nodes and let $\tilde{g}_a$ and $\tilde{g}_a'$ be the reproducing kernels of $D(N;a_0)$ and $D(N;a_0')$ respectively with $a \in X \setminus \{a_0, a_0'\}$. Then*
(1) $\tilde{g}_a = \tilde{g}_a' + \tilde{g}_{a_0'} - \tilde{g}_a'(a_0)$.
(2) $\tilde{g}_a' = \tilde{g}_a + \tilde{g}_{a_0}' - \tilde{g}_a(a_0')$.

Proof. By symmetry, we have only to show (1). Denote by ψ_a the right hand side of (1). For any $u \in D(N;a_0)$, we have $u - u(a_0') \in D(N;a_0')$ and

$$\begin{aligned}(\psi_a, u) &= (\tilde{g}_a', u - u(a_0')) + (\tilde{g}_{a_0'}, u)\\ &= u(a) - u(a_0') + u(a_0') = u(a),\end{aligned}$$

and hence ψ_a is a reproducing kernel of $D(N;a_0)$. By the uniqueness of $\tilde{g}_a$, we conclude that $\psi_a = \tilde{g}_a$. □

Theorem 6.1. *The definition of Kuramochi boundary points of N does not depend on the choice of a_0 in the sense that every equivalent class of sequences of nodes near the boundary of N is the same.*

Proof. Let a_0 and a_0' be two fixed nodes and assume that $\{x_j\}$ is a fundamental sequence converging to ξ with respect to a_0. Let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N such that $\{a_0, a_0'\} \subset X_1$. Denote by $\tilde{g}_a$ (resp. $\tilde{g}_a'$) the Kuramochi function of N with pole at a with respect to a_0 (resp. a_0'). By Lemma 6.2(2), we have

$$\tilde{g}_{x_j}'(x) - \tilde{g}_{a_0}'(x) = \tilde{g}_{x_j}(x) - \tilde{g}_{x_j}(a_0') \to \tilde{g}_\xi(x) - \tilde{g}_\xi(a_0')$$

as $n \to \infty$ for each $x \in X$. Therefore $\{x_j\}$ is a fundamental sequence with respect to a_0'. Consequently two equivalent fundamental sequences with respect to a_0 are fundamental sequences with respect to a_0' and equivalent to each other.

Next we show the converse. Let $\{x_j\}$ and $\{x_j'\}$ be fundamental sequences which determine distinct boundary points ξ and ξ' with respect to a_0. Assume that $\{x_j\}$ and $\{x_j'\}$ determine the same boundary point ξ_0 with respect to a_0'. Then by Lemma 6.2(1)

$$\begin{aligned}|\tilde{g}_\xi(x) - \tilde{g}_{\xi'}(x)| &= \lim_{j\to\infty} |\tilde{g}_{x_j}(x) - \tilde{g}_{x_j'}(x)|\\ &= \lim_{j\to\infty} |\tilde{g}_{x_j}'(x) - g_{x_j'}'(x) - \tilde{g}_{x_j}'(a_0) + \tilde{g}_{x_j'}'(a_0)|\\ &= |\tilde{g}_{\xi_0}'(x) - \tilde{g}_{\xi_0}'(x) - \tilde{g}_{\xi_0}'(a_0) + \tilde{g}_{\xi_0}'(a_0)| = 0\end{aligned}$$

for all $x \in X$. Therefore $\tilde{g}_\xi(x) = \tilde{g}_{\xi'}(x)$ for all $x \in X$, and hence $\xi = \xi'$. This is a contradiction. □

Several examples of Kuramochi boundaries were given in [7].

7. Integral representation of HS_0 and SHS_0 functions

A measure on a compact Hausdorff space means a positive Radon measure on it. We say that a sequence $\{\mu_n\}$ of measures on a compact Hausdorff space Ω *converges vaguely* to a measure μ on Ω if

$$\int f d\mu_n \to \int f d\mu$$

as $n \to \infty$ for every real-valued continuous function f on Ω.

The following lemma is well-known (cf. [1]):

Lemma 7.1. *Let $\{\mu_n\}$ be a sequence of measures on a compact Hausdorff space Ω. If the set $\{\mu_n(\Omega)\}$ of total masses is bounded, then there exists a subsequence of $\{\mu_n\}$ which converges vaguely to some measure on Ω.*

Results on measure theory which we use in this paper are found in [1].

Notice that ∂N may not be a countable set (cf. [7]). Therefore a measure on $\widetilde{X}$ can not be identified with a function on $\widetilde{X}$ in general.

For a measure μ on $\widetilde{X}$, the Kuramochi potential $\tilde{G}\mu$ of μ is defined by

$$\tilde{G}\mu(x) := \int \tilde{g}_z(x) d\mu(z)$$

if the integral is finite for all $x \in X$. Namely, we only consider the case where $\tilde{G}\mu \in L(X)$.

First we have

Theorem 7.1. *Every SHS_0 function v is expressed as a Kuramochi potential of a measure on $\widetilde{X}$, and vice versa.*

Proof. By Theorem 4.4(Riesz decomposition theorem), an SHS_0 function v can be written in the form: $v = \tilde{G}\mu + u$, where $\mu(x) := -\Delta v(x)$ for $x \in X_0$, $\mu(a_0) = 0$ and u is an HS_0 function. By regarding μ as a measure on $\widetilde{X}$, we have only to show that every HS_0 function can be expressed as a Kuramochi potential of a measure on $\widetilde{X}$. Let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N and put $A_n := X_n - X_{n-1}$. Since u and u_{A_n} are harmonic on $X_{n-1} \setminus \{a_0\}$ and $u_{A_n} = u$ on $A_n \cup \{a_0\}$, we have $u_{A_n} = u$ on X_n by the minimum principle. There exists $\nu_n \in L^+(X)$ such that $u_{A_n} = \tilde{G}\nu_n$ and $\nu(X \setminus A_n) = 0$ by Theorem 4.3. Notice that

$$\nu_n(\widetilde{X}) = \nu_n(X) = \Delta\tilde{G}\nu_n(a_0) = \Delta u(a_0) < \infty$$

for all n. We regard ν_n as a measure on $\widetilde{X}$. Since the set of the total mass of ν_n is bounded, we can extract a vaguely convergent subsequence of $\{\nu_n\}$ (cf. [1]). Denote the subsequence $\{\nu_n\}$ again and let ν be the limit. By Lemma 7.1, we see that $\nu(\{x\}) = 0$ for every $x \in X_0$, so that ν is a measure on ∂N. We have by Lemma 7.1

$$u_{A_n}(x) = \sum\nolimits_{\xi\in\widetilde{X}} \tilde{g}_\xi(x)\nu_n(\xi) = \int \tilde{g}_\xi(x) d\nu_n(\xi) \to \int \tilde{g}_\xi(x) d\nu(\xi)$$

as $n \to \infty$, since $\tilde{g}_\xi(x)$ is a continuous function of ξ for each $x \in X_0$. Since $u_{A_n} = u$ on X_n, we obtain $u = \tilde{G}\nu$.

