Optimality conditions and the basic constraint qualification for quasiconvex programming

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Abstract

In this paper, we consider optimality conditions and a constraint qualification for quasiconvex programming. To the purpose, we introduce a generator and a new subdifferential for quasiconvex functions by using Penot and Volle's theorem.

Keywords:
quasiconvex programming, optimality condition, constraint qualification, subdifferential

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1. Introduction

We consider the following minimization programming problem:

$$\begin{cases} \text{minimize } f(x), \\ \text{subject to } g_i(x) \le 0, \forall i \in I, \end{cases}$$

where I is an arbitrary set, f and g are extended real-valued functions from locally convex Hausdorff topological vector space X. When f and g_i are convex and $x_0 \in A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\}$, the following equivalence relation holds under some constraint qualifications:

$$f(x_0) = \inf_{x \in A} f(x) \iff \exists \lambda \in \mathbb{R}_+^{(I)} \text{ s.t. } 0 \in \partial f(x_0) + \sum_{i \in I} \lambda_i \partial g_i(x_0),$$

where $\mathbb{R}_+^{(I)} = \{\lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \ge 0, \{i \in I \mid \lambda_i \ne 0\} : \text{finite}\}$. The research of constraint qualifications for this optimality condition have been studied by many researchers. Recently, the basic constraint qualification (the BCQ) was proposed by Li, Ng and Pong [6]. The BCQ is said to be the weakest constraint qualification for this optimality condition because the BCQ and this optimality condition are equivalent.

The purpose of this paper is to generalize the result of [6] for quasiconvex programming. In quasiconvex optimization, Penot and Volle [8] reported an interesting result whereby a lower semi-continuous quasiconvex function consists of a supremum of some family of lower semi-continuous quasiaffine functions. This result is fundamental and useful for our purpose.

In the present paper, we consider optimality conditions and the basic constraint qualification for quasiconvex programming. By using Penot and Volle's theorem, we introduce a notion called "generator" and a new subdifferential for quasiconvex functions, and investigate generalized results reported in previous studies.

The remainder of the present paper is organized as follows. In Section 2, we introduce Penot and Volle's theorem, and introduce a notion "generator" for quasiconvex functions. In Section 3, we define a new subdifferential for quasiconvex functions and investigate an optimality condition for quasiconvex programming with a set constraint by using the subdifferential. In Section 4, we define a new constraint qualification called the basic constraint qualification for quasiconvex programming (the Q-BCQ), and consider an optimality condition for quasiconvex programming with inequality constraints. Also, we prove that the Q-BCQ is the weakest constraint qualification for this optimality condition. Finally, in Section 5, we emphasize the usefulness of our results in this paper.

2. Preliminaries

Let X be a locally convex Hausdorff topological vector space. In addition, let X^* be the continuous dual space of X, and let $\langle x^*, x\rangle$ denote the value of a functional $x^* \in X^*$ at $x \in X$. Given a set $Y \subset X^*$, we denote the weak*-closure, the interior, the convex hull, and the conical hull generated by Y, by $\mathrm{cl}Y$, $\mathrm{int}Y$, $\mathrm{co}Y$, and $\mathrm{cone}Y$, respectively. For convex subset A of X, the tangent cone and the normal cone of A at $z_0 \in A$ is denoted by $T_A(z_0) = \mathrm{cl} \cup_{\lambda > 0} \frac{A - z_0}{\lambda}$, and $N_A(z_0) = \{x^* \in X^* \mid \forall y \in A, \langle x^*, y - z_0\rangle \leq 0\}$. The indicator function δ_A and the support function σ_A of A are respectively defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ \infty & \textit{otherwise}, \end{cases}$$

and

$$\sigma_A(x^*) = \sup_{x \in A} \langle x^*, x \rangle \text{ for each } x^* \in X^*.$$

