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On Set Containment Characterization and Constraint Qualification for Quasiconvex Programming

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Abstract Dual characterizations of the containment of a convex set with quasiconvex inequality constraints are investigated. A new Lagrange-type duality and a new closed cone constraint qualification are described, and it is shown that this constraint qualification is the weakest constraint qualification for the duality.

Keywords set containment · quasiconvex constraints · quasiaffine functions · constraint qualification

Mathematics Subject Classification (2000) 90C26 · 26B25

1 Introduction

Set containment problems consist of characterizing the inclusion $A \subset B$, where $A = \{x \in X \mid \forall i \in I, f_i(x) \leq 0\}$ and $B = \{x \in X \mid \forall j \in J, h_j(x) \leq 0\}$. Motivated by general non-polyhedral knowledge-based data classification, a number of researchers have studied the set containment characterization; see [1–5]. Jeyakumar [3] established the set containment characterization for convex programming, under the convexity of f_i, $i \in I$ and the linearity (or the concavity) of h_j, $j \in J$, using the epigraph of the Fenchel conjugate of a convex function. Suzuki and Kuroiwa [5] established the set containment characterization using the H-quasiconjugate and the R-quasiconjugate, when each function is

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quasiconvex. Although the class of quasiconvex functions is wider than that of convex functions, the convex characterization of Jeyakumar [3] cannot be shown by the quasiconvex characterization of Suzuki and Kuroiwa [5].

Recently, constraint qualifications have been investigated using set containment characterizations. In convex optimization, a constraint qualification is an essential ingredient of the elegant and powerful duality theory. The best-known constraint qualifications are the interior point conditions, also known as the Slater-type constraint qualifications. Often, however, such constraint qualifications are not satisfied for problems that arise in applications. The lack of a constraint qualification can cause theoretical and numerical difficulties in applications. For convex programs, Jeyakumar, Dinh, and Lee [6] developed the closed cone constraint qualification, involving epigraphs and extending the Slater-type conditions. Constraint qualifications involving epigraphs have been used extensively in various studies; see [7–11]. Such constraint qualifications concern Jeyakumar's set containment characterization.

In the present paper, we consider set containment characterizations and the new closed cone constraint qualification for quasiconvex programming. Furthermore, we show a duality theorem for quasiconvex programming with the new constraint qualification. We define a notion of generator for quasiconvex functions through Penot and Volle's characterization in [12], and investigate the set containment characterization and the constraint qualification.

The remainder of the present paper is organized as follows. In Section 2, we introduce Penot and Volle's characterization and define a generator of quasiconvex functions. Furthermore, we introduce Jeyakumar's set containment characterizations and Lagrange duality theorem for convex programming. Section 3 introduces set containment characterizations for quasiconvex programming, from which we define a new closed cone constraint qualification in Section 4.

2 Preliminaries: Generator of Quasiconvex Function

Let X be a locally convex Hausdorff topological vector space, let X^* be the continuous dual space of X, and let $\langle x^*, x\rangle$ denote the value of a functional $x^* \in X^*$ at $x \in X$. Given a set $S \subset X^*$, we denote the weak*-closure, the convex hull, and the conical hull generated by S, by $\mathrm{cl}\, S$, $\mathrm{conv}\, S$, and $\mathrm{cone}\, S$, respectively. Throughout the paper, let f be a function from X to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, \infty]$. Here, f is said to be proper iff for all $x \in X$, $f(x) > -\infty$ and there exists $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\mathrm{dom}\, f$, that is, $\mathrm{dom}\, f := \{x \in X \mid f(x) \in \mathbb{R}\}$. The epigraph of f is $\mathrm{epi}\, f := \{(x, r) \in X \times \mathbb{R} \mid x \in \mathrm{dom}\, f, f(x) \le r\}$, and f is said to be convex iff $\mathrm{epi}\, f$ is convex. The Fenchel conjugate of f, $f^* : X^* \to \overline{\mathbb{R}}$, is defined as $f^*(u) := \sup_{x\in \mathrm{dom}\, f}\{\langle u, x\rangle - f(x)\}$. Remember that f is said to be quasiconvex iff for all x_1, $x_2 \in X$ and $\alpha \in]0, 1[$,

$$f((1-\alpha)x_1 + \alpha x_2) \le \max\{f(x_1), f(x_2)\}.$$

Define level sets of f with respect to a binary relation $\diamond$ on $\mathbb{R}$ as

$$\operatorname{lev}(f, \diamond, \alpha) := \{x \in X \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. Then, f is quasiconvex iff for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(f, \leq, \alpha)$ is a convex set, or equivalently, for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(f, <, \alpha)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true.