To prove the converse, let $u = \tilde{G}\nu$ be a Kuramochi potential of a measure ν on ∂N, i.e., $\nu(X_0) = 0$. Since $\tilde{g}_\xi(a_0) = 0$ for all $\xi \in \widetilde{X}$, we have

$$u(a_0) = \int \tilde{g}_\xi(a_0) d\nu(\xi) = 0.$$

For $x \in X_0$ and $\xi \in \partial N$, we have $\Delta\tilde{g}_\xi(x) = 0$, so that

$$\Delta u(x) = \int_{\xi\in\partial N} \Delta\tilde{g}_\xi(x) d\nu(\xi) = 0.$$

To prove that u is an HS$_0$ function, it suffices to show that $u_A \le u$ for every finite subset A of X_0. We have by Theorem 2.4 and Lemma 5.1

$$\begin{aligned} u_A(x) &= \sum\nolimits_{z\in A} u(z)\omega_A^x(z) \\ &= \sum\nolimits_{z\in A} \omega_A^x(z) \int_{\xi\in\partial N} \tilde{g}_\xi(z) d\nu(\xi) \\ &= \int \sum\nolimits_{z\in A} \tilde{g}_\xi(z)\omega_A^x(z) d\nu(\xi) \\ &= \int (\tilde{g}_\xi)_A(x) d\nu(\xi) \le \int \tilde{g}_\xi(x) d\nu(\xi) = u(x). \quad \square \end{aligned}$$

Corollary 7.1. *If μ is a measure on $\widetilde{X}$ such that $S_\mu \subset \partial N$, then $v := \tilde{G}\mu$ is an HS$_0$ function.*

Proof. It suffices to note that

$$\Delta v(x) = \int \Delta\tilde{g}_\xi(x) d\mu(\xi) = 0$$

for $x \in X_0$, since $\tilde{g}_\xi$ is harmonic on X_0. $\square$

Definition 7.1. *For an SHS function v and an infinite subset A of X_0, we put $v_A := v_A^*$ in the rest of this paper. In this sense, we put*

$$(\tilde{G}_A\mu)(x) := \int (\tilde{g}_z)_A(x) d\mu(z).$$

For an infinite subset A of X_0, denote by A^a the *closure* of A in the space $\widetilde{X}$. Denote by S_μ the support of a measure μ on $\widetilde{X}$.

As an extension of Theorem 4.3, we have

Theorem 7.2. *Let v be an SHS function and let A be an infinite subset of X_0. Then there exists a measure μ on $\widetilde{X}$ such that $v_A = \tilde{G}\mu$ and $S_\mu \subset A^a$.*

Proof. Let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N, and put $A_n := A \cap X_n$. By Theorem 4.3 there exists $\nu_n \in L^+(X)$ such that $v_{A_n} = \tilde{G}\nu_n$ and $S_{\nu_n} \subset A_n$. We have by Lemma 3.3

$$\nu_n(\widetilde{X}) = \Delta\tilde{G}\nu_n(a_0) = \Delta v_{A_n}(a_0) \le \Delta v_A(a_0) < \infty.$$

Namely, $\{\nu_n(\widetilde{X})\}$ is bounded. There exists a subsequence of $\{\nu_n\}$ which converges vaguely to a measure. Denote by $\{\nu_n\}$ the subsequence again and let ν be the limit. Then $S_\nu \subset A^a$. Since $\tilde{g}_z(x)$ is a continuous function of z in $\widetilde{X}$ for each $x \in X_0$,

$$v_{A_n}(x) = \tilde{G}\nu_n(x) = \int \tilde{g}_z(x)d\nu_n(z) \to \int \tilde{g}_z(x)d\nu(z) = \tilde{G}\mu(x)$$

as $n \to \infty$ for each $x \in X_0$. By definition, $\{v_{A_n}\}$ converges pointwise to v_A, and hence $v_A(x) = \tilde{G}\nu(x)$. □

Definition 7.2. *We say that μ is an associated measure of v_A if $v_A = \tilde{G}\mu$ holds.*

Remark 7.1. *Let v be an SHS function and let μ be an associated measure of v_A. Then*

$$\mu(\widetilde{X}) = \Delta\tilde{G}\mu(a_0) = \Delta v_A(a_0).$$

As a generalization of Lemma 3.1, we have

Theorem 7.3. *Let μ be a measure on $\widetilde{X}$. Then $(\tilde{G}\mu)_A = \tilde{G}_A\mu$ for every subset A of X_0.*

Proof. Let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N and put $A_n := A \cap X_n$. Then $\{(\tilde{g}_z)_{A_n}\}$ converges increasingly to $(\tilde{g}_z)_A$ as $n \to \infty$. By Lemma 3.1 and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}(\tilde{G}\mu)_A = \lim_{n\to\infty}(\tilde{G}\mu)_{A_n} &= \lim_{n\to\infty}(\tilde{G}_{A_n})\mu \\ &= \lim_{n\to\infty}\int(\tilde{g}_z)_{A_n}(x)d\mu(z) \\ &= \int(\tilde{g}_z)_A(x)d\mu(z) = \tilde{G}_A\mu. \quad \square\end{aligned}$$

For a closed subset F of ∂N, we introduce the reduced function v_F of an SHS function v onto $\widetilde{X} \setminus F$.

Definition 7.3. *For a closed subset F of ∂N, let us put*

$$d(x, F) := \min\{d(x, \xi); \xi \in F\}.$$

Lemma 7.2. *Let v be an SHS function and F be a closed subset of ∂N and put $F_{(m)} := \{x \in X_0; d(x, F) \leq 1/m\}$. Then $\{v_{F_{(m)}}\}$ converges decreasingly to an HS_0 function. Denote the limit by v_F and call it the reduced function of v to $\widetilde{X} \setminus F$.*

Proof. Since $F_{(m+1)} \subset F_{(m)}$, we have

$$0 \leq v_{F_{(m+1)}} \leq v_{F_{(m)}}$$

by Theorem 5.1(3). Thus $\{v_{F_{(m)}}\}$ converges decreasingly to an SHS$_0$ function by Theorem 4.1. Since $v_{F_{(m)}}$ is harmonic on $X_0 \setminus F_{(m)}$, its limit is harmonic on X_0. □

Proposition 7.1. *Let u and v be SHS functions and let E and F be closed subsets of ∂N. Then:*
(1) $v_F \leq v$ *on* X.
(2) $v_{\partial N} = v$ *if* v *is an* HS_0 *function.*
(3) $(u+v)_F = u_F + v_F$.
(4) *If* $E \subset F$, *then* $v_E \leq v_F$.

Proof. (1) We have $v_{F_{(m)}} \leq v$ by Lemma 5.1, so that $v_F \leq v$.

(2) Let $F = \partial N$. Then $\widetilde{X} \setminus F_{(m)}$ is a finite subset of X_0 and $v_{F_{(m)}} = v$ on $F_{(m)}$. Since $v - v_{F_{(m)}}$ is harmonic on $X_0 \setminus F_{(m)}$ and is equal to 0 on $F_{(m)} \cup \{a_0\}$, we see by the maximum principle that $v_{F_{(m)}} = v$ on X_0, and hence $v_{\partial N} = v$ on X.