Throughout the present paper, let f be a function from X to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Here, f is said to be proper if for all $x \in X$, $f(x) > -\infty$ and there exists $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\mathrm{dom} f$, that is, $\mathrm{dom} f = \{x \in X \mid f(x) < +\infty\}$. The epigraph of f, $\mathrm{epi} f$, is defined as $\mathrm{epi} f = \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\}$, and f is said to be convex if $\mathrm{epi} f$ is convex. In addition, the Fenchel conjugate of f, $f^* : X^* \to \overline{\mathbb{R}}$, is defined as $f^*(v) = \sup\{\langle v, x \rangle - f(x) \mid x \in \mathrm{dom} f\}$. Remember that f is said to be quasiconvex if for all x, $y \in X$ and $\alpha \in (0, 1)$,

$$f((1-\alpha)x + \alpha y) \leq \max\{f(x), f(y)\}.$$

Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \alpha) = \{x \in X \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. Then, f is quasiconvex if and only if for any $\alpha \in \mathbb{R}$, $L(f, \leq, \alpha)$ is a convex set, or equivalently, for any $\alpha \in \mathbb{R}$, $L(f, <, \alpha)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true.

It is well known that a proper lsc convex function consists of a supremum of some family of affine functions. In the case of quasiconvex functions, a similar result was also proved by Penot and Volle [8]. First, we introduce a notion of quasiaffine function. A function f is said to be quasiaffine if quasiconvex and quasiconcave. It is worth noting that f is lsc quasiaffine if and only if there exists $k \in Q$ and $w \in X^*$ such that $f = k \circ w$, where $Q = \{h : \mathbb{R} \to \overline{\mathbb{R}} \mid h \text{ is lsc and non-decreasing}\}$. By using a notion of quasiaffine function, Penot and Volle proved that f is lsc quasiconvex if and only if there exists $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ such that $f = \sup_{i \in I} k_i \circ w_i$. This result indicates that a lsc quasiconvex function f consists of a supremum of some family of lsc quasiaffine functions. Based on this result, in [11], we define a notion of generator for quasiconvex functions, that is, $G = \{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ is said to be a generator of f if $f = \sup_{i \in I} k_i \circ w_i$. Because Penot and Volle's result, all lsc quasiconvex functions have at least one generator. Also, when f is a proper lsc convex function, $B_f = \{(k_v, v) \mid v \in \mathrm{dom} f^*, k_v(t) = t - f^*(v), \forall t \in \mathbb{R}\} \subset Q \times X^*$ is a generator of f. Actually, for all $x \in X$,

$$f(x) = f^{**}(x) = \sup\{\langle v, x \rangle - f^*(v) \mid v \in \mathrm{dom} f^*\} = \sup_{v \in \mathrm{dom} f^*} k_v(\langle v, x \rangle).$$

We call the generator B_f "the basic generator" of convex function f. The basic generator is very important with respect to the comparison of convex and quasiconvex programming.

Moreover, we introduce a generalized notion of inverse function of $h \in Q$. The following function h^{-1} is said to be the hypo-epi-inverse of h:

$$h^{-1}(a) = \inf\{b \in \mathbb{R} \mid a < h(b)\} = \sup\{b \in \mathbb{R} \mid h(b) \leq a\}.$$

If h has an inverse function, then the inverse and the hypo-epi-inverse of h are the same, in detail see [8]. In the present paper, we denote the hypo-epi-inverse of h by h^{-1}. Also, we denote the lower left-hand Dini derivative of $h \in Q$ at t by $D_{-}h(t)$, that is $D_{-}h(t) = \liminf_{\varepsilon \to 0-} \frac{h(t+\varepsilon)-h(t)}{\varepsilon}$. A function h is said to be lower left-hand Dini differentiable if $D_{-}h(t)$ is finite for all $t \in \mathbb{R}$.

3. Subdifferential and an optimality condition

In this section, we introduce a new subdifferential for quasiconvex function, and by using this subdifferential, we investigate an optimality condition for quasiconvex programming with a set constraint.

At first, we introduce the new subdifferential for quasiconvex functions.

Definition 1. *Let f be a lsc quasiconvex function with a generator $G = \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*$, and assume that k_s is lower left-hand Dini differentiable for all $s \in S$. Then, we define the subdifferential of f at x_0 with respect to G as follows:*

$$\partial_G f(x_0) = \mathrm{clco}\{D_{-}k_s(\langle w_s, x_0\rangle)w_s \mid s \in S(x_0)\},$$

where $S(x_0) = \{s \in S \mid f(x_0) = k_s \circ w_s(x_0)\}$.