A function f is said to be quasiaffine iff f is quasiconvex and quasiconcave. Furthermore, f is lsc quasiaffine iff there exist $k \in Q$ and $w \in X^*$ such that $f = k \circ w$, where $Q := \{k : \mathbb{R} \to \overline{\mathbb{R}} \mid k \text{ is lsc and non-decreasing}\}$; see [12].

We introduce the following characterization theorem of the quasiconvex function.

Theorem 2.1 *[12] Let f be a function from X to $\overline{\mathbb{R}}$. Then, the following statements are equivalent:*

(i) f is lsc quasiconvex,
(ii) there exist a set I, $\{k_i\} \subset Q$ and $\{w_i\} \subset X^*$, such that $f = \sup_{i \in I} k_i \circ w_i$.

Theorem 2.1 indicates that a lsc quasiconvex function consists of a supremum of some family of lsc quasiaffine functions. We define the notion of generator of quasiconvex functions.

Definition 2.1 Let f be a function from X to $\overline{\mathbb{R}}$. A subset $\{(k_i, w_i) \mid i \in I\}$ of $Q \times X^*$ is said to be a generator of f iff $f = \sup_{i \in I} k_i \circ w_i$.

By using Theorem 2.1, we can prove that all quasiconvex functions have at least one generator. For example, when $f \in \Gamma_0(X)$, the set of all proper lsc convex functions from X to $\overline{\mathbb{R}}$, $B_f := \{(k_v, v) \mid v \in \operatorname{dom} f^*\} \subset Q \times X^*$, where $k_v(t) = t - f^*(v)$, is a generator of f. Actually, for all $x \in X$,

$$f(x) = f^{**}(x) = \sup\{\langle v, x\rangle - f^*(v) \mid v \in \operatorname{dom} f^*\} = \sup_{v \in \operatorname{dom} f^*} k_v(\langle v, x\rangle).$$

We call the generator B_f "the basic generator" of the convex function f.

Moreover, Penot and Volle [12] studied generalized concepts of the inverse of non-decreasing functions. Since now, the set of non-decreasing functions on $\mathbb{R}$ is denoted by G; that is, $G := \{k : \mathbb{R} \to \overline{\mathbb{R}} \mid k \text{ is non-decreasing}\}$. A function $l : \mathbb{R} \to \overline{\mathbb{R}}$ is said to be the hypo-epi-inverse of $k \in G$ iff for any r, s in $\mathbb{R}$, $s \leq l(r) \iff k(s) \leq r$; it is known that the hypo-epi-inverse of k is uniquely defined. We will denote the hypo-epi-inverse of k by k^{-1}, because, if k has an inverse function, then the inverse function and the hypo-epi-inverse of k are the same. Furthermore, for any $k \in Q$, we can calculate k^{-1}, as follows:

$$k^{-1}(r) = \inf\{t \in \mathbb{R} \mid r < k(t)\} = \sup\{s \in \mathbb{R} \mid k(s) \leq r\}.$$

In [3], the following set containment characterizations were proved.

Theorem 2.2 *[3] Let I be an arbitrary set, and, for each $i \in I$, let g_i be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$. In addition, let $\{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\}$ be a non-empty set, $u \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Then, (i) and (ii) given below are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\} \subset \{x \in \mathbb{R}^n \mid \langle u, x\rangle \leq \alpha\}$,
(ii) $(u, \alpha) \in \operatorname{cl} \operatorname{cone} \operatorname{conv} \bigcup_{i \in I} \operatorname{epi} g_i^*$.