(3) It suffices to notice that $(u+v)_{F_{(m)}} = u_{F_{(m)}} + v_{F_{(m)}}$ by Theorem 5.1(1).

(4) follows easily from Theorem 5.1(3). □

Similarly to Theorem 7.2, we have

Theorem 7.4. *Let v be an SHS function and let F be a closed subset of ∂N. Then there exists a measure μ such that $S_\mu \subset F$ and*

$$v_F(x) = \int \tilde{g}_\xi(x) d\mu(\xi)$$

holds for every $x \in X_0$.

Proof. Let $F_{(m)} := \{x \in X_0; d(x, F) \leq 1/m\}$. By Theorem 7.2, there exists a measure μ_m such that $v_{F_{(m)}} = \tilde{G}\mu_m$ and $S_{\mu_m} \subset (F_{(m)})^a$. By Remark 7.1

$$\mu_m(\widetilde{X}) = \Delta v_{F_{(m)}}(a_0) \leq \Delta v_{F_{(1)}}(a_0) < \infty,$$

so that $\{\mu_m(\widetilde{X})\}$ is bounded. There exists a vaguely convergent subsequence of $\{\mu_m\}$. Denote it again by $\{\mu_m\}$ and let μ be its limit. For each $x \in X$, we have by Lemma 7.1

$$v_F(x) = \lim_{m \to \infty} v_{F_{(m)}}(x) = \lim_{m \to \infty} \tilde{G}\mu_m(x) = \tilde{G}\mu(x) \quad \square .$$

Corollary 7.2. $\mu(\widetilde{X}) = \Delta \tilde{G}\mu(a_0) = \Delta v_F(a_0)$.

Similarly to Theorem 7.3, we have

Theorem 7.5. *Let μ be a measure on $\widetilde{X}$ and let F be a closed subset of ∂N. Then $(\tilde{G}\mu)_F = \tilde{G}_F\mu$.*

Proof. Let $F_{(m)} := \{x \in X_0; d(x,F) \le 1/m\}$. Then $\{(\tilde{g}_z)_{F_{(m)}}\}$ converges increasingly to $(\tilde{g}_z)_F$ as $m \to \infty$. By Theorem 7.3 and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}(\tilde{G}\mu)_F = \lim_{m\to\infty}(\tilde{G}\mu)_{F_{(m)}} &= \lim_{m\to\infty}\tilde{G}_{F_{(m)}}\mu \\ &= \lim_{m\to\infty}\int(\tilde{g}_z)_{F_{(m)}}(x)d\mu(z) \\ &= \int(\tilde{g}_z)_F(x)d\mu(z) = \tilde{G}_F\mu. \quad \square\end{aligned}$$

Example 7.1. Let N and a_0 be the same as in Example 3.1 and let $F = \partial N$. Let $\{k_m\}$ be an increasing subsequence of $\{k\}$ such that $F_{(m)} = \{x_k; k \ge k_m\}$. We have $1_{F_{(m)}}(x_k) = R_k/R_{k_m}$ for $k = 0, 1, 2, \cdots, k_m - 1$ and $1_{F_{(m)}}(x_k) = 1$ for $k \ge k_m$. Letting $m \to \infty$, we obtain $1_F(x_k) = R_k/R$ if $R < \infty$ and $1_F = 0$ if $R = \infty$.

Since $\tilde{g}_\xi$ is an HS_0 function and $\partial N = \{\xi\}$, we have $(\tilde{g}_\xi)_{\{\xi\}} = \tilde{g}_\xi$.

8. Classification of boundary points

Similarly to Theorem 5.2, we have

Theorem 8.1. *Let v be a DSHS function, let F be a closed subset of ∂N and let $F_{(m)} := \{x \in X_0; d(x,F) \le 1/m\}$. Then $D(v_{F_{(m)}} - v_F) \to 0$ as $m \to \infty$.*

Proof. For $m < n$, we have $F_{(n)} \subset F_{(m)}$ and

$$0 \le D(v_{F_{(n)}} - v_{F_{(m)}}) = D(v_{F_{(m)}}) - D(v_{F_{(n)}}),$$

by Corollary 5.3. It follows that $\{D(v_{F_{(n)}})\}$ has a limits and that $\{v_{F_{(n)}}\}$ forms a Cauchy sequence in $D(N; a_0)$. Our assertion follows from the fact that $\{v_{F_{(n)}}\}$ converges pointwise to v_F. $\square$

Corollary 8.1. *$v_F \in D(N; a_0)$ for a DSHS function v and a closed subset F of ∂N.*

Theorem 8.2. *Let v be a DSHS function and F be a closed subset of ∂N. Then $(v_F)_F = v_F$ holds.*

Proof. Let $F_{(m)} := \{x \in X_0; d(x,F) \le 1/m\}$ and put $u_n := v_{F_{(n)}} - v_F$. For $m < n$, $u_n \in \mathcal{D}_{F_{(m)}}(u_n)$, since $u_n \in D(N; a_0)$. We have $D((u_n)_{F_{(m)}}) \le D(u_n) \to 0$ as $n \to \infty$ by Theorems 2.1 and 8.1. On the other hand, we have by Theorem 2.2

$$D((u_n)_{F_{(m)}}) = D((v_{F_{(n)}})_{F_{(m)}} - (v_F)_{F_{(m)}}) = D(v_{F_{(n)}} - (v_F)_{F_{(m)}}).$$

Letting $n \to \infty$, wo obtain $D(v_F - (v_F)_{F_{(m)}}) = 0$. Hence $v_F = (v_F)_{F_{(m)}}$ for all m, which implies $v_F = (v_F)_F$. $\square$

Corollary 8.2. $(1_F)_F = 1_F$.

In order to study the relation: $(v_F)_F = v_F$ for an SHS function v and a closed subset F of ∂N, we prepare

Lemma 8.1. *Let u be an HS function, let F be a closed subset of ∂N such that $1_F = 0$ and let $F_{(m)} := \{x \in X_0; d(x, F) \leq 1/m\}$. Then*

$$(u_{F_{(m)}})_A - (u_F)_A \leq u_{F_{(m)}} - u_F \tag{8.1}$$

for every finite subset A of X_0.

Proof. Let A be a finite subset of X_0, let n be a large number such that $n > m$ and $F_{(n)} \cap A = \emptyset$ and let $\{N_k\}(N_k =< X_k, Y_k >)$ be an exhaustion of N. We put $B_m^{(k)} = B_m := F_{(m)} \cap X_k$ and $M := \max\{u(x); x \in A\}$ and prove the following inequality:

$$(u_{B_m^{(k)}})_A - (u_{B_n^{(k)}})_A \leq u_{B_n^{(m)}} - u_{B_n^{(k)}} + M1_{F_{(n)} \cap X_k}. \tag{8.2}$$

For simplicity, we put

$$f := u_{B_m} - u_{B_n},\ h := f - f_A + M1_{B_n}$$

and show that $h \geq 0$ on X.