This subdifferential is a generalized notion of the subdifferential for convex functions. Actually, if f is a convex function with the basic generator B_f, then

$$\begin{aligned}
\partial_{B_f} f(x_0) &= \mathrm{clco}\{D_{-}k_v(\langle v, x_0\rangle)v \mid v \in \mathrm{dom} f^*, f(x_0) = k_v(\langle v, x_0\rangle)\} \\
&= \mathrm{clco}\{v \mid v \in \mathrm{dom} f^*, f(x_0) = \langle v, x_0\rangle - f^*(v)\} \\
&= \partial f(x_0).
\end{aligned}$$

Also, if f is Gâteaux differentiable at x_0, k_s are differentiable at $\langle w_s, x_0\rangle$ for all $s \in S(x_0)$, and $S(x_0) \neq \emptyset$, then we can check $\partial_G f(x_0) = \{f'(x_0)\}$. Actually, for all $s \in S(x_0)$ and $d \in X$,

$$\begin{aligned}\langle f'(x_0), d\rangle &= \lim_{t\to 0} \frac{f(x_0+td)-f(x_0)}{t} \\ &\geq \lim_{t\to 0} \frac{k_s \circ w_s(x_0+td) - k_s \circ w_s(x_0)}{t} \\ &= \langle k_s'(\langle w_s, x_0\rangle), d\rangle \,.\end{aligned}$$

Similarly, we can prove that $\langle f'(x_0), -d\rangle \geq \langle k_s'(\langle w_s, x_0\rangle)w_s, -d\rangle$, that is, $f'(x_0) = k_s'(\langle w_s, x_0\rangle)w_s$.

Next, we show a necessary condition for a minimizer of a certain quasiconvex function in a closed convex set.

Theorem 1. *Let A be a closed convex subset of X, f be a lsc quasiconvex function with a generator $G = \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*$. Assume that k_s is lower left-hand Dini differentiable for all $s \in S$ and at least one of the following holds:*

(i) S is finite and k_s is continuous for all $s \in S$,

(ii) X is a Banach space, S is a compact topological space, $s \mapsto w_s$ is continuous on S to $(X^*, \|\cdot\|)$, $(s,t) \mapsto k_s(t)$ is usc on $S \times \mathbb{R}$, and $(s,t) \mapsto D_- k_s(t)$ is continuous on $S \times \mathbb{R}$.

If x_0 is a local minimizer of f in A then,

$$0 \in \partial_G f(x_0) + N_A(x_0).$$

PROOF. At first, we show that $\partial_G f(x_0)$ is w^*-compact. It is clear when the condition (i) holds. If the condition (ii) holds, then $S(x_0)$ is compact because $S(x_0) = \{s \in S \mid f(x_0) \leq k_s \circ w_s(x_0)\}$ and $s \mapsto k_s \circ w_s(x_0)$ is usc on S. Thus, $\{D_- k_s(\langle w_s, x_0\rangle)w_s \mid s \in S(x_0)\}$ is bounded since $s \mapsto w_s$ is continuous on S and $(s,t) \mapsto D_- k_s(t)$ is continuous on $S \times \mathbb{R}$. Hence, $\partial_G f(x_0)$ is w^*-compact by the Banach-Alaoglu theorem.

Now we assume that $0 \notin \partial_G f(x_0) + N_A(x_0)$. Since $\partial_G f(x_0) + N_A(x_0)$ is w^*-closed, we can find $d_0 \in X \setminus \{0\}$ satisfying

$$\langle y^*, d_0\rangle < 0 \leq \langle -x^*, d_0\rangle \,,$$

for all $y^* \in \partial_G f(x_0)$ and $x^* \in N_A(x_0)$. If $s \in S(x_0)$, then $D_- k_s(\langle w_s, x_0\rangle) > 0$ and $\langle w_s, d_0\rangle < 0$ since $D_- k_s(\langle w_s, x_0\rangle) w_s \in \partial_G f(x_0)$ and k_s is non-decreasing. From this, we have $\sup_{s\in S(x_0)} \langle w_s, d_0\rangle < 0$ and $d \mapsto \sup_{s\in S(x_0)} \langle w_s, d\rangle$ is usc. Indeed, it is clear when the condition (i) holds. If the condition (ii) holds, we can check them since $S(x_0)$ is compact and $s \mapsto w_s$ is continuous on S.

