# SET CONTAINMENT CHARACTERIZATION WITH STRICT AND WEAK QUASICONVEX INEQUALITIES

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Abstract. Dual characterizations of the containment of a convex set, defined by infinite quasiconvex constraints, in an evenly convex set, and in a reverse convex set, defined by infinite quasiconvex constraints, are provided. Notions of quasiconjugate for quasiconvex functions, $\lambda$-quasiconjugate and $\lambda$-semiconjugate, play important roles to derive the characterizations of the set containments.

## 1. Introduction

The set containment problem consists of characterizing the inclusion $A \subset B$, where $A = \{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq 0\}$ and $B = \{x \in \mathbb{R}^n \mid \forall j \in J, h_j(x) \leq 0\}$. Motivated by general non-polyhedral knowledge-based data classification, the set containment characterization have been studied by many researchers, see [1, 2, 4, 5, 9]. The first characterizations were given by Mangasarian [5] for linear systems and for systems involving differentiable convex functions, with $I$ and $J$ finite. These dual characterizations are provided in terms of Farkas' Lemma and the duality theorems of convex programming problems. Jeyakumar [4] established the set containment characterization with $I$ an arbitrary set and $J$ a finite set, assuming the convexity of $f_i$ for each $i \in I$, and the linearity (or the concavity) of $h_j$ for each $j \in J$, so that $A$ is a closed convex set and $B$ is a closed convex set (or a reverse convex set, respectively). These dual characterizations are provided in terms of the epigraph of the Fenchel conjugate of a convex function. Also, Goberna and Rodríguez [2] provided characterizations of the set containment for linear systems containing strict inequalities and weak inequalities as well as equalities. Furthermore, Goberna, Jeyakumar and Dinh [1] characterized set containments with convex inequalities which can be either weak or strict. These dual characterizations are also provided by the Fenchel conjugate.

It is well known that the Fenchel conjugate plays very important roles to consider dual problems of convex minimization problems. Similar researches of conjugates of quasiconvex functions have been studied. But the epigraph of a quasiconvex function is no longer convex. This causes a fundamental difference between convex and quasiconvex duality. For general quasiconvex functions, we

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have to use extra-parameters to obtain dual representations. For example, the $\lambda$-quasiconjugate ($\lambda \in \mathbb{R}$), by Greenberg and Pierskalla [3], has an extra-parameter, and plays an important role in quasiconvex optimization and in the theory of surrogate duality corresponding to that of the Fenchel conjugate in convex optimization and Lagrangian duality. Singer [7, 8] introduced the $\lambda$-semiconjugate which also has an extra-parameter, and studied the level set of the $\lambda$-semiconjugate and quasiconvex optimization. If we want to avoid the extra-parameter, then we often need to restrict the class of quasiconvex functions. Thach [10, 11] established two dualities without the extra-parameter for a general quasiconvex minimization (maximization) problem by using concepts of $H$-quasiconjugate and $R$-quasiconjugate.

More recently, Suzuki and Kuroiwa [9] established the set containment characterization with $I$ a finite set and $J$ an arbitrary set, assuming the quasiconvexity of $f_i$ for each $i \in I$, the linearity (or the quasiconcavity) of $h_j$ for each $j \in J$, and inequalities in $A$ are strict and in $B$ are strict (or weak, respectively). These dual characterizations are provided in terms of level sets of $H$-quasiconjugate and $R$-quasiconjugate functions. Furthermore, we studied some properties of $H$-quasiconjugate.

In this paper, we show set containment characterizations, assuming that all $f_i$ are quasiconvex, all $h_j$ are linear, $I$ and $J$ are possibly infinite, and the inequality in $A$ and $B$ can be either weak or strict. Furthermore, we consider a reverse convex system (i.e., all $f_i$ are quasiconvex and all $h_j$ are quasiconcave), containing both weak and strict inequalities. These dual characterizations are provided in terms of level sets of $\lambda$-quasiconjugate and $\lambda$-semiconjugate, especially 1, $-1$-quasiconjugate, 1-semiconjugate.

## 2. Notation and Preliminaries

Throughout this paper, let $f$ be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Remember that $f$ is said to be quasiconvex if for all $x_1$, $x_2 \in \mathbb{R}^n$ and $\alpha \in (0, 1)$,

$$f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Define

$$L(f, \diamond, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. Symbol $\diamond$ means any binary relation on $\mathbb{R}$. Then $f$ is quasiconvex if and only if for any $\alpha \in \mathbb{R}$,

$$L(f, \leq, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$$

is a convex set, or equivalently, for any $\alpha \in \mathbb{R}$,

$$L(f, <, \alpha) = \{x \in \mathbb{R}^n \mid f(x) < \alpha\}$$

is a convex set. We know that any convex function is quasiconvex. The converse is not true.

Given a set $S \subset \mathbb{R}^n$, we shall denote by $\mathrm{int}S$, $\mathrm{cl}S$ and $\mathrm{co}S$ the interior, the closure and the convex hull generated by $S$ respectively.

The following Proposition, which concerns the closure of the intersection of a family of convex sets, plays an important role in this paper.

**Proposition 1.** Let $I$ be an arbitrary set, and $S_i$ be a convex subset of $\mathbb{R}^n$ for each $i \in I$. If $\mathrm{int} \cap_{i\in I} S_i$ is nonempty, then $\mathrm{cl} \cap_{i\in I} S_i = \cap_{i\in I} \mathrm{cl} S_i$.