Theorem 2.3 *[3] Let I be an arbitrary set, and, for each $i \in I$, let g_i be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$, $\{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\} \neq \emptyset$. In addition, let h be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, (i) and (ii) given below are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\} \subset \{x \in \mathbb{R}^n \mid h(x) \geq 0\}$,
(ii) $(0, 0) \in \operatorname{epi} h^* + \operatorname{cl} \operatorname{cone} \operatorname{conv} \bigcup_{i \in I} \operatorname{epi} g_i^*$.

In [10], Farkas-Minkowski (FM) was investigated. Let I be an index set, for each $i \in I$, let $g_i \in \Gamma_0(X)$. The convex system $\{g_i(x) \leq 0 \mid i \in I\}$ is said to be FM iff the characteristic cone

$$\operatorname{cone} \operatorname{conv} \bigcup_{i \in I} \operatorname{epi} g_i^*$$

is w^*-closed. Furthermore, the following Lagrange duality theorem was proved.

Theorem 2.4 *[10] Let I be an index set, for each $i \in I$, let $g_i \in \Gamma_0(X)$. Assume that $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\} \neq \emptyset$. Then, the following statements are equivalent:*

(i) $\{g_i(x) \leq 0 \mid i \in I\}$ *is FM,*
(ii) for all $v \in X^*$, $\inf_{x \in A} \langle v, x\rangle = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in X} \{\langle v, x\rangle + \sum_{i \in I} \lambda_i g_i(x)\}$,
(iii) for all $h \in \Gamma_0(X)$ *with* $\operatorname{dom} h \cap A \neq \emptyset$, *where* $\operatorname{epi} h^* + \operatorname{epi} \delta_A^*$ *is* w^**-closed,*

$$\inf_{x \in A} h(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in X} \Big\{h(x) + \sum_{i \in I} \lambda_i g_i(x)\Big\},$$

where, $\mathbb{R}_+^{(I)} := \{\lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \geq 0, \{i \in I \mid \lambda_i \neq 0\}$ *is finite*$\}$.

3 Set Containment Characterization

In this section, we show two set containment characterizations for quasiconvex constraints. First, we present a characterization of a containment of a convex set, defined by a quasiconvex constraint, in a closed half-space.

Theorem 3.1 *Let f be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with a generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$, $u \in X^*$, and α, $\beta \in \mathbb{R}$. Assume that there exists $i_0 \in I$ such that $k_{i_0}^{-1}(\beta) \in \mathbb{R}$ and $\{x \in X \mid f(x) \leq \beta\} \neq \emptyset$. Then, (i) and (ii) given below are equivalent:*

(i) $\{x \in X \mid f(x) \leq \beta\} \subset \{x \in X \mid \langle u, x\rangle \leq \alpha\}$,
(ii) $(u, \alpha) \in \operatorname{cl} \operatorname{cone} \operatorname{conv} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$.

Proof Put $g_i : X \to \mathbb{R}$, $g_i(x) = w_i(x) - k_i^{-1}(\beta)$, then

$$A = \{x \in X \mid \forall i \in I, k_i \circ w_i(x) \leq \beta\} = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}.$$

Since g_i is continuous and linear, we can show the following equation by using Theorem 2.2:

$$\operatorname{epi} \delta_A^* = \operatorname{cl} \operatorname{cone} \operatorname{conv} \bigcup_{i \in I} \operatorname{epi} g_i^*.$$

Clearly $\operatorname{epi} g_i^* = \{w_i\} \times [k_i^{-1}(\beta), \infty[$; this completes the proof. □

A generator of the quasiconvex function, obtained in Theorem 2.1, is not unique, and any lsc quasiconvex function has infinite generators. However, the set in condition (ii) of Theorem 3.1 does not depend on the generator of the function f. When subsets $\{(k_i, w_i) \mid i \in I\}$ and $\{(l_j, u_j) \mid j \in J\}$ of $Q \times X^*$ are generators of f, then, by using Theorem 3.1, the following three conditions can be proved to be equivalent:

(i) $\{x \in X \mid f(x) \leq \beta\} \subset \{x \in X \mid \langle u, x\rangle \leq \alpha\}$,

(ii) $(u, \alpha) \in \operatorname{cl} \operatorname{cone} \operatorname{conv} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$,

(iii) $(u, \alpha) \in \operatorname{cl} \operatorname{cone} \operatorname{conv} \bigcup_{j \in J} \{(u_j, \delta) \in X^* \times \mathbb{R} \mid l_j^{-1}(\beta) \leq \delta\}$.

