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FUNCTIONAL FREENESS FOR THE BERMAN CLASS $K_{m,n}$ OF OCKHAM ALGEBRAS

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Abstract. In this paper we show that an algebra $\Omega(m,n)$ is functionally free for the Berman class $K_{m,n}$ of Ockham algebras, that is, for any two polynomials f and g, they are identically equal in $K_{m,n}$ if and only if $f = g$ holds in $\Omega(m,n)$. This result can be applied to the well-known algebras, e.g., Boolean, de Morgan, Kleene, Stone, Bunge algebras, and so on.

1. Introduction

It is well known that, in order to show whether two polynomials f and g are identically equal in the class of Boolean algebras, we only calculate the values of polynomials in the typical Boolean algebra $2 = \{0,1\}$. If their values are always identical then they are equal as polynomials otherwise not. The property is called a functional freeness of Boolean algebras. There are results about the properties of other algebras, e.g., de Morgan, Kleene algebras ([1],[2]). The classes of these algebras are subvarieties of the Berman class $K_{m,n}$ of Ockham algebras. In this paper we think about the functional freeness of the algebras in the Berman class $K_{m,n}$ and show that the algebra $\Omega(m,n)$ defined below is functional free for the Berman class $K_{m,n}$. From the result we can deduce the properties of the other algebras (e.g., Boolean, de Morgan, Kleene, Stone, Bunge algebras) without difficulty.

2. Algebras in $K_{m,n}$

We shall define algebras in the Berman class $K_{m,n}$ of Ockham algebras. Let m and n be intergers such that $m \geq 1$ and $n \geq 0$. An algebraic structure $L = (L; \wedge, \vee, N, 0, 1)$ of type $(2,2,1,0,0)$ is called an Ockham algebra when

(1) $(L; \wedge, \vee, N, 0, 1)$ is a bounded distributive lattice;

(2) $N : L \to L$ is a map satisfying the following conditions

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(c1) $N0 = 1, N1 = 0$

(c2) $N(x \wedge y) = Nx \vee Ny,\ N(x \vee y) = Nx \wedge Ny$

An algebra in the Berman class $K_{m,n}$ is the Ockham algebra satisfying the condition

(c3) $N^{2m+n}x = N^n x$,

where $N^n x$ is defined recursively as $N^0 x = x, N^{n+1}x = N(N^n x)$.

We have many examples of the algebras in the Berman class. We list some familiar algebras which form the subvarieties of the Berman class $K_{m,n}$.

(a) $K_{1,0}$: It is the class of de Morgan algebras.

(b) $K_{1,0}$ with the condition $x \wedge Nx = 0$: This is the class of Boolean algebras.

(c) $K_{1,0}$ with the condition $x \wedge Nx \leq y \vee Ny$: The class of Kleene algebras [2].

(d) $K_{1,1}$ with the condition $x \wedge Nx = 0$: The class of Stone algebras [3].

(e) $K_{1,1}$ with the condition $x \vee Nx = 1$: The class of Bunge algebras [9].

Now we define an algebraic structure $\Omega(m, n)$ which is in the Berman class $K_{m,n}$. The algebra plays an important role to prove the functional freeness for the class of algebras.

For brevity we put $k = 2m + n$. Let $\Omega(m, n) = \{(x_1, x_2, ..., x_k) | x_i \in \{0, 1\}\}$. We denote an element of $\Omega(m, n)$ by x and the i-th factor by $(x)_i = x_i$. For $x, y \in \Omega(m, n)$, we define $x = y$ if every factor of the elements is identical, that is, $(x)_i = (y)_i$ hence $x_i = y_i$ for every i $(1 \leq i \leq k)$. We introduce the operations $\wedge, \vee$, and N in the set $\Omega(m, n)$. If no confusion arises then we denote $\Omega(m, n)$ simply by Ω.

For every $x = (x_i)_i, y = (y_i)_i \in \Omega$, we define

$$(x \wedge y)_i = \begin{cases} \min\{x_i, y_i\} = x_i \cdot y_i & \text{if } i \text{ is odd} \\ \max\{x_i, y_i\} = x_i + y_i - x_i \cdot y_i & \text{if } i \text{ is even} \end{cases}$$

$$(x \vee y)_i = \begin{cases} \max\{x_i, y_i\} = x_i + y_i - x_i \cdot y_i & \text{if } i \text{ is odd} \\ \min\{x_i, y_i\} = x_i \cdot y_i & \text{if } i \text{ is even} \end{cases}$$

$Nx = (x_2, x_3, ..., x_k, x_{n+1})$, that is,

$$(Nx)_i = \begin{cases} x_{i+1} & \text{if } i \neq k \\ x_{n+1} & \text{if } i = k \end{cases}$$

We set the special elements $0 = (0, 1, 0, 1, ...)$ and $1 = (1, 0, 1, 0, ...)$.