*Proof.* We may assume that $x \in \cap_{i\in I} \mathrm{cl} S_i$. Since $\mathrm{int} \cap_{i\in I} S_i \neq \emptyset$, there exists $z \in \mathrm{int} \cap_{i\in I} S_i$. Then for each $i \in I$, $\{(1-\alpha)x + \alpha z \mid \alpha \in (0,1]\} \subset \mathrm{int} S_i$, because $S_i$ is convex and $z \in \mathrm{int} S_i$. Therefore $\{(1-\alpha)x + \alpha z \mid \alpha \in (0,1]\} \subset \cap_{i\in I} S_i$, i.e., $x \in \mathrm{cl} \bigcap_{i\in I} S_i$. The converse is clear. $\square$

A subset $S$ of $\mathbb{R}^n$ is said to be evenly convex if it is the intersection of some family of open halfspaces. Clearly, any open convex set and any closed convex set are evenly convex, and every evenly convex set is convex. A subset $S$ of $\mathbb{R}^n$ is said to be $H$-evenly convex if it is the intersection of some family of open halfspaces, and each open halfspace contains 0. It is clear that a nonempty set $S$ is $H$-evenly convex if and only if $S$ is evenly convex and contains 0. Clearly, the whole space is evenly convex and $H$-evenly convex. Also, we define the empty set is evenly convex and $H$-evenly convex by convention. The evenly convex hull of $S$, denoted by $\mathrm{eco}S$, is the smallest evenly convex set which contains $S$. The $H$-evenly convex hull of $S$, denoted by $\mathrm{Heco}S$, is the smallest $H$-evenly convex set which contains $S$. Note that $\mathrm{co}S \subset \mathrm{eco}S \subset \mathrm{cl\ co}S$ and these differences are slight because $\mathrm{cl\ co}S = \mathrm{cl\ eco}S$. Moreover if $S$ is nonempty, then $\mathrm{Heco}S = \mathrm{eco}(S \cup \{0\})$.

Next, we introduce some notions that are used in the latter half of this paper.

**Definition 1.** Let $S$ be a nonempty subset of $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$. We define polar sets as follows.

$$S^{*(<,\alpha)} = \{v \in \mathbb{R}^n \mid \forall x \in S, \langle v, x\rangle < \alpha\},$$
$$S^{*(\leq,\alpha)} = \{v \in \mathbb{R}^n \mid \forall x \in S, \langle v, x\rangle \leq \alpha\}.$$

Clearly, if $\alpha > 0$ then $S^{*(<,\alpha)}$ is $H$-evenly convex, $(S^{*(<,\alpha)})^{*(<,\alpha)}$ is $\mathrm{Heco}S$, $S^{*(\leq,\alpha)}$ is closed $H$-evenly convex, and $(S^{*(\leq,\alpha)})^{*(\leq,\alpha)}$ is $\mathrm{clHeco}S$. Moreover, for all $\alpha \in \mathbb{R}$,

$$\left(\bigcup_{i\in I} S_i\right)^{*(\leq,\alpha)} = \bigcap_{i\in I}\left(S_i^{*(\leq,\alpha)}\right) \text{ and } \left(\bigcup_{i\in I} S_i\right)^{*(<,\alpha)} = \bigcap_{i\in I}\left(S_i^{*(<,\alpha)}\right).$$

Also, we introduce following two Propositions without proof.

**Proposition 2.** Let $I$ be an arbitrary set, $S_i$ be a nonempty $H$-evenly convex subset of $\mathbb{R}^n$ for each $i \in I$, and $\alpha > 0$.

Then,

$$\left(\bigcap_{i\in I} S_i\right)^{*(<,\alpha)} = \mathrm{Heco} \bigcup_{i\in I}\left(S_i^{*(<,\alpha)}\right),$$

furthermore, if $S_i$ is closed for each $i \in I$, then

$$\left(\bigcap_{i\in I} S_i\right)^{*(\leq,\alpha)} = \mathrm{clHeco}\bigcup_{i\in I}\left(S_i^{*(\leq,\alpha)}\right).$$

**Proposition 3.** Let $S$ be a nonempty subset of $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then, following statements hold.
(i) $(\mathrm{cl\ co}S)^{*(\leq,\alpha)} = S^{*(\leq,\alpha)}$ and $(\mathrm{eco}S)^{*(<,\alpha)} = S^{*(<,\alpha)}$,
(ii) if $\alpha > 0$ then $(\mathrm{clHeco}S)^{*(\leq,\alpha)} = S^{*(\leq,\alpha)}$ and $(\mathrm{Heco}S)^{*(<,\alpha)} = S^{*(<,\alpha)}$.

Next, we define some notions of a function. A function $f$ is said to be evenly quasiconvex if $L(f, \leq, \alpha)$ is evenly convex for all $\alpha \in \mathbb{R}$. A function $f$ is said to be strictly evenly quasiconvex if $L(f, <, \alpha)$ is evenly convex for all $\alpha \in \mathbb{R}$. Clearly, every evenly quasiconvex function is quasiconvex, every lower semicontinuous (lsc) quasiconvex function is evenly quasiconvex and every upper semicontinuous (usc) quasiconvex function is strictly evenly quasiconvex. It is easy to show that every strictly evenly quasiconvex function is evenly quasiconvex, but the converse is not generally true, see [9]. A function $f$ is said to be $H$-evenly quasiconvex if $L(f, \leq, \alpha)$ is $H$-evenly convex for all $\alpha \in \mathbb{R}$. A function $f$ is said to be strictly $H$-evenly quasiconvex if $L(f, <, \alpha)$ is $H$-evenly convex for all $\alpha \in \mathbb{R}$. In [9], we showed that $f$ is $H$-evenly quasiconvex if and only if $f$ is evenly quasiconvex and $f(0) = \min_{x\in\mathbb{R}^n} f(x)$.

Next, we introduce two notions of quasiconjugate. In this paper, we characterize set containments by using these quasiconjugates.

**Definition 2** ([3])**.** The $\lambda$-quasiconjugate of $f$ is the function $f^{\nu}_{\lambda} : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f^{\nu}_{\lambda}(u) = \lambda - \inf\{f(x) \mid \langle u, x\rangle \geq \lambda\},\ \forall u \in \mathbb{R}^n.$$

**Definition 3** ([7])**.** The $\lambda$-semiconjugate of $f$ is the function $f^{\theta}_{\lambda} : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f^{\theta}_{\lambda}(u) = \lambda - \inf\{f(x) \mid \langle u, x\rangle > \lambda\},\ \forall u \in \mathbb{R}^n.$$

Singer [7] defined the $\lambda$-semiconjugate in the following form,

$$f^{\theta}_{\lambda}(u) = \lambda - 1 - \inf\{f(x) \mid \langle u, x\rangle > \lambda - 1\},\ \forall u \in \mathbb{R}^n.$$

But we redefine the $\lambda$-semiconjugate in this paper.

We can check easily that $f^{\nu}_{\lambda}$ is $H$-evenly quasiconvex and $f^{\theta}_{\lambda}$ is lsc $H$-evenly quasiconvex if $\lambda > 0$ in the similar way of [7, 11].