Moreover, the set in the above condition (ii) is equal to $\operatorname{epi} \delta_A^*$, that is

$$\operatorname{epi} \delta_A^* = \operatorname{cl} \operatorname{cone} \operatorname{conv} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\},$$

where $A = \operatorname{lev}(f, \leq, \beta)$, and δ_A is the indicator function of A. This equation is very important to define the newly proposed CCCQ in the way described in [10].

In the present paper, we consider the quasiconvex set containment characterization with singular constraint $\{x \in X \mid f(x) \leq \beta\}$. When we consider multiple constraints, that is, $\{x \in X \mid \forall i \in I,\ g_i(x) \leq \beta\}$, we can obtain the singular constraint $\{x \in X \mid \sup_{i \in I} g_i \leq \beta\}$, in which the generator of $\sup_{i \in I} g_i$ is the union of generators of g_i.

Next, we present a characterization of the containment of a convex set, defined by a quasiconvex constraint, in a reverse convex set, as defined by a convex constraint.

Theorem 3.2 *Let f be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with a generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$, and let $g \in \Gamma_0(X)$. In addition, let $\beta \in \mathbb{R}$. Assume that $A = \{x \in X \mid f(x) \leq \beta\} \neq \emptyset$, $\operatorname{dom} g \cap A \neq \emptyset$, $\operatorname{epi} g^* + \operatorname{epi} \delta_A^*$ is w^*-closed and that there exists $i_0 \in I$ such that $k_{i_0}^{-1}(\beta) \in \mathbb{R}$. Then, (i) and (ii) are equivalent:*

(i) $\{x \in X \mid f(x) \leq \beta\} \subset \{x \in X \mid g(x) \geq 0\}$,

(ii) $(0, 0) \in \operatorname{epi} g^* + \operatorname{cl} \operatorname{cone} \operatorname{conv} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$.

Proof Assume that the condition (ii) holds. Based on the above assumption and Theorem 3.1, we can show that

$$(0,0) \in \operatorname{epi} g^* + \operatorname{epi} \delta_A^* = \operatorname{epi} (g + \delta_A)^*;$$

see [13]. This shows $\inf_{x \in A} g(x) \geq 0$, that is, condition (i) holds. The inverse implication is similar. □

4 Closed Cone Constraint Qualification

In optimization, research on constraint qualification is very important, and, in convex optimization, closed cone constraint qualification has been investigated extensively; see [6,7,10,11]. In the present paper, we investigate a new closed cone constraint qualification for quasiconvex programming.

Definition 4.1 Let f be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$, and $A = \{x \in X \mid f(x) \leq 0\} \neq \emptyset$. Assume that there exists $i_0 \in I$ such that $k_{i_0}^{-1}(0) \in \mathbb{R}$. Then, the quasiconvex system $\{f(x) \leq 0\}$ satisfies the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ) with respect to $\{(k_i, w_i) \mid i \in I\}$ iff

$$\operatorname{cone}\operatorname{conv} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(0) \leq \delta\}$$

is w^*-closed.

As a consequence of Theorem 3.1, $\{f(x) \leq 0\}$ satisfies the Q-CCCQ with respect to $\{(k_i, w_i) \mid i \in I\}$ iff the alternative form of the Q-CCCQ, that is

$$\operatorname{epi} \delta_A^* \subset \operatorname{cone}\operatorname{conv} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(0) \leq \delta\},$$

holds.