Clearly the structure $\Omega = (\Omega; \wedge, \vee, N, 0, 1)$ is a bounded lattice.

Lemma 1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

$N(x \wedge y) = Nx \vee Ny$

$N(x \vee y) = Nx \wedge Ny$

Proof. We show only the first case. The other cases can be proved similarly.

If i is odd, then the left-hand side is
$(x \wedge (y \vee z))_i$
$= \min\{x_i, \max\{y_i, z_i\}\}$
$= x_i \cdot (y_i + z_i - y_i \cdot z_i)$
$= x_i \cdot y_i + x_i \cdot z_i - x_i \cdot y_i \cdot z_i.$

On the right-hand side is

$$\begin{aligned} &((x \wedge y) \vee (x \wedge z))_i \\ &= \max\{\min\{x_i, y_i\}, \min\{x_i, z_i\}\} \\ &= \max\{x_i \cdot y_i, x_i \cdot z_i\} \\ &= x_i \cdot y_i + x_i \cdot z_i - (x_i)^2 \cdot y_i \cdot z_i, \text{ since } (x_i)^2 = x_i, \\ &= x_i \cdot y_i + x_i \cdot z_i - x_i \cdot y_i \cdot z_i. \end{aligned}$$

We can also show the equality in case of i being even. Therefore we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

The above indicates that the structure $\Omega = (\Omega; \wedge, \vee, N, 0, 1)$ is the Ockham algebra.

It is neccessary to show that the structure Ω is in the Berman class $K_{m,n}$. Before doing so, we think about the i-th factor of $N^p x$ for every integer $p \geq 1$. To see each factor of the element $N^p x$, when x is denoted by $(x_1, x_2, ..., x_k)$, we consider an infinite sequence of factors of x:

$x_1, x_2, ..., x_k, x_{k+1}(= x_{n+1}), x_{k+2}(= x_{n+2}), ..., x_{k+2m}(= x_k), x_{k+2m+1}(= x_{n+1}),$
$x_{k+2m+2}(= x_{n+2}), ..., x_{k+2m+2m}(= x_k), x_{k+2m+2m+1}(= x_{n+1}), ...$

It follows that $(N^p x)_i = x_{i+p}$ for every i. In general the j-th term of the sequence is obtained as follows. When j is denoted as $2m \cdot \alpha + \beta$ $(0 \leq \alpha, 0 \leq \beta < 2m)$, the j-th term x_j is x_β. Using the fact, we can show the next lemma.

Lemma 2. *For every $x \in \Omega$ and $i \leq k$, we have $N^k x = N^n x$*

Proof. It is sufficient to show that $(N^k x)_i = (N^n x)_i$ for every $i \leq k$. By the argument above, we have that $(N^k x)_i = x_{i+k}$ and $(N^n x)_i = x_{i+n}$. If we denote $i + n = 2m \cdot a + b$ $(0 \leq a, 0 \leq b < 2m)$, since $i + k = 2m \cdot (a + 1) + b$, then we have $(N^k x)_i = x_{i+k} = x_b = x_{i+n} = (N^n x)_i$. This means that $(N^k x)_i = (N^n x)_i$ and hence that $N^k x = N^n x$.

Consequently we can conclude that the structure $\Omega = (\Omega; \wedge, \vee, N, 0, 1)$ is in the Berman class $K_{m,n}$.

3. Functional freeness

In this section we show that the algebra Ω is functionally free for the Berman class $K_{m,n}$ of Ockham algebras. In general, an algebra A is said to be functionally free for a non-empty class C of algebras provided that the following condition is satisfied: any two polynomials are identically equal in A iff they are identically equal in each algebra in C. For example, (1) two-element Boolean algebra $2 = \{0, 1\}$ is functionally free for the class B of all Boolean algebras, (2) three-element Kleene algebra $3 = \{0, a, 1\}$ is so for the class K of all Kleene algebras, and (3) four-element de Morgan algebra $M = \{0, a, b, 1\}$ is so for the class M of all de Morgan algebras.

We define polynomials before proving the functional freeness of Ω.

Let $V = \{p_1, p_2, ...\}$ be the set of varibles. We define polynomials as follows.

1. Every variable $p_n \in V$ is a polynomial;
2. If f and g are polynomials, then so are $f \wedge g$, $f \vee g$, and Nf.