Therefore, there exists U_{d_0} a neighborhood of d_0 such that $\langle w_s, d\rangle < 0$ for all $s \in S(x_0)$ and $d \in U_{d_0}$. Since x_0 is a local minimizer of f in A, there exists U_{x_0} a neighborhood of x_0 such that for all $x \in U_{x_0} \cap A$, $f(x_0) \le f(x)$.

Also $d_0 \in T_A(x_0) = \mathrm{cl} \cup_{\lambda>0} \frac{A-x_0}{\lambda}$ because $\langle x^*, d_0\rangle \le 0$ for all $x^* \in N_A(x_0)$. Then there exist $d_1 \in U_{d_0}$, $\lambda_0 > 0$ and $x_1 \in A$ such that $d_1 = \frac{x_1-x_0}{\lambda_0}$. Put $x_n = (1-\frac{1}{n})x_0 + \frac{1}{n}x_1 = x_0 + \frac{\lambda_0}{n}d_1$, then $x_n \in A \cap U_{x_0}$ for large enough n, therefore $f(x_0) \le f(x_n)$.

If the condition (i) holds, since S is finite, we can find $s_0 \in S$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $s_0 \in S(x_{n_i})$ for all $i \in \mathbb{N}$, and we have $s_0 \in S(x_0)$ because f and $k_{s_0} \circ w_{s_0}$ are continuous. For large enough $i \in \mathbb{N}$, $k_{s_0} \circ w_{s_0}(x_{n_i}) = f(x_{n_i}) \ge f(x_0) = k_{s_0} \circ w_{s_0}(x_0)$, and then,

$$\frac{k_{s_0}(\langle w_{s_0}, x_0\rangle + \frac{\lambda_0}{n_i}\langle w_{s_0}, d_1\rangle) - k_{s_0}(\langle w_{s_0}, x_0\rangle)}{\frac{\lambda_0}{n_i}\langle w_{s_0}, d_1\rangle} \le 0,$$

since $d_1 \in U_{d_0}$. Therefore $D_- k_{s_0}(\langle w_{s_0}, x_0\rangle) \le 0$, it is contradiction.

If the condition (ii) holds, all $S(x_n)$ are not empty because S is compact and $s \mapsto k_s \circ w_s(x_n)$ is usc on S. Let $\{s_n\}$ be a sequence satisfying $s_n \in S(x_n)$ for all $n \in \mathbb{N}$. Then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $\{s_{n_i}\}$ converges to some $s_0 \in S$. Therefore

$$\begin{aligned} f(x_0) &\le \liminf_{i\to\infty} f(x_{n_i}) \\ &\le \limsup_{i\to\infty} k_{s_{n_i}} \circ w_{s_{n_i}}(x_{n_i}) \\ &\le k_{s_0} \circ w_{s_0}(x_0) \\ &\le f(x_0), \end{aligned}$$

that is, $s_0 \in S(x_0)$. Then, for sufficiently large $i \in \mathbb{N}$, $k_{s_{n_i}} \circ w_{s_{n_i}}(x_{n_i}) = f(x_{n_i}) \ge f(x_0) \ge k_{s_{n_i}} \circ w_{s_{n_i}}(x_0)$ and $\langle w_{s_{n_i}}, d_1\rangle < 0$, because $s_0 \in S(x_0)$, $d_1 \in U_{d_0}$ and $\{w_{s_{n_i}}\}$ converges w_{s_0}. From this and $k_{s_{n_i}}$ is non-decreasing, $k_{s_{n_i}}$ is constant on interval $[\langle w_{s_{n_i}}, x_0\rangle + \frac{\lambda_0}{n_i}\langle w_{s_{n_i}}, d_1\rangle, \langle w_{s_{n_i}}, x_0\rangle]$ and hence we have $D_- k_{s_{n_i}}(\langle w_{s_{n_i}}, x_0\rangle) = 0$. Finally we obtain $D_- k_{s_0}(\langle w_{s_0}, x_0\rangle) = 0$, but this is a contradiction.