Theorem 4.1 *Let f be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with a generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$. Assume that $A = \{x \in X \mid f(x) \leq 0\} \neq \emptyset$. Then, the following statements are equivalent:*

(i) $\{f(x) \leq 0\}$ *satisfies the Q-CCCQ w.r.t.* $\{(k_i, w_i) \mid i \in I\}$,

(ii) for all $v \in X^*$,

$$\inf_{x \in A} \langle v, x \rangle = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in \mathbb{R}^n} \left\{ \langle v, x \rangle + \sum_{i \in I} \lambda_i (\langle w_i, x \rangle - k_i^{-1}(0)) \right\},$$

(iii) for all $h \in \Gamma_0(X)$ *with* $\operatorname{dom} h \cap A \neq \emptyset$ *and* $\operatorname{epi} h^* + \operatorname{epi} \delta_A^*$ *is* w^**-closed,*

$$\inf_{x \in A} h(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in \mathbb{R}^n} \left\{ h(x) + \sum_{i \in I} \lambda_i (\langle w_i, x \rangle - k_i^{-1}(0)) \right\}.$$

Proof Put $g_i : X \to \mathbb{R}$, $g_i(x) = w_i(x) - k_i^{-1}(0)$; then

$$A = \{x \in X \mid \forall i \in I, k_i \circ w_i(x) \leq 0\} = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\},$$

and we can check that (i) is equivalent to "$\{g_i(x) \leq 0\}$ is FM". Since g_i is continuous and linear, then, by using Theorem 2.4, we can prove that (i), (ii) and (iii) are equivalent. □

In the rest of this section, we compare the Q-CCCQ with FM by Venn diagram. At first, we show an example where FM is satisfied.

Example 4.1 Let $X = \mathbb{R}$ and $f(x) = |x|$. Then, we can check that f is convex and f^* is as follows:

$$f^*(v) = \begin{cases} 0, & \text{if } v \in [-1, 1], \\ \infty, & \text{if } v \notin [-1, 1]. \end{cases}$$

Furthermore,

$$\operatorname{cone}\operatorname{conv}\operatorname{epi} f^* = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\},$$

that is, FM is satisfied.

Next, we show an example of convex programming where Q-CCCQ is satisfied, but FM is not.

Example 4.2 Let X be a normed space, $f(x) = \frac{1}{2}\|x\|^2$. Then, we can see that f is convex, $f^*(v) = \frac{1}{2}\|v\|^2$, and

$$\operatorname{cone}\operatorname{conv}\operatorname{epi} f^* = (X^* \times]0, \infty[) \cup \{(0, 0)\},$$

that is, FM is not satisfied. However, we can choose a generator, which satisfies the Q-CCCQ.

Let $S_* = \{w \in X^* \mid \|w\| = 1\}$, and $k \in Q$ as follows:

$$k(t) = \begin{cases} \frac{1}{2}t^2, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then, $f = \sup_{w \in S_*} k \circ w$, $k^{-1}(0) = 0$, and the equality

$$\operatorname{cone}\operatorname{conv} \bigcup_{w \in S_*} \{(w, \delta) \mid k^{-1}(0) \leq \delta\} = X^* \times [0, \infty[$$

holds, that is, $\{f(x) \leq 0\}$ satisfies the Q-CCCQ w.r.t. $\{(k, w) \mid w \in S_*\}$.

Example 4.2 indicates that the Q-CCCQ and a representation of constraint functions by using the notion of generator are useful to convex programming. We give the following example which shows that the Q-CCCQ is satisfied in a quasiconvex, but not convex, problem.

Example 4.3 Let $X = \mathbb{R}^n$, $a \in \mathbb{R}^n$, and $f(x) = \sqrt{\|x-a\|} - 2$. Then, f is quasiconvex with the following generator: let $S_* = \{w \in \mathbb{R}^n \mid \|w\| = 1\}$, $k_w \in Q$ as follows:

$$k_w(t) = \begin{cases} \sqrt{t - \langle w, a \rangle} - 2, & \text{if } t > \langle w, a \rangle, \\ 0, & \text{if } t \leq \langle w, a \rangle. \end{cases}$$

Then, $f = \sup_{w \in S_*} k_w \circ w$, $k_w^{-1}(0) = \langle w, a \rangle$, and the equality

$$\operatorname{epi} \delta_A^* = \operatorname{cone} \operatorname{conv} \bigcup_{w \in S_*} \{(w, \delta) \mid k_w^{-1}(0) \leq \delta\}$$

holds, that is, $\{f(x) \leq 0\}$ satisfies the Q-CCCQ w.r.t. $\{(k_w, w) \mid w \in S_*\}$.