Let L be an arbitrary algebra. The map $v : V \to L$ is called a valuation on L. The valuation v is extended uniquely to v^* of all the polynomials as follows; For any polynomials f and g,

(v1) $v^*(p_n) = v(p_n)$ for every $p_n \in V$
(v2) $v^*(f \wedge g) = v^*(f) \wedge v^*(g)$
(v3) $v^*(f \vee g) = v^*(f) \vee v^*(g)$
(v4) $v^*(Nf) = N(v^*(f))$

Hence the value $v^*(f)$ of the polynomial f is determined by the values of p_n which are components of f. We note that the symbols $\wedge$, $\vee$, and N of the right-hand side of the equations are in L. If no confusion arises we denote v^* by v simply.

We say that f and g are identically equal in L (or simply $f = g$ in L) if $v^*(f) = v^*(g)$ for every valuation v on L. We also say that f and g are identically equal in the class C of algebras (or simply $f = g$ in C) when $f = g$ holds in every algebra L in C. In the following, we shall show that $f = g$ in $K_{m,n}$ iff $f = g$ in Ω. Therefore, to investigate whether $f = g$ holds or not in the class $K_{m,n}$ of Ockham algebras, it is sufficient only to calculate the values $v^*(f)$ and $v^*(g)$ for all valuations v on Ω.

Proposition 1. *Let D be any bounded distributive lattice and $a, b \in D$. If $a \neq b$, then there is a prime filter P of D such that $a \in P$ but $b \notin P$.*

Proof. This is a well-known fact about distributive lattices. Hence we omit the proof. See [5].

In general if there is a partition of a set then we can introduce an equivalence relation on it. Let P be a prime filter of $L \in K_{m,n}$. The set L can be divided into 2^k subsets by P as follows:

$$L_{111...1} = \{x | x \in P, Nx \in P, N^2x \in P, ..., N^{k-1}x \in P\}$$
$$L_{101...1} = \{x | x \in P, Nx \notin P, N^2x \in P, ..., N^{k-1}x \in P\}$$
$$...$$
$$L_{000...0} = \{x | x \notin P, Nx \notin P, N^2x \notin P, ..., N^{k-1}x \notin P\}$$

Thus we can define an equivalence relation $\sim_P$ on L as

$$\sim_P \ni (x, y) \Longleftrightarrow \exists L_{s_1 s_2 ... s_k}(x, y \in L_{s_1 s_2 ... s_k}) \text{ , where } s_i \in \{0, 1\}.$$

This means that

$(x, y) \in \sim_P$ iff $\forall i(N^i x \in P \Leftrightarrow N^i y \in P)$.

We say $\sim_P$ an induced equivalence relation by P. For that relation we can show the next lemma.

Lemma 3. *If P is a prime filter of L, then the induced relation $\sim_P$ by P is a congruent relation on L.*

Proof. We have to prove that for any $(x, y), (a, b) \in \sim_P$

(1) $(x \wedge a, y \wedge b) \in \sim_P$;

(2) $(x \vee a, y \vee b) \in \sim_P$;

(3) $(Nx, Ny) \in \sim_P$.

From the fact $N^k x = N^n x$, it is clear that the condition (3) holds. We only show the case of (1).

We simply denote an element $x \in L$ as a sequence of 0 and 1 as follows: $x = x_1 x_2 x_3 \ldots x_k$, where x_i is deined by

$$x_k = \begin{cases} 1 & \text{if } N^i x \in P \\ 0 & \text{if } N^i x \notin P \end{cases}$$

By definition of $\sim_P$, we have

$(x, y) \in \sim_P$ iff $\forall i(x_i = y_i)$.

Hence it is sufficient to show that $\forall i((x \wedge a)_i = (y \wedge b)_i)$ when $x_i = y_i$ and $a_i = b_i$ for all i.

Since P is the prime filter, we have that

$(x \wedge y)_i = \min\{x_i, y_i\}$ if i is even

$(x \wedge y)_i = \max\{x_i, y_i\}$ if i is odd.

Thus if i is even then it follows that

$(x \wedge a)_i = \min\{x_i, a_i\} = \min\{y_i, b_i\} = (y \wedge b)_i$.

In case of i odd, we also obtain that

$(x \wedge a)_i = \max\{x_i, a_i\} = \max\{y_i, b_i\} = (y \wedge b)_i$.

Therefore in either case we can conclude that $\forall i((x \wedge a)_i = (y \wedge b)_i)$, that is, $(x \wedge a, y \wedge b) \in \sim_P$.

The other case (2) can be proved similarly. This means that $\sim_P$ is the congruence relation on L.