## 3. Containment of a convex set in an evenly convex set

In this section, we present characterizations of the containment of a convex set, defined by infinite quasiconvex constraints, in an evenly convex set, i.e., let $I$, $J$, $S$, $W$ be arbitrary sets, $f_i$ and $g_j$ be quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each

$i \in I$ and $j \in J$, $v_s \in \mathbb{R}^n$ and $\alpha_s \in \mathbb{R}$ for each $s \in S$, $u_w \in \mathbb{R}^n$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Then, we show the characterization of $A \subset B$, where

$$\begin{aligned} A &= \{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\}, \\ B &= \{x \in \mathbb{R}^n \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s, \forall w \in W, \langle u_w, x\rangle \leq \gamma_w\}. \end{aligned}$$

In the beginning, we show a result of the containment when $|J| = |S| = 1$ and $I = W = \emptyset$.

**Theorem 1.** Let $g$ be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $v \in \mathbb{R}^n \setminus \{0\}$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Then, following conditions (i), (ii) and (iii) are equivalent.

(i) $L(g, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\}$,
(ii) $v \in (L(g, <, \beta))^{*(<,\alpha)}$,
(iii) $v \in L(g^{\nu}_{\alpha}, \leq, \alpha - \beta)$.

*Proof.* It is clear that (i) and (ii) are equivalent. Assume that $L(g, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\}$, then the implication $g(x) < \beta$ implies $\langle v, x\rangle < \alpha$, or equivalently, $\langle v, x\rangle \geq \alpha$ implies $g(x) \geq \beta$ holds. This shows

$$g^{\nu}_{\alpha}(v) = \alpha - \inf\{g(x) \mid \langle v, x\rangle \geq \alpha\} \leq \alpha - \beta.$$

Conversely, if $g^{\nu}_{\alpha}(v) \leq \alpha - \beta$, then $\inf\{g(x) \mid \langle v, x\rangle \geq \alpha\} \geq \beta$. Therefore the implication $\langle v, x\rangle \geq \alpha$ implies $g(x) \geq \beta$, or $g(x) < \beta$ implies $\langle v, x\rangle < \alpha$ holds. This derives $L(g, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\}$. □

Next, we show the set containment characterization, assuming that all $g_j$ ($j \in J$) are strictly evenly quasiconvex, $J$ and $S$ are possibly infinite, and $I$ and $W$ are empty.

**Theorem 2.** Let $J$, $S$ be arbitrary sets, $\beta \in \mathbb{R}$, $g_j$ be a strictly evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $j \in J$, and $v_s \in \mathbb{R}^n$ and $\alpha_s \in (0, \infty)$ for each $s \in S$. Assume that $g_j(0) < \beta$ for each $j \in J$. Then, following conditions (i) and (ii) are equivalent.

(i) $\{x \in \mathbb{R}^n \mid \forall j \in J, g_j(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s\}$,
(ii) $\forall s \in S, \dfrac{v_s}{\alpha_s} \in \mathrm{Heco} \bigcup_{j \in J} L((g_j)^{\nu}_1, \leq, 1 - \beta)$.

*Proof.* It is clear that (i) and

$$\forall s \in S,\ \frac{v_s}{\alpha_s} \in \left(\bigcap_{j \in J} L(g_j, <, \beta)\right)^{*(<,1)}$$

are equivalent. By using the assumption, $L(g_j, <, \beta)$ is a $H$-evenly convex set for each $j \in J$. Therefore, by using Proposition 2, for all $s \in S$, $\frac{v_s}{\alpha_s} \in \mathrm{Heco} \bigcup_{j \in J}(L(g_j, <, \beta))^{*(<,1)}$. Furthermore, by using Theorem 1, for all $s \in S$, $\frac{v_s}{\alpha_s} \in \mathrm{Heco} \bigcup_{j \in J} L((g_j)^{\nu}_1, \leq, 1 - \beta)$. □

In the following theorem, we show the set containment characterization, assuming that all $f_i$ $(i \in I)$ are evenly quasiconvex, $I$ and $S$ are possibly infinite, and $J$ and $W$ are empty.

**Theorem 3.** Let $I$ and $S$ be arbitrary sets, $\beta \in \mathbb{R}$, $f_i$ be a evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$, and $v_s \in \mathbb{R}^n$ and $\alpha_s \in (0, \infty)$ for each $s \in S$. Assume that $f_i(0) \leq \beta$ for each $i \in I$. Then, following conditions (i) and (ii) are equivalent.

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq \beta\} \subset \{x \in \mathbb{R}^n \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s\}$,

(ii) $\forall s \in S,\ \dfrac{v_s}{\alpha_s} \in \operatorname{Heco} \bigcup_{i \in I} L((f_i)^\nu_1, <, 1-\beta)$.

*Proof.* It is clear that (i) and

$$\forall s \in S,\ \frac{v_s}{\alpha_s} \in \left(\bigcap_{i \in I} L(f_i, \leq, \beta)\right)^{*(<,1)}$$

are equivalent. By using the assumption, $L(f_i, \leq, \beta)$ is a $H$-evenly convex set for each $i \in I$. Therefore, by using Proposition 2, for all $s \in S$, $\frac{v_s}{\alpha_s} \in \operatorname{Heco} \bigcup_{i \in I} (L(f_i, \leq, \beta))^{*(<,1)}$. Because $L(f_i, \leq, \beta) = \bigcap_{\varepsilon > 0} L(f_i, \leq, \beta + \varepsilon)$, by using Proposition 2 again, for all $s \in S$, $\frac{v_s}{\alpha_s} \in \operatorname{Heco} \bigcup_{i \in I} \operatorname{Heco} \bigcup_{\varepsilon > 0} (L(f_i, \leq, \beta + \varepsilon))^{*(<,1)}$. Furthermore, $\bigcup_{\varepsilon > 0} (L(f_i, \leq, \beta + \varepsilon))^{*(<,1)} = \bigcup_{\varepsilon > 0} (L(f_i, <, \beta + \varepsilon))^{*(<,1)}$, so we can prove that $\frac{v_s}{\alpha_s} \in \operatorname{Heco} \bigcup_{i \in I} \bigcup_{\varepsilon > 0} L((f_i)^\nu_1, \leq, 1 - \beta - \varepsilon)$ for all $s \in S$ by using Theorem 1. Therefore, for all $s \in S$, $\frac{v_s}{\alpha_s} \in \operatorname{Heco} \bigcup_{i \in I} L((f_i)^\nu_1, <, 1-\beta)$. The converse is similar. □

Next, we show the set containment characterization, assuming that all $f_i$ $(i \in I)$ are evenly quasiconvex, all $g_j$ $(j \in J)$ are strictly evenly quasiconvex, $I$, $J$ and $S$ are arbitrary sets and $W$ is empty.