By Theorems 2.2 and 2.3, $f_A = (u_{B_m})_A - (u_{B_n})_A$ and

$$f_{A \cup B_n} = (u_{B_m})_{A \cup B_n} - (u_{B_n})_{A \cup B_n} \leq u_{B_m} - u_{B_n} = f,$$

so that we have

$$\begin{aligned} h_{A \cup B_n} &= f_{A \cup B_n} - (f_A)_{A \cup B_n} + M(1_{B_n})_{A \cup B_n} \\ &= f_{A \cup B_n} - f_A + M1_{B_n} \\ &\leq f - f_A + M1_{B_n} = h. \end{aligned}$$

By this inequality and Theorem 2.5, we have

$$h(x) \geq h_{A \cup B_n}(x) \geq \beta := \min\{h(z); z \in A \cup B_n\}$$

for all $x \in X_0$. Now we have only to show that $\beta \geq 0$. For $z \in A$, $h(z) = M1_{B_n}(z) \geq 0$. We consider the case where $z \in B_n$. By the relation $f \leq u_{B_m} \leq u \leq M$ on A, we have by Theorem 2.5

$$f_A(x) \leq \max\{f(z); z \in A\} \leq M$$

on X. Since $f \geq 0$ by Theorem 5.1(3), we have for $z \in B_n$

$$h(z) \geq -f_A(z) + M \geq 0,$$

and hence $\beta \geq 0$ and (8.2) holds.

Letting $k \to \infty$ in (8.2), we obtain

$$(u_{F_{(m)}})_A - (u_{F_{(n)}})_A \leq u_{F_{(m)}} - u_{F_{(n)}} + M1_{F_{(n)}}. \tag{8.3}$$

Next letting $n \to \infty$ in (8.3), we have by Lemmas 2.1 and 7.2

$$(u_{F_{(m)}})_A - (u_F)_A \leq u_{F_{(m)}} - u_F + M1_F = u_{F_{(m)}} - u_F,$$

since $1_F = 0$. This completes the proof of (8.1). □

Theorem 8.3. *Let v be an SHS function and let F be a closed subset of ∂N. If $1_F = 0$, then $(v_F)_F = v_F$ holds.*

Proof. Since v_F is an HS_0 function by Lemma 7.2, $(v_F)_F \le v_F$. We show the converse inequality. By Theorem 4.4, v can be decomposed in the form: $v = \tilde{G}\mu + u$, where $\mu(x) = -\Delta v(x)$ on X_0, $\mu(a_0) = 0$ and u is an HS function. We have by Proposition 7.1(3) and Theorem 7.5,

$$v_F = \tilde{G}_F\mu + u_F, \quad (v_F)_F = \sum\nolimits_{z\in X}((\tilde{g}_z)_F)_F\mu(z) + (u_F)_F.$$

Since $\tilde{g}_z$ is DSHS function for $z \in X_0$, we have $((\tilde{g}_z)_F)_F = (\tilde{g}_z)_F$ by Theorem 8.2. Therefore it suffices to show that $u_F \le (u_F)_F$. Let $F_{(m)} := \{x \in X_0; d(x, F) \le 1/m\}$. By Lemma 8.1, we have

$$(u_{F_{(m)}})_A - (u_F)_A \le u_{F_{(m)}} - u_F$$

for every finite subset A of X_0. Let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N and take $A = B_m^{(k)} := F_{(m)} \cap X_k$ as in Lemma 8.1. Letting $k \to \infty$, we have

$$(u_{F_{(m)}})_{B_m^{(k)}} \to (u_{F_{(m)}})_{F_{(m)}} = u_{F_{(m)}}$$

by Theorem 5.1 and $(u_F)_{B_m^{(k)}} \to (u_F)_{F_{(m)}}$. Therefore we have $(u_F)_{F_{(m)}} \ge u_F$. By letting $m \to \infty$, we obtain $(u_F)_F \ge u_F$. □

Definition 8.1. *For each $\xi \in \partial N$, set*

$$\alpha(\xi) := \Delta(\tilde{g}_\xi)_{\{\xi\}}(a_0).$$

Theorem 8.4. *For $\xi \in \partial N$, $\alpha(\xi) = 1$ or $\alpha(\xi) = 0$.*

Proof. For simplicity, we put $\tilde{g} := \tilde{g}_\xi$ and $F := \{\xi\}$. Then $\tilde{g}_F = \alpha(\xi)\tilde{g}$ by Theorem 7.4 and Corollary 7.2. If $1_F = 0$, we have by Theorem 8.3

$$\tilde{g}_F = (\tilde{g}_F)_F = (\alpha(\xi)\tilde{g})_F = \alpha(\xi)^2\tilde{g},$$

so that $\alpha(\xi)(1 - \alpha(\xi)) = 0$, i.e., $\alpha(\xi) = 1$ or $\alpha(\xi) = 0$.

If $1_F > 0$, then $1_F = c\tilde{g}$ with $c > 0$ by Theorem 7.4. By Corollary 8.2,we have

$$\tilde{g}_F = \frac{1}{c}(1_F)_F = \frac{1}{c}1_F = \tilde{g}.$$

Since $\tilde{g}_F = \alpha(\xi)\tilde{g}$, we conclude that $\alpha(\xi) = 1$. □

Corollary 8.3. *According as $\alpha(\xi) = 0$ or 1, $(\tilde{g}_\xi)_{\{\xi\}} = 0$ or $\tilde{g}_\xi$.*

Definition 8.2. *We set*

$$(\partial N)_0 := \{\xi \in \partial N; \alpha(\xi) = 0\},$$

$$(\partial N)_1 := \{\xi \in \partial N; \alpha(\xi) = 1\}.$$

We call an element of $(\partial N)_1$(resp. $(\partial N)_0$) a Kuramochi minimal (resp. non-minimal) (boundary) point.

We prove

Theorem 8.5. $(\partial N)_0$ *is an F_σ-set.*

Proof. Let $m \in Z^+, m \geq 1$. Define δ_m as the set of all $\xi \in \partial N$ with the following property:

$$\alpha_B(\xi) := \Delta(\tilde{g}_\xi)_B(a_0) \leq \frac{1}{2}$$

for every infinite subset B of X_0 such that B^a contains a neighborhood of ξ and

$$B \subset U(\xi, 1/m) := \{z \in \widetilde{X}; d(z, \xi) < \frac{1}{m}\}.$$

First we show $(\partial N)_0 = \cup_{m=1}^{\infty} \delta_m$. Let $\xi \in (\partial N)_0$, let $F := \{\xi\}$ and let $A_{(k)} := \{x \in X_0; d(x, \xi) \leq 1/k\}$ and put $\tilde{g} := \tilde{g}_\xi$. Then $(\tilde{g})_F = \lim_{k\to\infty}(\tilde{g})_{A_{(k)}}$ by Lemma 7.2. Since $(\tilde{g})_F = 0$ by Corollary 8.3, we have

$$\lim_{k\to\infty} \alpha_{A_{(k)}}(\xi) = \lim_{k\to\infty} \Delta(\tilde{g})_{A_{(k)}}(a_0) = \Delta(\tilde{g})_F(a_0) = 0.$$

There exists k_0 such that $\alpha_{A_{(k)}}(\xi) \leq 1/2$ for all $k \geq k_0$. For any infinite subset B of X_0 such that $\xi \in B^a$ and $B \subset U(\xi, 1/k)$, we have $B \subset A_{(k)}$ and

$$\alpha_B(\xi) \leq \alpha_{A_{(k)}}(\xi) \leq 1/2$$

for $k \geq k_0$, so that $\xi \in \delta_k$ for $k \geq k_0$.