On the other hand, in separable Banach space, a similar result was introduced when S is compact, f_s are locally Lipschitz, $f = \sup_{s\in S} f_s$, and some assumption hold in [9]. If condition (ii) holds and k_s are differentiable, then $k_s \circ w_s$ are locally Lipschitz. However, in Theorem 1, we assume that X is a usual Banach space and k_s are only lower left-hand Dini differentiable, thus, Theorem 1 is not a direct consequence of the result in [9]. Also, if f is a proper lsc convex function with basic generator B_f and $\mathrm{dom} f^*$ is compact, then condition (ii) holds. For this reason, it seems that condition (ii) is not so strong for quasiconvex programming.

4. The basic constraint qualification

In this Section, we define a new constraint qualification and consider an optimality condition for quasiconvex programming with inequality constraints, and we prove that the new constraint qualification is the weakest constraint qualification for the optimality condition.

We introduce the following new constraint qualification.

Definition 2. *Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, for each $i \in I$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i, $T = \{t = (i,j) \mid i \in I, j \in J_i\}$, $T(x) = \{t \in T \mid k_t(\langle w_t, x\rangle) = 0$, $k_t^{-1}(0) = \langle w_t, x\rangle\}$, and $A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\}$.*

The family $\{g_i \mid i \in I\}$ is said to satisfy the basic constraint qualification for quasiconvex programming (the Q-BCQ) with respect to $\{(k_t, w_t) \mid t \in T\}$ at $x \in A$ if

$$N_A(x) = \mathrm{coneco} \bigcup_{t\in T(x)} \{w_t\}.$$

We can check that one inclusion always holds. Indeed, for each $t \in T(x)$ and $y \in A$, $\langle w_t, y\rangle \le \langle w_t, x\rangle$ because $\langle w_t, x\rangle = k_t^{-1}(0)$. Furthermore, $N_A(x)$ is a convex cone, this shows that $N_A(x) \supset \mathrm{coneco} \bigcup_{t\in T(x)}\{w_t\}$. Therefore, the Q-BCQ is equivalent to the following inclusion

$$N_A(x) \subset \mathrm{coneco} \bigcup_{t\in T(x)} \{w_t\}.$$

In the following theorem, we show an optimality condition for quasiconvex programming and the Q-BCQ is the weakest constraint qualification for this optimality condition. Recall $\Gamma_0(X)$, the set of all proper lsc convex functions.

Let $Q_F(X)$ be the set of all quasiconvex functions which have a finite and lower left-hand Dini differentiable generator, that is,

$$Q_F(X) = \left\{ \sup_{s\in S} k_s \circ w_s \;\middle|\; \begin{array}{l} \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*, \ S : \text{finite}, \\ \forall s \in S, k_s : \text{continuous and lower left-hand Dini diff.} \end{array} \right\}.$$

Theorem 2. *Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, for each $i \in I$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i, $T = \{t = (i, j) \mid i \in I, j \in J_i\}$, $T(x) = \{t \in T \mid k_t(\langle w_t, x\rangle) = 0, k_t^{-1}(0) = \langle w_t, x\rangle\}$, $A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\}$ and $x_0 \in A$. Then, the following statements (i), (ii), (iii) and (iv) are equivalent:*

(i) $\{g_i(x) \le 0 \mid i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x_0,

(ii) for each $v \in X^*$, x_0 is a minimizer of v in A if and only if there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $\lambda_t = 0$ for all $t \in T \setminus T(x_0)$, the complementarity condition, and

$$-v = \sum_{t\in T} \lambda_t w_t,$$

(iii) for each $f \in \Gamma_0(X)$ with $\mathrm{dom} f \cap A \neq \emptyset$ and $\mathrm{epi} f^* + \mathrm{epi}\delta_A^*$ is w^*-closed, x_0 is a minimizer of f in A if and only if there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $\lambda_t = 0$ for all $t \in T \setminus T(x_0)$, and

$$0 \in \partial f(x_0) + \sum_{t\in T} \lambda_t w_t,$$

(iv) for all $f \in Q_F(X)$ with a generator G, if x_0 is a local minimizer of f in A, then, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $\lambda_t = 0$ for all $t \in T \setminus T(x_0)$, and

$$0 \in \partial_G f(x_0) + \sum_{t\in T} \lambda_t w_t.$$

PROOF. We now first prove (i) implies (iii). By the assumption of f, the subdifferential sum formula holds, that is,

$$\partial(f + \delta_A)(x_0) = \partial f(x_0) + \partial\delta_A(x_0).$$

Because $\partial\delta_A(x_0) = N_A(x_0)$ and condition (i) holds,

$$x_0 \text{ minimizes } f \text{ on } A \iff 0 \in \partial f(x_0) + \mathrm{coneco} \bigcup_{t\in T(x_0)} \{w_t\},$$

this shows that (iii) holds.