**Theorem 4.** Let $I$, $J$ and $S$ be arbitrary sets, $\beta \in \mathbb{R}$, $f_i$ be a evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$, $g_j$ be a strictly evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $j \in J$, and $v_s \in \mathbb{R}^n$ and $\alpha_s \in (0, \infty)$ for each $s \in S$. Assume that $f_i(0) \leq \beta$ and $g_j(0) < \beta$ for each $i \in I$ and $j \in J$. Then, following conditions (i) and (ii) are equivalent.

(i) $\{x \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\} \subset \{x \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s\}$,

(ii) $\forall s \in S$,

$$\frac{v_s}{\alpha_s} \in \operatorname{Heco}\left[\left(\bigcup_{i \in I} L((f_i)^\nu_1, <, 1-\beta)\right) \bigcup \left(\bigcup_{j \in J} L((g_j)^\nu_1, \leq, 1-\beta)\right)\right].$$

*Proof.* The proof is similar to Theorem 2 and 3. □

In the following theorem, we show the result of the characterizing set containment when $|J| = |W| = 1$ and $I = S = \emptyset$, by using $\lambda$-semiconjugate.

**Theorem 5.** Let $g$ be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $u \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Then, following conditions (i), (ii) and (iii) are equivalent.

(i) $L(g, <, \beta) \subset \{x \in \mathbb{R}^n \mid \langle u, x\rangle \leq \gamma\}$,
(ii) $u \in (L(g, <, \beta))^{*(\leq, \gamma)}$,
(iii) $u \in L(g^\theta_\gamma, \leq, \gamma - \beta)$.

*Proof.* It is clear that (i) and (ii) are equivalent. We may assume that $L(g, <, \beta) \subset \{x \in \mathbb{R}^n \mid \langle u, x\rangle \leq \gamma\}$, then the implication $g(x) < \beta$ implies $\langle u, x\rangle \leq \gamma$, or equivalently, $\langle u, x\rangle > \gamma$ implies $g(x) \geq \beta$ holds. This shows

$$g^\theta_\gamma(u) = \gamma - \inf\{g(x) \mid \langle u, x\rangle > \gamma\} \leq \gamma - \beta.$$

Conversely, if $g^\theta_\gamma(u) \leq \gamma - \beta$, then $\inf\{g(x) \mid \langle u, x\rangle > \gamma\} \geq \beta$. Therefore the implication $\langle u, x\rangle > \gamma$ implies $g(x) \geq \beta$, or $g(x) < \beta$ implies $\langle u, x\rangle \leq \gamma$ holds. This derives $L(g, <, \beta) \subset \{x \mid \langle u, x\rangle \leq \gamma\}$. □

Next, we show the set containment characterization, assuming that $g_j$ $(j \in J)$ are quasiconvex, $J$ and $W$ are arbitrary sets, and $I$ and $S$ are empty.

**Theorem 6.** Let $J$ and $W$ be arbitrary sets, $\beta \in \mathbb{R}$, $g_j$ be a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $j \in J$, and $u_w \in \mathbb{R}^n$ and $\gamma_w \in (0, \infty)$ for each $w \in W$. Assume that $g_j(0) < \beta$ for each $j \in J$ and $\mathrm{int}\{x \in \mathbb{R}^n \mid \forall j \in J, g_j(x) < \beta\}$ is nonempty. Then, following conditions (i) and (ii) are equivalent.

(i) $\{x \in \mathbb{R}^n \mid \forall j \in J, g_j(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \forall w \in W, \langle u_w, x\rangle \leq \gamma_w\}$,
(ii) $\forall w \in W$, $\dfrac{u_w}{\gamma_w} \in \mathrm{clHeco} \bigcup_{j \in J} L((g_j)^\theta_1, \leq, 1 - \beta)$.

*Proof.* It is easy to show that (i) is equivalent to

$$\forall w \in W,\ \frac{u_w}{\gamma_w} \in \left(\bigcap_{j \in J} L(g_j, <, \beta)\right)^{*(\leq, 1)}.$$

Since $\mathrm{int}\{x \in \mathbb{R}^n \mid \forall j \in J, g_j(x) < \beta\}$ is nonempty, we can prove that

$$(\bigcap_{j \in J} L(g_j, <, \beta))^{*(\leq, 1)} = (\mathrm{cl} \bigcap_{j \in J} L(g_j, <, \beta))^{*(\leq, 1)} = (\bigcap_{j \in J} \mathrm{cl} L(g_j, <, \beta))^{*(\leq, 1)}$$

by using Proposition 1. From the assumption, $\mathrm{cl}L(g_j, <, \beta)$ is closed $H$-evenly convex for each $j \in J$. Therefore, by using Proposition 2, $\frac{u_w}{\gamma_w} \in \mathrm{clHeco} \bigcup_{j \in J} (\mathrm{cl}L(g_j, <, \beta))^{*(\leq, 1)}$ for all $w \in W$. Furthermore, by using Theorem 5, $\frac{u_w}{\gamma_w} \in \mathrm{clHeco} \bigcup_{j \in J} L((g_j)^\theta_1, \leq, 1 - \beta)$ for all $w \in W$. The converse is similar. □

In the following theorem, we show the set containment characterization, assuming that all $f_i$ $(i \in I)$ are quasiconvex, $I$ and $W$ are arbitrary sets, and $J$ and $S$ are empty.

**Theorem 7.** Let $I$ and $W$ be arbitrary sets, $\beta \in \mathbb{R}$, $f_i$ be a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$, and $u_w \in \mathbb{R}^n$ and $\gamma_w \in (0, \infty)$ for each $w \in W$. Assume that $f_i(0) \leq \beta$ for each $i \in I$ and $\mathrm{int}\{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq \beta\}$ is nonempty. Then, following conditions (i) and (ii) are equivalent.