Conversely, if $\xi \in \delta_m$, then $\alpha(\xi) \leq \alpha_B(\xi) \leq 1/2$, where $B = A_{(m+1)}$ mentioned in the definition of δ_m. Since $\alpha(\xi) = 0$ or 1, we have $\alpha(\xi) = 0$, so that $\xi \in (\partial N)_0$. Therefore $(\partial N)_0$ is equal to the countable union of δ_m.

Next we prove that δ_m is closed. Let $\xi_j \in \delta_m$ and assume that $\xi_j \to \xi_0$ as $j \to \infty$. Put $\tilde{g}_j := \tilde{g}_{\xi_j}$ and $\tilde{g}_0 := \tilde{g}_{\xi_0}$. Take B in such a way that B^a contains a neighborhood of ξ_0 and $B \subset U(\xi_0, 1/m)$. There is a j_0 such that $\xi_j \in B^a$ for every $j \geq j_0$. Since $\xi_j \in \delta_m$, $\alpha_B(\xi_j) \leq 1/2$ for every $j \geq j_0$. Let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N and put $B_n = B \cap X_n$. Since $(\tilde{g}_0)_{B_n}$ converges increasingly to $(\tilde{g}_0)_B$ as $n \to \infty$ and these functions take value 0 at a_0, we see by Remark 1.1 that $\Delta(\tilde{g}_0)_{B_n}(a_0)$ converges increasingly to $\Delta(\tilde{g}_0)_B(a_0)$. Thus, for any $\epsilon > 0$, there exists a large n such that

$$\Delta(\tilde{g}_0)_B(a_0) - \epsilon < \Delta(\tilde{g}_0)_{B_n}(a_0).$$

Since $\Delta(\tilde{g}_j)_{B_n}(a_0) \le \Delta(\tilde{g}_j)_B(a_0)$, we have

$$\begin{aligned}\frac{1}{2} \ge \limsup_{j\to\infty} \alpha_B(\xi_j) &= \limsup_{j\to\infty} \Delta(\tilde{g}_j)_B(a_0) \\ &\ge \liminf_{j\to\infty} \Delta(\tilde{g}_j)_B(a_0) \\ &\ge \lim_{j\to\infty} \Delta(\tilde{g}_j)_{B_n}(a_0) \quad \text{by Lemma 2.1} \\ &= \Delta(\tilde{g}_0)_{B_n}(a_0).\end{aligned}$$

Therefore, $\Delta(\tilde{g}_0)_B(a_0) < 1/2 + \epsilon$. Since ϵ is arbitrary, we obtain $\alpha_B(\xi_0) = \Delta(\tilde{g}_0)_B(a_0) \le 1/2$, and hence $\xi_0 \in \delta_m$. □

9. Canonical representation

Lemma 9.1. *Let v be an SHS function and let F_n be a sequence of closed subsets of ∂N such that $F_n \subset F_{n+1}$ and $F = \cup_{n=1}^{\infty} F_n$ is closed. If $v_{F_n} = 0$ for each n, then $v_F = 0$.*

Proof. Let $x_0 \in X_0$ and take $\epsilon > 0$. For each F_n, there exists a subset B_n of X_0 such that $x_0 \notin B_n$, B_n^a is a closed neighborhood of F_n and $v_{B_n}(x_0) < \epsilon/2^n$. Then

$$\cup_{n=1}^{\infty} B_n^a \supset \cup_{n=1}^{\infty} F_n = F.$$

There exist $B_1, \cdots, B_q$ such that $B_1^a \cup \cdots \cup B_q^a$ is a closed neighborhood of F. Let $F_{(m)} := \{x \in X_0; d(x, F) \le 1/m\}$. If m is large, $F_{(m)} \subset B := B_1 \cup \cdots \cup B_q$. Then we have by Theorem 5.1(2)

$$v_F \le v_{F_{(m)}} \le v_B \le \sum_{n=1}^{q} v_{B_n},$$

so that $v_F(x_0) \le \epsilon$. Since ϵ is arbitrary, $v_F(x_0) = 0$. Since v_F is harmonic on X_0 and attains its minimum at x_0, we have $v_F = 0$ by the minimum principle. □

Lemma 9.2. *Let v be an SHS function and assume that $(v_F)_F = v_F$ for every closed subset F of $(\partial N)_0$. Then $v_F = 0$ for every closed subset F of $(\partial N)_0$.*

Proof. Let F be a closed subset of $(\partial N)_0$. Then F is equal to the countable union of the sets $\{\delta_m\}$ defined in the proof of Theorem 8.5. Let E be a closed subset of δ_m with diameter less than $1/(2m)$ and let B be a subset of X_0 such that B^a is a closed neighborhood of E and B^a is contained in $1/(2m)$-neighborhood of E. Then B^a is contained in the $(1/m)$-neighborhood of any point of E. Hence $\alpha_B(\xi) \le 1/2$ for every $\xi \in E \subset \delta_m$. On the other hand, $v_E = \tilde{G}\mu$ by Theorem

7.4, where μ is a measure such that $\mu(\widetilde{X} \setminus E) = 0$. Since $(v_E)_E = v_E$ by our assumption, we have by Proposition 7.1 and Remark 1.1

$$\begin{aligned} \mu(E) = \Delta\tilde{G}\mu(a_0) &= \Delta v_E(a_0) = \Delta(v_E)_E(a_0) \\ &= \Delta(\tilde{G}\mu)_E(a_0) \leq \Delta(\tilde{G}\mu)_B(a_0) \\ &= \Delta(\tilde{G}_B)\mu(a_0) = \int \Delta(\tilde{g}_\xi)_B(a_0)d\mu(\xi) \\ &= \int \alpha_B(\xi)d\mu(\xi) \leq \frac{1}{2}\mu(E). \end{aligned}$$

It follows that $\mu(E) = 0$, i.e., $\mu = 0$ and hence $v_E = 0$. Since δ_m can be divided into a finite number of closed sets with diameter less than $1/(2m)$, we see by Lemma 9.1 that $v_{\delta_m} = 0$. Since $F \cap \delta_m$ is closed and $F = \cup_{m=1}^{\infty}(F \cap \delta_m)$, we have $v_F = 0$ by Lemma 9.1. □

Theorem 9.1. *Let v be an SHS function. Then $v_F = 0$ for any closed subset F of $(\partial N)_0$.*

Proof. By Corollary 8.2, $(1_F)_F = 1_F$ for every closed subset F of ∂N, so that $1_F = 0$ for every closed subset F of $(\partial N)_0$ by Lemma 9.2. On account of Theorem 8.3, we have $(v_F)_F = v_F$, and hence $v_F = 0$ by Lemma 9.2. □

Theorem 9.2. *Let v be an SHS function and let F be a closed subset of ∂N. Then there exists a measure μ such that $\mu(\widetilde{X} \setminus (F \cap (\partial N)_1) = 0$ and*

$$v_F(x) = \int_{F \cap (\partial N)_1} \tilde{g}_\xi(x)d\mu(\xi).$$

Proof. Let $F_{(m)} := \{x \in X_0; d(x, F) \leq 1/m\}$ and let μ_m and μ be the same as in the proof of Theorem 7.4. Then $\{\mu_n\}$ converges vaguely to μ, $S_\mu \subset F$ and

$$v_F(x) = \int_F \tilde{g}_\xi(x)d\mu(\xi).$$

It suffices to show that $\mu((\partial N)_0) = 0$.