Next, it is clear that (iii) implies (ii) and (iv) implies (ii).

We now prove that (ii) implies (i). We want to show that if $x^* \in N_A(x_0)$ then $x^* \in \operatorname{coneco}\bigcup_{t\in T(x_0)}\{w_t\}$. Let $x^* \in N_A(x_0)$. Because $x^* \in N_A(x_0)$, $\delta_A^*(x^*) = \langle x^*, x_0\rangle$. Therefore, x_0 minimizes $-x^*$ on A. Then by using condition (ii), there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $x^* = \sum_{t\in T}\lambda_t w_t \in \operatorname{coneco}\bigcup_{t\in T(x_0)}\{w_t\}$.

Finally, by using Theorem 1, we can prove (i) implies (iv). This completes the proof.

In Theorem 2, $Q_F(X)$ corresponds to the condition (i) of Theorem 1. In the following theorem, we define $Q_C(X)$ which corresponds to the condition (ii) of Theorem 1 as follows,

$$Q_C(X) = \left\{ \sup_{s\in S} k_s \circ w_s \;\middle|\; \begin{array}{l} \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*,\ S : \text{compact}, \\ s \mapsto w_s : \text{continuous}, (s,t) \mapsto k_s(t) : \text{usc}, \\ D_-k_s(t) \in \mathbb{R} \text{ and } (s,t) \mapsto D_-k_s(t) : \text{continuous}. \end{array} \right\}.$$

Theorem 3. *Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, for each $i \in I$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i, $T = \{t = (i,j) \mid i \in I, j \in J_i\}$, $T(x) = \{t \in T \mid k_t(\langle w_t, x\rangle) = 0,\ k_t^{-1}(0) = \langle w_t, x\rangle\}$, $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$ and $x_0 \in A$. Assume that X is a Banach space, then the following statements (v) is equivalent to the statements (i), (ii), (iii) and (iv) in Theorem 2.*

(v) for all $f \in Q_C(X)$ with a generator $G = \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*$, if x_0 is a local minimizer of f in A, then, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $\lambda_t = 0$ for all $t \in T \setminus T(x_0)$, and

$$0 \in \partial_G f(x_0) + \sum_{t\in T}\lambda_t w_t.$$

PROOF. By using Theorem 1, we can prove (i) implies (v). Also, it is clear that (v) implies (ii).

Lastly in this section, we investigate a relation between Q-BCQ and a previous result. In [6], the basic constraint qualification for convex programming was provided. Let $\{g_i \mid i \in I\}$ be a family of proper lsc convex function from

X to $\overline{\mathbb{R}}$, then, the family $\{g_i \mid i \in I\}$ is said to satisfy the basic constraint qualification (the BCQ) at $x \in A$ if

$$N_A(x) = \text{coneco} \bigcup_{i \in I(x)} \partial g_i(x),$$

where $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$ and $I(x) = \{i \in I \mid g_i(x) = 0\}$.

If $\{g_i \mid i \in I\}$ is a family of proper lsc convex function with the basic generator, $T = \{(i, v) \mid i \in I, v \in \text{dom} g_i^*\}$, then, for all $x \in A$, we can check

$$\bigcup_{(i,v) \in T(x)} \{v\} = \bigcup_{i \in I(x)} \partial g_i(x),$$

that is, the BCQ and the Q-BCQ w.r.t. the basic generator are equivalent.

Furthermore, we can prove the following theorem in [6] by using Theorem 2.