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq \beta\} \subset \{x \in \mathbb{R}^n \mid \forall w \in W, \langle u_w, x\rangle \leq \gamma_w\}$,

(ii) $\forall w \in W$, $\dfrac{u_w}{\gamma_w} \in \mathrm{clHeco} \bigcup_{i \in I} L((f_i)^\theta_1, <, 1 - \beta)$.

*Proof.* It is clear that (i) is equivalent to for all $w \in W$, $\frac{u_w}{\gamma_w} \in (\bigcap_{i \in I} L(f_i, \leq, \beta))^{*(\leq,1)}$. By the similar way in Theorem 6, we can prove that $\frac{u_w}{\gamma_w} \in (\bigcap_{i\in I} \mathrm{cl}L(f_i, \leq, \beta))^{*(\leq,1)}$ for all $w \in W$. From the assumption, $\mathrm{cl}L(f_i, \leq, \beta)$ is a closed $H$-evenly convex set for each $j \in J$. Therefore, by using Proposition 2, $\frac{u_w}{\gamma_w} \in \mathrm{clHeco}\bigcup_{i\in I}(\mathrm{cl}L(f_i, \leq, \beta))^{*(\leq,1)}$ for all $w \in W$. Also, by using Proposition 1 again,

$$\mathrm{clHeco}\bigcup_{i\in I}\left(\mathrm{cl}\bigcap_{\varepsilon>0} L(f_i, <, \beta + \varepsilon)\right)^{*(\leq,1)} = \mathrm{clHeco}\bigcup_{i\in I}\left(\bigcap_{\varepsilon>0} \mathrm{cl}L(f_i, <, \beta + \varepsilon)\right)^{*(\leq,1)}.$$

By using the assumption, for each $\varepsilon > 0$, $\mathrm{cl}L(f_i, <, \beta + \varepsilon)$ is a closed $H$-evenly convex set. Therefore, by using Proposition 2, for all $w \in W$,

$$\frac{u_w}{\gamma_w} \in \mathrm{clHeco}\bigcup_{i\in I}\bigcup_{\varepsilon>0}(\mathrm{cl}L(f_i, <, \beta + \varepsilon))^{*(\leq,1)}.$$

Furthermore,

$$\mathrm{clHeco}\bigcup_{i\in I}\bigcup_{\varepsilon>0}(\mathrm{cl}L(f_i, <, \beta + \varepsilon))^{*(\leq,1)} = \mathrm{clHeco}\bigcup_{i\in I}\bigcup_{\varepsilon>0}(L(f_i, <, \beta + \varepsilon))^{*(\leq,1)},$$

and by using Theorem 5, we can prove that for all $w \in W$,

$$\frac{u_w}{\gamma_w} \in \mathrm{clHeco}\bigcup_{i\in I}\bigcup_{\varepsilon>0} L((f_i)^\theta_1, \leq, 1 - \varepsilon - \beta),$$

i.e., $\frac{u_w}{\gamma_w} \in \mathrm{clHeco}\bigcup_{i\in I} L((f_i)^\theta_1, <, 1 - \beta)$. □

Next, we show the set containment characterization, assuming that $f_i$ and $g_j$ are quasiconvex for each $i \in I$ and $j \in J$, $I$, $J$ and $W$ are arbitrary sets, and $S$ is empty.

**Theorem 8.** Let $I$, $J$ and $W$ be arbitrary sets, $\beta \in \mathbb{R}$, $f_i$ and $g_j$ be quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$ and $j \in J$, and $u_w \in \mathbb{R}^n$ and $\gamma_w \in (0, \infty)$ for each $w \in W$. Assume that $f_i(0) \leq \beta$ for each $i \in I$, $g_j(0) < \beta$ for each $j \in J$, and $\mathrm{int}\{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\}$ is nonempty. Then, following conditions (i) and (ii) are equivalent.

(i) $\{x \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\} \subset \{x \mid \forall w \in W, \langle u_w, x\rangle \leq \gamma_w\}$,

(ii) $\forall w \in W$,

$$\frac{u_w}{\gamma_w} \in \mathrm{clHeco}\left[\left(\bigcup_{i\in I} L((f_i)_1^\theta, <, 1-\beta)\right) \bigcup \left(\bigcup_{j\in J} L((g_j)_1^\theta, \leq, 1-\beta)\right)\right].$$

*Proof.* The proof is similar to Theorem 6 and 7. □

In the following theorem, we show the set containment characterization, assuming that all $f_i$ $(i \in I)$ are evenly quasiconvex, all $g_j$ $(j \in J)$ are strictly evenly quasiconvex, and $I$, $J$, $S$ and $W$ are arbitrary sets.

**Theorem 9.** Let $I$, $J$, $S$ and $W$ be arbitrary sets, $\beta \in \mathbb{R}$, $f_i$ be a evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$, $g_j$ be a strictly evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $j \in J$, $v_s \in \mathbb{R}^n$ and $\alpha_s \in (0, \infty)$ for each $s \in S$, and $u_w \in \mathbb{R}^n$ and $\gamma_w \in (0, \infty)$ for each $w \in W$. Assume that $f_i(0) \leq \beta$ for each $i \in I$, $g_j(0) < \beta$ for each $j \in J$ and $\mathrm{int}\{x \in \mathbb{R}^n \mid f_i(x) \leq \beta, i \in I, g_j(x) < \beta, j \in J\}$ is nonempty. Then, following conditions (i) and (ii) are equivalent.