Let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N. By Theorem 4.3, there exists $\mu_m^{(n)} \in L^+(X)$ such that $S_{\mu_m^{(n)}} \subset F_m^n := F_{(m)} \cap X_n$ and $v_{F_m^n} = \tilde{G}\mu_m^{(n)}$. As in the proof of Theorem 7.2, we may assume that $\{\mu_m^{(n)}\}$ converges vaguely to μ_m as $n \to \infty$.

Let $\epsilon > 0$ be given and fix $x_0 \in X_0$. For δ_k defined in the proof of Theorem 8.5, we have $v_{\delta_k} = 0$ by Theorem 9.1. There exists a closed neighborhood E of δ_k in $\widetilde{X}$ such that $v_B(x_0) < \epsilon$ with $B := E \cap X_0$.

Define $\nu_m^{(n)}$ by $\nu_m^{(n)} = \mu_m^{(n)}$ on B and $\nu_m^{(n)} = 0$ on $X \setminus B$. We may assume that $\{\nu_m^{(n)}\}$ converges vaguely to a measure ν_m such that $S_{\nu_m} \subset (B \cap F_{(m)})^a$ as

$n \to \infty$ and that $\{\nu_m\}$ converges vaguely to a measure ν such that $S_\nu \subset B^a \cap F$ as $m \to \infty$. Since $S_{\nu_m^{(n)}} \subset B$, we have by Theorem 3.2(1)

$$\begin{aligned}\tilde{G}\nu_m^{(n)} = \sum\nolimits_{z\in B} \tilde{g}_z \nu_m^{(n)}(z) &= \sum\nolimits_{z\in B} (\tilde{g}_z)_B \nu_m^{(n)}(z) = (\tilde{G}\nu_m^{(n)})_B \\ &\leq (v_{F_m^n})_B \leq v_B,\end{aligned}$$

since $\tilde{G}\nu_m^{(n)} \leq \tilde{G}\mu_m^{(n)} \leq v$. By letting $n \to \infty$ and then $m \to \infty$, we obtain

$$\tilde{G}\nu \leq v_B \quad \text{on} \quad X_0. \tag{9.1}$$

Now we proceed the proof of $\mu((\partial N)_0) = 0$. By (9.1), we have

$$\int_{\delta_k} \tilde{g}_\xi(x_0) d\nu(\xi) \leq \tilde{G}\nu(x_0) \leq v_B(x_0) < \epsilon.$$

Since ϵ is arbitrary and $\tilde{g}_\xi(x_0) > 0$ on δ_k, we see that $\nu(\delta_k) = 0$. Therefore

$$\nu((\partial N)_0) = \nu(\sum_{k=1}^{\infty} \delta_k) \leq \sum_{k=1}^{\infty} \nu(\delta_k) = 0.$$

Finally we show that $\mu(\delta_k) = 0$ for each k.

For any $\epsilon > 0$, there exists a continuous function f on $\widetilde{X}$ such that $0 \leq f \leq 1$ on $\widetilde{X}$, $f = 1$ on δ_k, $S_f \subset E$ and

$$|\mu(\delta_k) - \int f d\mu| < \epsilon, \quad |\nu(\delta_k) - \int f d\nu| < \epsilon.$$

Since $\mu_m^{(n)} = \nu_m^{(n)}$ on $B = E \cap X_0$, we have

$$\sum\nolimits_{x\in X} f(x)\mu_m^{(n)}(x) = \sum\nolimits_{x\in X} f(x)\nu_m^{(n)}(x).$$

Letting $n \to \infty$ and then $m \to \infty$, we obtain $\int f d\mu = \int f d\nu$. Therefore $|\mu(\delta_k) - \nu(\delta_k)| < \epsilon$. Since ϵ is arbitrary, we have $\mu(\delta_k) = \nu(\delta_k) = 0$. Hence $\mu((\partial N)_0) = 0$. $\square$

Corollary 9.1. *Let u be an HS_0 function and let μ be the associated measure of u. Then*

$$u(x) = \int_{(\partial N)_1} \tilde{g}_\xi(x) d\mu(\xi).$$

Proof. By Proposition 7.1 and Theorem 9.2, we have

$$u(x) = u_{\partial N}(x) = \int_{(\partial N)_1} \tilde{g}_\xi(x) d\mu(\xi). \quad \square$$

We call the measure μ in Corollary 9.1 a canonical measure of u and the representation a *canonical representation*.

Theorem 9.3. *Let v be an SHS function and F be a closed subset of ∂N.*
(1) If E is a closed subset of ∂N and if $F \subset E$, then $(v_F)_E = v_F$.
(2) If B is an infinite subset of X_0 such that B^a is a closed neighborhood of F, then $(v_F)_B = v_F$.

Proof. (1) Let $\xi \in F \cap (\partial N)_1$. By Corollary 8.3 and Proposition 7.1(1),(4), we have

$$\tilde{g}_\xi = (\tilde{g}_\xi)_{\{\xi\}} \leq (\tilde{g}_\xi)_E \leq \tilde{g}_\xi$$

and hence $(\tilde{g}_\xi)_E = \tilde{g}_\xi$. By Theorem 9.2, $v_F = \tilde{G}\mu$ with $S_\mu \subset F$ and $\mu((\partial N)_0) = 0$. By Theorem 7.5

$$\begin{aligned} (v_F)_E = (\tilde{G}\mu)_E &= \tilde{G}_E\mu = \int_{F\cap(\partial N)_1} (\tilde{g}_\xi)_E d\mu(\xi) \\ &= \int_{F\cap(\partial N)_1} \tilde{g}_\xi d\mu(\xi) = \tilde{G}\mu = v_F. \end{aligned}$$

(2) Since v_F is an HS$_0$ function, $(v_F)_B \leq v_F$. Let $F_{(m)} := \{x \in X_0; d(x,F) \leq 1/m\}$. Then $F_{(m)} \subset B$ for large m and we have

$$v_F = \lim_{m\to\infty} v_{F_{(m)}} \leq v_B.$$

In particular, we have $(\tilde{g}_\xi)_F \leq (\tilde{g}_\xi)_B$. We have by Theorem 7.3 and the above observation,

$$\begin{aligned} (v_F)_B = (\tilde{G}\mu)_B &= \tilde{G}_B\mu = \int (\tilde{g}_\xi)_B d\mu(\xi) \\ &\geq \int (\tilde{g}_\xi)_F d\mu(\xi) = (\tilde{G})_F\mu \\ &= (\tilde{G}\mu)_F = (v_F)_F = v_F. \end{aligned}$$

Therefore $(v_F)_B = v_F$. □

10. Minimal functions

Definition 10.1. *Let u be an HS_0 function. We say that u is minimal if $v = cu$ whenever v and $u - v$ are HS_0 functions.*

Lemma 10.1. *Let u be a minimal function. If u is expressed as*

$$u(x) = \int_B \tilde{g}_z(x) d\mu(z)$$

with a Borel subset B of ∂N and a measure μ, then μ is a point measure at some $\xi_0 \in B$ with total mass $c = \Delta u(a_0)$.