Corollary 1. *[6] $\{g_i \mid i \in I\}$ be a family of proper lsc convex functions from X to $\overline{\mathbb{R}}$, Assume that $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\} \neq \emptyset$ and $x_0 \in A$. Then, the following statements are equivalent:*

(i) $\{g_i(x) \leq 0 \mid i \in I\}$ satisfies the BCQ at x_0,

(ii) for all $v \in X^*$, x_0 is a minimizer of v in A if and only if there exists $\lambda \in \mathbb{R}_+^{(I(x_0))}$ such that

$$-v \in \sum_{i \in I(x_0)} \lambda_i \partial g_i(x_0),$$

(iii) for all $f \in \Gamma_0(X)$ with $\text{dom} f \cap A \neq \emptyset$ and $\text{epi} f^* + \text{epi} \delta_A^*$ is w^*-closed, x_0 is a minimizer of f in A if and only if there exists $\lambda \in \mathbb{R}_+^{(I(x_0))}$ such that

$$0 \in \partial f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial g_i(x_0).$$

Also, we can prove that the conditions (i), (ii) and (iii) in Theorem 2 are equivalent by using Corollary 1. Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, for each $i \in I$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i, and $T = \{t = (i, j) \mid i \in I, j \in J_i\}$. Then, $A =$

$\{x \in X \mid \forall t \in T, w_t(x) - k_t^{-1}(0) \leq 0\}$ and $\partial(w_t - k_t^{-1}(0)) = \{w_t\}$ for all $t \in T$. Since $w_t - k_t^{-1}(0)$ is a continuous linear function, we can prove a equivalence relation of the conditions (i), (ii) and (iii) in Theorem 2 by using Corollary 1. Hence, we can see that (i), (ii) and (iii) of Theorem 2 and Corollary 1 are equivalent. However, (iv) of Theorem 2 and (v) of Theorem 3 are new results which concern quasiconvex programming and we can consider problems whose objective function is quasiconvex by using Theorem 2 and 3.

5. Usefulness of our results

In this section, we emphasize the usefulness of optimality conditions and Q-BCQ by some examples. At first, we show the following quasiconvex programming problem that Theorem 2 is used effectively.

Example 1. *Let $X = \mathbb{R}^2$, $I = \{1, 2\}$, $g_1(x) = -(x_1-2)^3$, $g_2(x) = -(x_2-1)^5$ and $f(x) = \sqrt{|x_1 - 1| + |x_2 - 1|}$, then, f, g_1 and g_2 are continuous quasiconvex, and $A = \{x \in \mathbb{R}^2 \mid x_1 \geq 2, x_2 \geq 1\}$. Also, $G_1 = \{(k_1, (-1, 0)) \mid k(a) = (a+2)^3\}$ is a generator of g_1, $G_2 = \{(k_2, (0, -1)) \mid k_2(a) = (a+1)^5\}$ is a generator of g_2 and $G_0 = \{(h_1, (1, 1)), (h_2, (-1, 1)), (h_3, (-1, -1)), (h_4, (1, -1))\}$ is a generator of f, where h_1 be a function from $\mathbb{R}$ to $\mathbb{R}$ as follows:*

$$h_1(a) = \begin{cases} \sqrt{a-2} & a \geq 2, \\ 0 & \textit{otherwise.} \end{cases}$$

and $h_2(a) = h_4(a) = h_1(a + 2)$, $h_3(a) = h_1(a + 4)$ for all $a \in \mathbb{R}$. We can check easily that the Q-BCQ w.r.t. $G_1 \cup G_2$ is satisfied at each point of A. We observe whether there exist $x \in A$ and $\lambda \in \mathbb{R}^2_+$ satisfying $0 \in \partial_{G_0} f(x) + \lambda_1(-1, 0) + \lambda_2(0, -1)$ and the complementarity condition or not. If $x \in \mathrm{int}A$, then, $\partial_{G_0} f(x) = \{\frac{1}{2\sqrt{x_1+x_2-2}}(1, 1)\}$ and $g_i(x) \neq 0$ $(i \in I)$, this implies $\lambda = 0$ if the complementarity condition holds. Hence, the optimality condition is not satisfied. If $x \in \{y \mid y_1 = 2, y_2 > 1\}$, then, $\lambda_2 = 0$ if the complementarity condition holds. Also, $\partial_{G_0} f(x) = \{\frac{1}{2\sqrt{x_1+x_2-2}}(1, 1)\}$, that is, the optimality condition is not satisfied. If $x \in \{y \mid y_1 > 2, y_2 = 1\}$, then, $\partial_{G_0} f(x) = \mathrm{clco}\left\{\frac{1}{2\sqrt{x_1+x_2-2}}(1, 1), \frac{1}{2\sqrt{x_1-x_2}}(1, -1)\right\}$ and $\lambda_1 = 0$ if the complementarity condition holds, that is, the optimality condition is not satisfied. If $x = (2, 1)$, then,

$$\partial_{G_0} f(x) = \mathrm{clco}\{D_- h_1(\langle(1, 1), x_0\rangle)(1, 1), D_- h_4(\langle(1, -1), x_0\rangle)(1, -1)\}$$

$$\begin{aligned} &= \operatorname{clco}\left\{\frac{1}{2}(1,1), \frac{1}{2}(1,-1)\right\} \\ &= \left\{v \in \mathbb{R}^2 \;\middle|\; v_1 = \frac{1}{2}, v_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\}. \end{aligned}$$