(i) $A \subset B$,

(ii) $\forall s \in S$,

$$\frac{v_s}{\alpha_s} \in \mathrm{Heco}\left[\left(\bigcup_{i\in I} L((f_i)_1^\nu, <, 1-\beta)\right) \bigcup \left(\bigcup_{j\in J} L((g_j)_1^\nu, \leq, 1-\beta)\right)\right],$$

$\forall w \in W$,

$$\frac{u_w}{\gamma_w} \in \mathrm{clHeco}\left[\left(\bigcup_{i\in I} L((f_i)_1^\theta, <, 1-\beta)\right) \bigcup \left(\bigcup_{j\in J} L((g_j)_1^\theta, \leq, 1-\beta)\right)\right]$$

where

$$\begin{aligned} A &= \{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\}, \\ B &= \{x \in \mathbb{R}^n \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s, \forall w \in W, \langle u_w, x\rangle \leq \gamma_w\}. \end{aligned}$$

*Proof.* The proof is similar to Theorem 2, 3, 6 and 7. □

## 4. Containment of a convex set in a reverse convex set

In this section, we present characterizations of the containment of a convex set, defined by infinite quasiconvex constraints, in a reverse convex set, defined by infinite quasiconvex constraints, i.e., let $I$, $J$, $W$ be arbitrary sets, $f_i$ and $g_j$ be quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$ and for each $j \in J$, $k_w$ be a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Then, we show the characterization of $A \subset B$, where

$$\begin{aligned} A &= \{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\}, \\ B &= \{x \in \mathbb{R}^n \mid \forall w \in W, k_w(x) \geq \gamma_w\}. \end{aligned}$$

In the beginning, we show the result of the containment when $|J| = |W| = 1$ and $I$ is empty.

**Theorem 10.** Let $g$ be a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $k$ be a usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\gamma \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Assume that $L(g, <, \beta)$ and $L(k, <, \gamma)$ are nonempty. Then, following conditions (i), (ii) and (iii) are equivalent.

(i) $L(g, <, \beta) \subset L(k, \geq, \gamma)$,
(ii) $L(g, <, \beta) \bigcap L(k, <, \gamma) = \emptyset$,
(iii) there exists $\alpha \in \mathbb{R}$ such that
$0 \in L(g^{\theta}_{\alpha}, \leq, \alpha - \beta) \setminus \{0\} + L(k^{\nu}_{-\alpha}, \leq, -\alpha - \gamma) \setminus \{0\}$.

*Proof.* It is clear that (i) and (ii) are equivalent. We may assume that the condition (ii) holds. Then, there exists $v \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that for all $x \in L(k, <, \gamma)$ and $y \in L(g, <, \beta)$,

$$\langle v, x \rangle > \alpha \geq \langle v, y \rangle ,$$

since $g$ is quasiconvex and $k$ is usc quasiconvex. Clearly, $v \in (L(g, <, \beta))^{*(\leq, \alpha)}$ and $-v \in (L(k, <, \gamma))^{*(<, -\alpha)}$. By using Theorem 1 and Theorem 5, $v \in (L(g^{\theta}_{\alpha}, \leq, \alpha - \beta)$ and $-v \in L(k^{\nu}_{-\alpha}, \leq, -\alpha - \gamma)$. Therefore $0 \in L(g^{\theta}_{\alpha}, \leq, \alpha - \beta) \setminus \{0\} + L(k^{\nu}_{-\alpha}, \leq, -\alpha - \gamma) \setminus \{0\}$. The converse is similar. □

Next, we show the set containment characterization, assuming that all $g_j$ ($j \in J$) are quasiconvex, all $k_w$ ($w \in W$) are usc quasiconvex, $J$ and $W$ are arbitrary sets, and $I$ is empty.

**Theorem 11.** Let $J$ and $W$ be arbitrary sets, $g_j$ be a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $j \in J$, $k_w$ be a usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Assume that $0 \in \operatorname{int} \cap_{j \in J} L(g_j, <, \beta)$ and $\cap_{w \in W} L(k_w, <, \gamma_w)$ is nonempty. Then, following conditions (i) and (ii) are equivalent.

(i) $\cap_{j \in J} L(g_j, <, \beta) \subset \cap_{w \in W} L(k_w, \geq, \gamma_w)$,
(ii) $\forall w \in W$,

$$0 \in \left( \mathrm{clHeco} \cup_{j \in J} L((g_j)^{\theta}_{1}, \leq, 1 - \beta) \setminus \{0\} \right) + L((k_w)^{\nu}_{-1}, \leq, -1 - \gamma_w) \setminus \{0\}.$$

*Proof.* We may assume that the condition (i) is hold. Then, for all $w \in W$, $\cap_{j \in J} L(g_j, <, \beta) \bigcap L(k_w, <, \gamma_w) = \emptyset$. Since all $g_j$ are quasiconvex, $k_w$ are usc quasiconvex and $0 \in \operatorname{int} \cap_{j \in J} L(g_j, <, \beta)$, there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in L(k_w, <, \gamma_w)$ and $y \in \cap_{j \in J} L(g_j, <, \beta)$, $\langle v, x \rangle > 1 \geq \langle v, y \rangle$. By using Theorem 1 and Theorem 6, we can prove that $v \in \mathrm{clHeco} \cup_{j \in J} L((g_j)^{\theta}_{1}, \leq, 1 - \beta)$ and $-v \in L((k_w)^{\nu}_{-1}, \leq, -1 - \gamma_w)$, i.e., $0 \in (\mathrm{clHeco} \cup_{j \in J} L((g_j)^{\theta}_{1}, \leq, 1 - \beta) \setminus \{0\}) + L((k_w)^{\nu}_{-1}, \leq, -1 - \gamma_w) \setminus \{0\}$. The converse implication is similar. □

In the following theorem, we show the set containment characterization, assuming that $f_i$ ($i \in I$) are quasiconvex, $k_w$ ($w \in W$) are usc quasiconvex, $I$ and $W$ are arbitrary sets, and $J$ is empty.

**Theorem 12.** Let $I$ and $W$ be arbitrary sets, $f_i$ be a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$, $k_w$ be a usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Assume that $0 \in \operatorname{int} \cap_{i\in I} L(f_i, \leq, \beta)$ and $\cap_{w\in W} L(k_w, <, \gamma_w)$ is nonempty. Then, following conditions (i) and (ii) are equivalent.