When P is the prime filter of L, we define $L/_{\sim_P} = \{[x] \mid x \in L\}$ and $[x] = \{y \in L \mid x \sim_P y\}$. Since the relation $\sim_P$ is congruent on L, we can consistently define the oparations $\wedge, \vee$, and N on $L/_{\sim_P}$:

$[x] \wedge [y] = [x \wedge y]$

$[x] \vee [y] = [x \wedge y]$

$N[x] = [Nx]$

It is easy to show the next theorem.

Theorem 1. *(1) The structure* $L/_{\sim_P} = (L/_{\sim_P}; \wedge, \vee, N, [0], [1])$ *is in the Berman class.*

(2) The map $\eta : L \to L/_{\sim_P}$ *defined by* $\eta(x) = [x]$ *is a homomorphism.*

Lemma 4. *The map* $\xi : L/_{\sim_P} \to \Omega$ *is an embedding, where* ξ *is defined by* $\xi([x]) = (s_1, s_2, .., s_k)$ *if* $x \in L_{s_1 s_2 \ldots s_k}$.

Proof. It is clear that ξ is well-defined and an injection. We only show that ξ is a homomorphism, that is,

$$\xi([x] \wedge [y]) = \xi([x]) \wedge \xi([y])$$
$$\xi([x] \vee [y]) = \xi([x]) \vee \xi([y])$$
$$\xi(N[x]) = N(\xi([x])).$$

Since P is the prime filter, it follows that

$x \wedge y \in P$ iff $x \in P$ and $y \in P$,

$N(x \wedge y) \in P$ iff $Nx \in P$ or $Ny \in P$,

Hence we have $\xi([x] \wedge [y]) = \xi([x]) \wedge \xi([y])$. The other cases are proved similarly.

Theorem 2. Ω *is functionally free for the Berman class* $K_{m,n}$ *of Ockham algebras, that is,* $f = g$ *in* $K_{m,n}$ *if and only if* $f = g$ *in* Ω.

Proof. It is clear that a equation $f = g$ holds for polynomials f and g in $K_{m,n}$ then it holds in Ω. To prove the converse we suppose that $f = g$ does not hold in $K_{m,n}$. It is sufficient to indicate the existence of some algebra in $K_{m,n}$ and a valuation τ on it such that $\tau(f) \neq \tau(g)$.

By definition there are algebra $L \in K_{m,n}$ and a valuation $v : V \to L$ such that $v^*(f) \neq v^*(g)$. By Proposition 1, there is a prime filter P of L such that $v^*(f) \in P$ but $v^*(g) \notin P$. We devide L into 2^k subsets by use of P and take the congruent relation $\sim_P$ induced by P. That is, for every $x, y \in L$,

$$x \sim_P y \iff \exists L_{s_1 s_2 \ldots s_k} \text{ such that } x, y \in L_{s_1 s_2 .. s_k}.$$

We define a valuation $\tau : V \to \Omega$ by $\tau = \xi \circ \eta \circ v$, that is, $\tau(p_n) = \xi([v(p_n)])$.

It is clear from definition that for each polynomial h,

$$\tau^*(h) = \xi([v^*(h)]).$$

Since $v^*(f) \in P$ but $v^*(g) \notin P$, we have $[v^*(f)] \neq [v^*(g)]$. Since ξ is injective, it follows that $\xi([v^*(f)]) \neq \xi([v^*(g)])$. This means that $\tau^*(f) \neq \tau^*(g)$.

Thus the theorem can be proved completely.

References

1. A. Bialynicki-Birula, *Remarks on quasi-Boolean algebras*, Bull. Ac. Pol. Sc, **5** (1957), 615-619.
2. A. Bialynicki-Birula and H. Rasiowa, *On constructible falsity in the constructive logic with strong negation*, Colloquium Mathematicum, **6** (1958), 287-310.
3. T.S. Blyth and J.C. Varlet, *Ockham algebras*, Oxford Sicence Publications, 1994.

4. M. Dunn, *Relevance logic and entailment*, in Handbook of Philos. Logic **3**, 117-224, D.Gabbay and F. Guenthner eds., D. Reidel, 1987.
5. G.Gratzer, *Lattice theory : first concepts and distributive lattices*, Freeman, San Francisco, 1971.
6. M.Kondo, *Completeness theorem of Kleene logic*, Mem. Fac. Sci. Shimane univ., **29** (1995), 77-87.
7. M.Kondo, *Characterization theorem of 4-valued DeMorgan logic*, submitted.
8. J.Lukasiewicz, *On three-valued logic*, in Polish Logic, McCall, ed., Clarendon Press, Oxford, 1967.
9. J. P. Marquis, *Approximation and logic*, Notre Dame J. of Formal Logic, **32** (1992), 184-196.
10. H. Rasiowa, *An Algebraic Approch to Non-Classical Logics*, North-Holland, Amsterdam, 1974.

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