Proof. Since $\mu(B) > 0$, there exists a closed subset F_1 of B such that the diameter of F_1 is less than 1 and $\mu(F_1) > 0$. Let F_2 be a closed subset of F_1 with diameter less than 1/2 such that $\mu(F_2) > 0$. In this way, we obtain a sequence $\{F_j\}$ of closed subsets of B such that $\mu(F_j) > 0$, $F_j \subset F_{j-1}$ and the diameter of F_j is less than $1/j$. There exists a point $\xi_0 \in B$ such that $\xi_0 \in F_j$ for all j. Let

$$v := \int_{F_j} \tilde{g}_\xi d\mu(\xi).$$

Then

$$u - v = u - \int_{F_j} \tilde{g}_\xi d\mu(\xi) = \int_{B\setminus F_j} \tilde{g}_\xi d\mu(\xi),$$

so that v and $u-v$ are HS_0 functions by Corollary 7.1. Since u is minimal, there exists $c_j > 0$ such that $u = c_j v = \widetilde{G}\mu_j$, where μ_j is the restriction of $c_j\mu$ to F_j. By Remark 7.1,

$$\mu_j(\widetilde{X}) = \Delta u(a_0) = c.$$

Let μ_0 be the vague limit of a subsequence of $\{\mu_j\}$. Then we see that μ_0 is a point measure at ξ_0 and that $u = \widetilde{G}\mu_0 = c\tilde{g}_{\xi_0}$. Suppose that $\mu \neq \mu_0$. Then there exists a closed subset F of B such that $\xi_0 \notin F$ and $\mu(F) > 0$. By the same reasoning as above, we can find a point $\xi_1 \in B$ such that $u = c\tilde{g}_{\xi_1}$. Therefore, $\tilde{g}_{\xi_0} = \tilde{g}_{\xi_1}$ and hence $\xi_0 = \xi_1$. This is a contradiction. Thus $\mu = \mu_0$. □

Theorem 10.1. *(1) Let u be minimal and let F be a closed subset of ∂N. If $u_F > 0$ in X_0 and $u - u_F$ is an HS_0 function, then there exist a point $\xi_0 \in F \cap (\partial N)_1$ such that $u(x) = c\tilde{g}_{\xi_0}(x)$ with $c = \Delta u(a_0)$.*
(2) Any minimal function is a constant multiple of $\tilde{g}_\xi$ for some $\xi \in (\partial N)_1$.
(3) $\tilde{g}_\xi$ is minimal if and only if $\xi \in (\partial N)_1$.

Proof. (1) Since u_F and $u - u_F$ are HS_0 functions by our assumption, the minimality of u implies $u_F = c'u$ with a constant c'. The condition $u_F > 0$ implies that $c' > 0$. By Theorem 9.2 and Lemma 10.1, we have

$$u = \frac{1}{c'}u_F = \frac{1}{c'}\int_{F\cap(\partial N)_1} \tilde{g}_\xi d\mu(\xi) = c\tilde{g}_{\xi_0}$$

for some point $\xi_0 \in F \cap (\partial N)_1$.

(2) follows from Corollary 9.1 and Lemma 10.1.

(3) Let $\xi \in (\partial N)_1$ and suppose that v and $v' := \tilde{g}_\xi - v$ are HS_0 functions. By Corollary 8.3, $(\tilde{g}_\xi)_F = \tilde{g}_\xi$ with $F = \{\xi\}$. We have

$$v_F + v'_F = (\tilde{g}_\xi)_F = \tilde{g}_\xi = v + v'.$$

From $v_F \leq v$ and $v'_F \leq v'$, it follows that $v_F = v$ and $v'_F = v'$. By Theorem 7.4, $v = c\tilde{g}_\xi$, which implies that $\tilde{g}_\xi$ is minimal.

Conversely, assume that $\tilde{g}_\xi$ is minimal. By (2), there exist $\xi_0 \in (\partial N)_1$ and a constant c such that $\tilde{g}_\xi = c\tilde{g}_{\xi_0}$. Observing that

$$\Delta\tilde{g}_\xi(a_0) = \Delta\tilde{g}_{\xi_0}(a_0) = 1,$$

we see that $c = 1$, so that $\xi = \xi_0 \in (\partial N)_1$. □

11. Uniqueness of canonical representation

For a finite subset A of X_0 and $\xi \in \partial N$, there exists by Theorem 4.3 $\mu_{\xi,A} \in L^+(X)$ such that $(\tilde{g}_\xi)_A = \tilde{G}\mu_{\xi,A}$ and $\mu_{\xi,A}(z) = 0$ on $X \setminus A$.

Lemma 11.1. *For $f \in L_0(X)$ with $f(a_0) = 0$, let*

$$\Phi_f(\xi) := \sum_{z\in X} f(z)\mu_{\xi,A}(z).$$

Then $\Phi_f(\xi)$ is continuous on ∂N.

Proof. Let $\xi_1,\ \xi_2 \in \partial N$. By Theorem 3.1(4) and Theorem 2.5, we have

$$\begin{aligned} |\Phi_f(\xi_1) - \Phi_f(\xi_2)| &= |\sum_{z\in X}(-\sum_{a\in X}[\Delta f(a)]\tilde{g}_z(a))(\mu_{\xi_1,A}(z) - (\mu_{\xi_2,A}(z))| \\ &\leq \sum_{a\in X}|\Delta f(a)||\sum_{z\in X}\tilde{g}_z(a)(\mu_{\xi_1,A}(z) - \mu_{\xi_2,A}(z))| \\ &= \sum_{a\in X}|\Delta f(a)||(\tilde{g}_{\xi_1})_A(a) - (\tilde{g}_{\xi_2})_A(a)| \\ &= \sum_{a\in X}|\Delta f(a)||(\tilde{g}_{\xi_1} - \tilde{g}_{\xi_2})_A(a)| \\ &\leq \max_{a\in A}|\tilde{g}_{\xi_1} - \tilde{g}_{\xi_2}|\sum_{a\in X}|\Delta f(a)|. \end{aligned}$$

By definition, $d(\xi_1,\xi_2) \to 0$ if and only if $|\tilde{g}_{\xi_1}(x) - \tilde{g}_{\xi_2}(x)| \to 0$ for each $x \in X_0$. Thus $\Phi_f(\xi)$ is continuous on ∂N. □