(i) $\cap_{i\in I} L(f_i, \leq, \beta) \subset \cap_{w\in W} L(k_w, \geq, \gamma_w)$,
(ii) $\forall w \in W$,

$$0 \in \left(\mathrm{clHeco} \cup_{i\in I} L((f_i)_1^{\theta}, <, 1-\beta) \setminus \{0\}\right) + L((k_w)_{-1}^{\nu}, \leq, -1-\gamma_w) \setminus \{0\}.$$

*Proof.* We may assume that the condition (i) is hold. Then, $\cap_{i\in I} L(f_i, \leq, \beta) \bigcap L(k_w, <, \gamma_w) = \emptyset$ for all $w \in W$. Since all $f_i$ are quasiconvex, $k_w$ are usc quasiconvex and $0 \in \operatorname{int} \cap_{i\in I} L(f_i, \leq, \beta)$, there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in L(k_w, <, \gamma_w)$ and $y \in \cap_{i\in I} L(f_i, \leq, \beta)$, $\langle v, x\rangle > 1 \geq \langle v, y\rangle$. By using Theorem 1 and Theorem 7, we can prove that $v \in \mathrm{clHeco} \cup_{j\in J} L((f_i)_1^{\theta}, <, 1-\beta)$ and $-v \in L((k_w)_{-1}^{\nu}, \leq, -1-\gamma_w)$. The converse is similar. □

In the last theorem of this paper, we show the set containment characterization, assuming that $f_i$ and $g_j$ are quasiconvex for each $i \in I$ and $j \in J$, $k_w$ are usc quasiconvex for each $w \in W$, and $I$, $J$ and $W$ are arbitrary sets.

**Theorem 13.** Let $I$, $J$ and $W$ be arbitrary sets, $f_i$ and $g_j$ be quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$ and $j \in J$, $k_w$ be a usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Assume that $0 \in \operatorname{int}[(\cap_{i\in I} L(f_i, \leq, \beta)) \bigcap (\cap_{j\in J} L(g_j, <, \beta))]$ and $\cap_{w\in W} L(k_w, <, \gamma_w)$ is nonempty. Then, following conditions (i) and (ii) are equivalent.

(i) $(\cap_{i\in I} L(f_i, \leq, \beta)) \bigcap (\cap_{j\in J} L(g_j, <, \beta)) \subset \cap_{w\in W} L(k_w, \geq, \gamma_w)$,
(ii) $\forall w \in W$,

$$0 \in \left(\mathrm{clHeco}\left\{(\cup_{i\in I} L((f_i)_1^{\theta}, <, 1-\beta)) \bigcup (\cup_{j\in J} L((g_j)_1^{\theta}, \leq, 1-\beta))\right\} \setminus \{0\}\right)$$
$$+ L((k_w)_{-1}^{\nu}, \leq, -1-\gamma_w) \setminus \{0\}.$$

*Proof.* The proof is similar to Theorem 11 and 12.

□

Finally, we discuss results in Section 3 and 4. In Section 3, we show set containment characterizations in an evenly convex set, assuming that the inequalities in $A$ and $B$ can be either weak or strict. But, in Section 4, we show set containment characterizations in a reverse convex set, assuming that inequalities in $A$ can be either weak or strict and in $B$ are only weak. Hereinafter, we show that it is difficult to characterize the set containment characterization in a reverse convex set, assuming that the inequalities in $B$ are strict.

We consider the characterization of $A \subset B$, where $I$, $J$ and $S$ are arbitrary sets, $f_i$ and $g_j$ are quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$ and $j \in J$,

$h_s$ is a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\alpha_s \in \mathbb{R}$ for each $s \in S$, $\beta \in \mathbb{R}$, and

$$\begin{aligned} A &= \{x \in \mathbb{R}^n \mid f_i(x) \leq \beta, i \in I, g_j(x) < \beta, j \in J\}, \\ B &= \{x \in \mathbb{R}^n \mid h_s(x) > \alpha_s, s \in S\}. \end{aligned}$$

Assume that $J$ is empty, $f_i$ is lsc quasiconvex for all $i \in I$, $h_s$ is lsc quasiconvex and $L(h_s, \leq, \alpha_s)$ is bounded for all $s \in S$, and $0 \in \operatorname{int} \cap_{i\in I} L(f_i, \leq, \beta)$, then, following conditions (i) and (ii) are equivalent.

$$\begin{array}{ll} \text{(i)} & \cap_{i\in I} L(f_i, \leq, \beta) \subset \cap_{s\in S} L(h_s, >, \alpha_s), \\ \text{(ii)} & \forall s \in S, \exists v \in \mathbb{R}^n \setminus \{0\} \text{ s.t.} \\ & \forall x \in L(h_s, \leq, \alpha_s), \forall y \in \cap_{i\in I} L(f_i, \leq, \beta), \langle v, x\rangle > 1 \geq \langle v, y\rangle . \end{array}$$

Of course, we can rewrite the condition (ii) by using level sets of quasiconjugate functions. Assume that $I$ is empty, $|J| < \infty$, $g_j$ is usc quasiconvex for all $j \in J$, $h_s$ is quasiconvex for all $s \in S$, and $0 \in \operatorname{int} \cap_{j\in J} L(g_j, <, \beta)$, then, following conditions (i) and (ii) are equivalent.

$$\begin{array}{ll} \text{(i)} & \cap_{j\in J} L(g_j, <, \beta) \subset \cap_{s\in S} L(h_s, >, \alpha_s), \\ \text{(ii)} & \forall s \in S, \exists v \in \mathbb{R}^n \setminus \{0\} \text{ s.t.} \\ & \forall x \in L(h_s, \leq, \alpha_s), \forall y \in \cap_{j\in J} L(g_j, <, \beta), \langle v, x\rangle \geq 1 > \langle v, y\rangle . \end{array}$$

Hence, we can show the set containment characterization by using 1-quasiconjugate and $-1$-semiconjugate.

However, if $J$ is an arbitrary set, then $\cap_{j\in J} L(g_j, <, \beta)$ is not always open even if $g_j$ is usc quasiconvex for all $j \in J$. Therefore, if $g_j$ is usc quasiconvex for all $j \in J$, $h_s$ is lsc quasiconvex and $L(h_s, \leq, \alpha_s)$ is bounded for all $s \in S$, $0 \in \operatorname{int} \cap_{j\in J} L(g_j, <, \beta)$ and $\cap_{j\in J} L(g_j, <, \beta) \subset \cap_{s\in S} L(h_s, >, \alpha_s)$, then, for all $s \in S$, there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in \cap_{j\in J} L(g_j, <, \beta)$ and $y \in L(h_s, \leq, \alpha_s)$,

$$\langle v, x\rangle \geq 1 \geq \langle v, y\rangle .$$

Also, the above inequality does not imply that $\cap_{j\in J} L(g_j, <, \beta) \subset \cap_{s\in S} L(h_s, >, \alpha_s)$, and these assumptions of functions are the strongest one in this problem. Therefore, it is hard to characterize set containments by using quasiconjugate function.