Put $\lambda = \left(\frac{1}{2}, \frac{1}{2}\right)$, then, $0 \in \partial_{G_0} f(x) + \lambda_1(-1,0) + \lambda_2(0,-1)$. Therefore, $(2,1)$ satisfies the necessary condition for a local minimizer. In this case, the other $x \in A$ does not satisfy the optimality condition, hence $(2,1)$ is the global minimizer of f in A.

As stated above, Q-BCQ is used effectively for quasiconvex programming. At the same time, Q-BCQ and the notion of generator are useful for convex programming. Now we show the following example.

Example 2. *Let $X = \mathbb{R}^2$, $I = \{1\}$, $g(x) = (x_1 - x_2)^2$. Then, $A = \{y \mid y_1 = y_2\}$, for all $y \in A$, $N_A(y) = \{v \mid v_1 + v_2 = 0\}$, $I(y) = I$. Also, the BCQ is not satisfied at any point $y \in A$ because $\nabla g(y) = 0$. However, we can choose a suitable generator for satisfying the Q-BCQ. Let k be a function from $\mathbb{R}$ to $\mathbb{R}$ as follows:*

$$k(t) = \begin{cases} t^2 & t \geq 0, \\ 0 & \textit{otherwise}, \end{cases}$$

let $J = \{(k,(1,-1)), (k,(-1,1))\}$. Then, J is a generator of g. Furthermore, for all $y \in A$, $k(\langle(1,-1),(y_1,y_2)\rangle) = k(\langle(-1,1),(y_1,y_2)\rangle) = 0$, and

$$N_A(y) = \{v \mid v_1 + v_2 = 0\} = \operatorname{coneco}\bigcup\{(1,-1),(-1,1)\}.$$

Therefore the Q-BCQ w.r.t. J at y is satisfied.

Let $f(x) = (x_1 - 5)^2 + (x_2 - 3)^2$, then, f is a continuous convex function. Since Q-BCQ is satisfied, we can find a minimizer by using an optimality condition in this paper. We observe whether there exist $x \in A$ and $\lambda \in \mathbb{R}^2_+$ satisfying $0 \in \partial f(x) + \lambda_1(1,-1) + \lambda_2(-1,1)$ and the complementarity condition or not. We can check easily that $\partial f(x) = \{\nabla f(x)\} = \{(2(x_1 - 5), 2(x_2 - 3))\}$. If there exists λ satisfying the optimality condition, then, we can calculate $x = (4,4)$. Put $\lambda = (0,2)$, then $0 \in \partial f(x) + \lambda_1(1,-1) + \lambda_2(-1,1)$. By using Theorem 2, $(4,4)$ is the global minimizer.

Also, as stated in the comments below Corollary 1, we rewrite the constraint as a convex one which satisfies the BCQ. Let $g_1(x) = \langle(1,-1),x\rangle - k^{-1}(0) = \langle(1,-1),x\rangle$, and $g_2(x) = \langle(-1,1),x\rangle - k^{-1}(0) = \langle(-1,1),x\rangle$, *then we can check easily that* $A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0 (i = 1, 2)\}$*, and* $\{g_i(x) \leq 0 \mid i = 1, 2\}$ *satisfies the BCQ. By using Corollary 1,* $x \in A$ *is a minimizer of* f *in* A *if and only if there exists* $\lambda \in \mathbb{R}^2_+$ *such that* $0 \in \partial f(x) + \sum_{i=1,2} \lambda_i \partial g_i(x) = \partial f(x) + \lambda_1(1, -1) + \lambda_2(-1, 1)$*. Hence, by the similar way in the first half of this example, we can find the global minimizer* $(4, 4)$*.*

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