Lemma 11.2. *Let $\{N_n\}(N_n = < X_n, Y_n >)$ be an exhaustion of N and put $\mu_\xi^{(n)} = \mu_{\xi,X_n}$. If $\xi \in (\partial N)_1$, then $\{\mu_\xi^{(n)}\}$ converges vaguely to the unit point measure at ξ as $n \to \infty$.*

Proof. Since $\tilde{G}\mu_\xi^{(n)} = (\tilde{g}_\xi)_{X_n} = \tilde{g}_\xi$ on X_n, we see that $\tilde{G}\mu_\xi^{(n)}(x) \to \tilde{g}_\xi(x)$ as $n \to \infty$ for each $x \in X_0$. Noting that

$$\mu_\xi^{(n)}(X) = \Delta(\tilde{g}_\xi)_{X_n}(a_0) = \Delta\tilde{g}_\xi(a_0) = 1,$$

we see that there exists a subsequence of $\{\mu_\xi^{(n)}\}$ which converges vaguely to a measure μ_0. Denote by $\{\mu_\xi^{(n)}\}$ the subsequence again. Since $\tilde{g}_\xi(x)$ is a continuous function of ξ on $\widetilde{X}$, we have by Lemma 7.1 $\tilde{G}\mu_\xi^{(n)}(x) \to \tilde{G}\mu_0(x)$ as $n \to \infty$ for each $x \in X_0$. Thus $\tilde{g}_\xi = \tilde{G}\mu_0$. By Lemma 10.1 and Theorem 10.1(3), there exists a point $\xi_0 \in (\partial N)_1$ such that $\tilde{G}\mu_0 = \tilde{g}_{\xi_0}$, so that $\xi = \xi_0$. Thus μ_0 is the unit point measure at ξ. Furthermore, for any subsequence of $\{\mu_\xi^{(n)}\}$ which converges vaguely to a measure ν, we see by the above argument that $\nu = \mu_0$, and hence we conclude that $\{\mu_\xi^{(n)}\}$ converges vaguely to μ_0. □

Theorem 11.1. *Let u be an HS_0 function. If there exist measures μ and ν on ∂N such that $\mu((\partial N)_0) = \nu((\partial N)_0) = 0$ and $u = \tilde{G}\mu = \tilde{G}\nu$, then $\mu = \nu$. Namely, the canonical representation of an HS_0 function is unique.*

Proof. Let $\{\mu_\xi^{(n)}\}$ be the same as in Lemma 11.2. For μ, we define a sequence $\{\mu_n\}$ of functions in $L^+(X)$ by

$$\mu_n(z) = \int \mu_\xi^{(n)}(z)d\mu(\xi)$$

for each $z \in X_0$. This is possible because $\mu_\xi^{(n)}(z) = \Phi_{\varepsilon_z}(\xi)$ is a continuous function of ξ on ∂N for every $z \in X_0$ by Lemma 11.1. Similarly, we define $\{\nu_n\}$ for ν. We have by Theorem 7.3

$$\begin{aligned}\tilde{G}\mu_n(x) &= \sum\nolimits_{x\in X} \tilde{g}_z(x)\mu_n(z) = \sum\nolimits_{x\in X} \tilde{g}_z(x)\int \mu_\xi^{(n)}(z)d\mu(\xi) \\ &= \int \sum\nolimits_{x\in X} \tilde{g}_z(x)\mu_\xi^{(n)}(z)d\mu(\xi) = \int \tilde{G}\mu_\xi^{(n)}(x)d\mu(\xi) \\ &= \int (\tilde{g}_\xi)_{X_n} d\mu(\xi) = (\tilde{G}\mu)_{X_n}.\end{aligned}$$

Similarly, $\tilde{G}\nu_n = (\tilde{G}\nu)_{X_n}$, and hence $\tilde{G}\mu_n = \tilde{G}\nu_n$. Thus $\mu_n = \nu_n$ by Lemma 3.2. Let f be a continuous function on $\widetilde{X}$. By Lemma 11.2, $f_n(\xi) := \int f d\mu_\xi^{(n)}$ converges to $f(\xi)$ as $n \to \infty$. Observing that

$$\mu_\xi^{(n)}(X) = \Delta\tilde{G}\mu_\xi^{(n)}(a_0) = \Delta(\tilde{g}_\xi)_{X_n}(a_0) \leq \Delta\tilde{g}_\xi(a_0) = 1,$$

we have

$$|f_n(\xi)| \leq c\mu_\xi^{(n)}(X) \leq c \quad \text{with} \quad c := \max\{|f(z)|; z \in \widetilde{X}\} < \infty.$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \int f d\mu_n = \lim_{n\to\infty} \int (\int f d\mu_\xi^{(n)}) d\mu(\xi) = \lim_{n\to\infty} \int f_n(\xi)d\mu(\xi) = \int f d\mu.$$

Thus μ_n converges vaguely to μ. Similarly, ν_n converges vaguely to ν and hence $\mu = \nu$. □

By Theorem 9.2, we obtain

Corollary 11.1. *Let v be an SHS function and let F be a closed subset of ∂N. The canonical measure for v_F is supported by F.*

References

1. N. Bourbaki, Intégration, Chapitres 1, 2, 3 et 4, Hermann, Paris 1965.
2. C. Constantinescu and A. Cornea, Ideal Ränder Riemannscher Flächen, Berlin-Göttingen-Heidelberg, 1963.
3. R. J. Duffin, Discrete potential theory, Duke math. J. **20**(1953), 233–251.
4. T. Kayano and M. Yamasaki, Boundary limit of discrete Dirichlet potentials, Hiroshima Math. J. **14**(1984), 401–406.
5. R. S. Martin, Minimal positive harmonic function, Trans. Amer. Math. Soc. **49**(1941), 137–172.
6. A. Murakami and M. Yamasaki, Extremal problems with respect to ideal boundary components of an infinite network, Hiroshima Math. J. **19**(1989), 77–87.

7. A. Murakami, Kuramochi boundaries of infinite networks and extremal problems, Hiroshima Math. J. **24**(1994), 243–256.
8. A. Murakami, M. Yamasaki and Y. Yone-e, Some properties of reproducing kernels on an infinite network, Mem. Fac. Sci. Shimane Univ. **28**(1994), 1–8.
9. T. Nakamura and M. Yamasaki, Generalized extremal length of an infinite network, Hiroshima Math. J. **6**(1976), 95–111.
10. M. Ohtsuka, An elementary introduction of Kuramochi boundary, J. Sci. Hiroshima Univ. Ser. A-I Math. **28**(1964), 271–299.
11. P. Soardi, Potential theory on infinite networks, Lecture Notes in Mathematics 1590, Springer-Verlag, 1994.
12. M. Yamasaki: Extremum problems on an infinite network, Hiroshima Math. J. **5**(1975), 223–250.
13. M. Yamasaki, Discrete potentials on an infinite network, Mem. Fac. Shimane Univ. **13**(1979), 31–44.
14. M. Yamasaki, Ideal boundary limit of discrete Dirichlet functions, Hiroshima Math. J. **16**(1986), 353–360.

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