## 5. Application to quasiconvex minimization problem

In this section, we show that set containment characterizations in this paper is useful to consider quasiconvex minimization problem. Let $I$ be an arbitrary set, $f_i$ be a lsc quasiconvex function from $X$ to $\overline{\mathbb{R}}$ for each $i \in I$, $A = \{x \in X \mid \forall i \in I, f_i(x) \leq 0\}$, and $k$ be a usc quasiconvex function. Assume that $0 \in \operatorname{int} A$, and consider the following problem $(P)$,

$$(P) \begin{cases} \text{minimize } k(x), \\ \text{subject to } x \in A. \end{cases}$$

In [6], it was shown that all lsc quasiconvex functions are the supremum of some family of lsc quasi-affine functions, where a function is said to be quasi-affine if it is quasiconvex and quasiconcave. Furthermore, $f$ is lsc quasi-affine if and only if there exists $l \in Q$ and $v \in \mathbb{R}^n$ such that $f = l \circ v$, where $Q = \{l : \mathbb{R} \to \overline{\mathbb{R}} \mid l \text{ is lsc and non-decreasing}\}$. So, without loss of generality, we can assume that $f_i$ is a lsc quasi-affine function for each $i \in I$, i.e., there exist $\{(l_i, v_i) \mid i \in I\} \subset Q \times \mathbb{R}^n$ such that $f_i = l_i \circ v_i$ for each $i \in I$. Also, in [6], Penot and Volle studied the hypo-epi-inverse which is a generalized concept of the inverse of non-decreasing functions. The hypo-epi-inverse of $g \in Q$ is equal to $\sup\{s \in \mathbb{R} \mid g(s) \leq r\}$ for any $r$, and if $g$ has an inverse function, then the inverse and the hypo-epi-inverse of $g$ are the same. Hence, we denote the hypo-epi-inverse of $g$ by $g^{-1}$.

By using Theorem 12, for each $\gamma \in \mathbb{R}$, following conditions (i), (ii) and (iii) are equivalent.

(i) $\cap_{i\in I} L(f_i, \leq, 0) \subset L(k, \geq, \gamma)$,
(ii) $0 \in \text{clHeco} \cup_{i\in I} L((f_i)^\theta_1, <, 1) + L(k^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}$, and
(iii) $0 \in \text{clco}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\}) + L(k^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}$.

Actually, for each $i \in I$,

$$\begin{aligned} L((l_i \circ v_i)^\theta_1, <, 1) &= \{z \in \mathbb{R}^n \mid (l_i \circ v_i)^\theta_1(z) < 1\} \\ &= \{z \in \mathbb{R}^n \mid 1 - \inf\{l_i \circ v_i(x) \mid \langle z, x\rangle > 1\} < 1\} \\ &= \{z \in \mathbb{R}^n \mid \inf\{l_i \circ v_i(x) \mid \langle z, x\rangle > 1\} > 0\}. \end{aligned}$$

If $z \notin \mathbb{R}_+\{v_i\}$, it is clear that $\inf\{l_i \circ v_i(x) \mid \langle z, x\rangle > 1\} = \inf_{t\in\mathbb{R}} l_i(t) \leq 0$ because $S$ is nonempty. And if $z \in \mathbb{R}_+\{v_i\} \setminus \{0\}$, there exists $\lambda > 0$ such that $z = \lambda v_i$, so $\inf\{l_i \circ v_i(x) \mid \langle z, x\rangle > 1\} = l_i(\frac{1}{\lambda})$ because $l_i$ is nondecreasing. Also, it is clear that $\inf\{l_i \circ v_i(x) \mid \langle 0, x\rangle > 1\} = \infty$, hence we can prove that $L((l_i \circ v_i)^\theta_1, <, 1) = [0, \frac{1}{(l_i)^{-1}(0)})\{v_i\}$. Furthermore, $\text{clHeco} \cup_{i\in I} L((l_i \circ v_i)^\theta_1, <, 1) = \text{clHeco} \cup_{i\in I} [0, \frac{1}{(l_i)^{-1}(0)})\{v_i\} = \text{cleco}\{\cup_{i\in I}[0, \frac{1}{(l_i)^{-1}(0)})\{v_i\} \cup \{0\}\}$ because $\cup_{i\in I}[0, \frac{1}{(l_i)^{-1}(0)})\{v_i\}$ is nonempty. Also $\text{cleco}\{\cup_{i\in I}[0, \frac{1}{(l_i)^{-1}(0)})\{v_i\} \cup \{0\}\} = \text{clco}\{\cup_{i\in I}[0, \frac{1}{(l_i)^{-1}(0)})\{v_i\} \cup \{0\}\} = \text{clco}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\})$, so the above conditions (i), (ii) and (iii) are equivalent.

Clearly, $\inf_{x\in A} k(x) = \sup\{\gamma \in \mathbb{R} \mid \cap_{i\in I} L(f_i, \leq, 0) \subset L(k, \geq, \gamma)\}$. So, we can prove that

$$\inf_{x\in A} k(x) = \sup\{\gamma \mid 0 \in \text{clco}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\}) + L(k^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}\},$$

that is, we get this new duality problem of $(P)$,

$$(D) \begin{cases} \text{maximize } \gamma, \\ \text{subject to } 0 \in \text{clco}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\}) + L(k^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}. \end{cases}$$

The value of the dual problem $(D)$ is equal to $-\inf_{z\in T}(k^\nu_{-1}(z) + 1)$, where $T = -\text{clco}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\})$. Furthermore, $A^{*(\leq,1)} = \text{clco}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in$

$I\} \cup \{0\}) = -T$ and $k^{\nu}_{-1} + 1 = k^R$, which is defined in [11]. Since, $\inf_{x\in A} k(x) = -\inf_{z\in -A^{*(\leq,1)}} k^R(z)$, we can get another duality problem of $(P)$,

$$(D')\begin{cases} \text{minimize } k^R(z), \\ \text{subject to } z \in -A^{*(\leq,1)}, \end{cases}$$

This duality problem $(D')$ is similar to a duality problem of $(P)$ in [11].

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