We show the following example where Q-CCCQ is not satisfied in convex programming.

Example 4.4 Let $X = \mathbb{R}^2$, $I =]0, 1]$, $w_i = (-i, i-1)$, k_i be a function as follows:

$$k_i(t) = \begin{cases} it, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

and $f = \sup_{i \in I} k_i \circ w_i$. It is clear that $\{f(x) \leq 0\}$ is not satisfies Q-CCCQ w.r.t. $\{(k_i, w_i) \mid i \in I\}$. Even if $\mathcal{G} = \{(k, w) \mid k \in Q, w \in \mathbb{R}^2, k \circ w \leq f\}$, it is the biggest generator of f, then f does not satisfy the Q-CCCQ w.r.t. $\mathcal{G}$. Indeed, $\operatorname{epi}\sigma_A = \{(x, \alpha) \mid x \in -\mathbb{R}^2_+, \alpha \geq 0\}$, and

$$((0, -1), 0) \notin \operatorname{cone} \operatorname{conv} \{(w, \delta) \mid k^{-1}(0) \leq \delta, k \circ w \leq f\} + \{0\} \times [0, \infty[$$

since if $w = (0, -1)$ and $k \circ w \leq f$, then $k \leq 0$, that is $k^{-1}(0) = \infty$. Furthermore, we can check that FM is not satisfied since the basic generator is included in $\mathcal{G}$.

Finally, we show the following example where Q-CCCQ is not satisfied in quasiconvex programming.

Example 4.5 Let $X = \mathbb{R}^2$, $I =]0, 1]$, $w_i = (-i, i-1)$, k_i be a function as follows:

$$k_i(t) = \begin{cases} it, & \text{if } t > 0, \\ -1, & \text{if } t \leq 0, \end{cases}$$

and $f = \sup_{i \in I} k_i \circ w_i$. Then, f is quasiconvex. We can check similarly that Q-CCCQ w.r.t. $\{(k_i, w_i) \mid i \in I\}$ and Q-CCCQ w.r.t. $\mathcal{G}$ are not satisfied.

Figure 1 summarizes the results illustrated by the examples.

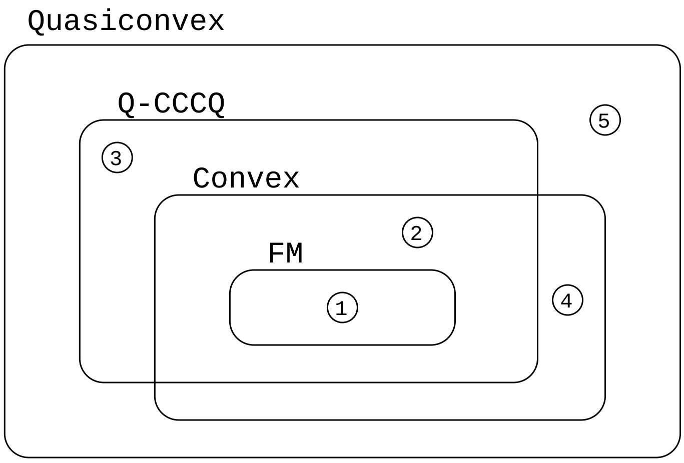

Fig. 1 Venn diagram of constraint qualifications

In the above diagram, we compare the Q-CCCQ with FM which is the weakest constraint qualification for Lagrange (strong) duality theorem in convex programming in order to focus on the relation between quasiconvex programming and convex programming. We can see similar scheme in [14, 15] by Pellegrini and Moldovan. Pellegrini and Moldovan investigated constraint qualifications and regularity conditions for Lagrange min-max duality theorem which is similar to, but different from, our problem.

5 Concluding Remarks

In this paper, we define the generator of quasiconvex functions by the result of Penot and Volle in [12], and investigate dual characterizations of the containment of a convex set with quasiconvex inequality constraints. We describe a new Lagrange-type duality, and the new closed cone constraint qualification, the Q-CCCQ, and we show that the Q-CCCQ is the weakest constraint qualification for the new Lagrange-type duality.

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