Necessary and sufficient conditions for some constraint qualifications in quasiconvex programming

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Abstract

In this paper, we investigate relations between constraint qualifications in quasiconvex programming. At first, we show a necessary and sufficient condition for the Q-CCCQ, and investigate some sufficient conditions for the Q-CCCQ. Also, we consider a relation between the Q-CCCQ and the Q-BCQ and we compare the Q-BCQ with some constraint qualifications.

Keywords: quasiconvex programming, constraint qualification

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1. Introduction

In mathematical programming, some optimality conditions were investigated. Also, constraint qualifications for these optimality conditions were studied, for example, linear independent constraint qualification (LICQ), Slater constraint qualification and Guignard constraint qualification. In convex programming, the following optimality condition is well known: let I be an arbitrary set, f and g_i be convex function, $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$ and $x_0 \in A$, then the following equivalence relation holds

$$x_0 \text{ is a minimizer of } f \text{ in } A \iff \exists \lambda \in \mathbb{R}_+^{(I)} \text{ s.t. } 0 \in \partial f(x_0) + \sum_{i \in I} \lambda_i \partial g_i(x_0),$$

where $\mathbb{R}_+^{(I)} = \{\lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \geq 0, \{i \in I \mid \lambda_i \neq 0\} : \text{finite}\}$. In [6], the basic constraint qualification (the BCQ) was investigated as the weakest constraint qualification for the above optimality condition. Also, in [3], Farkas Minkowski (FM) was investigated as the weakest constraint qualification for Lagrange duality.

In quasiconvex programming, we investigated the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ) [8] and the basic constraint qualification for quasiconvex programming (the Q-BCQ) [9] as a similar constraint qualification of FM and the BCQ. Actually, the Q-CCCQ and the Q-BCQ are the weakest constraint qualification for a certain duality and an optimality condition, respectively. In the research of constraint qualifications, discoveries of such constraint qualifications are very important.

However, in practice, it is difficult to check whether the Q-CCCQ and the Q-BCQ hold or not. Hence, some plain constraint qualifications is very important after all. For example, in convex programming, the Slater condition is a plain and sufficient constraint qualification for FM and the BCQ. However, in quasiconvex programming, the Slater condition is not a sufficient condition for the Q-CCCQ and the Q-BCQ. Hence, in this paper, we investigate some relations between constraint qualifications in quasiconvex programming, especially, we consider an equivalent condition of the Q-CCCQ and some sufficient or necessary conditions of the Q-CCCQ. Also, we investigate the Q-BCQ and some constraint qualifications.

The remainder of the present paper is organized as follows. In Section 2, we introduce some notation and preliminaries. In Section 3, we introduce an equivalent condition and some sufficient condition of the Q-CCCQ. In Section 4, we introduce a necessary condition for the Q-CCCQ. Finally, in Section 5, we compare the Q-BCQ with some constraint qualifications.

2. Preliminaries

Let X be a locally convex Hausdorff topological vector space. In addition, let X^* be the continuous dual space of X, and let $\langle x^*, x\rangle$ denote the value of a functional $x^* \in X^*$ at $x \in X$. Given a set $A^* \subset X^*$, we denote the w^*-closure, the convex hull, the boundary, and the conical hull generated by A^*, by $\mathrm{cl}A^*$, $\mathrm{co}A^*$, ∂A^*, and $\mathrm{cone}A^*$, respectively. The normal cone of $A \subset X$ at $z_0 \in A$ is denoted by $N_A(z_0) = \{x^* \in X^* \mid \forall y \in A, \langle x^*, y - z_0\rangle \leq 0\}$. The indicator function δ_A and the support function σ_A of A are respectively defined by

$$\delta_A(x) := \begin{cases} 0 & x \in A, \\ \infty & otherwise, \end{cases}$$

and

$$\sigma_A(x^*) := \sup_{x\in A} \langle x^*, x\rangle \text{ for each } x^* \in X^*.$$

Throughout the present paper, let f be a function from X to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Here, f is said to be proper if for all $x \in X$, $f(x) > -\infty$ and there exists $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\mathrm{dom} f$, that is, $\mathrm{dom} f = \{x \in X \mid f(x) < \infty\}$. The epigraph of f, $\mathrm{epi} f$, is defined as $\mathrm{epi} f = \{(x, r) \in X \times \mathbb{R} \mid x \in \mathrm{dom} f, f(x) \leq r\}$, and f is said to be convex if $\mathrm{epi} f$ is convex. In addition, the Fenchel conjugate of f, $f^* : X^* \to \overline{\mathbb{R}}$, is defined as $f^*(u) = \sup_{x \in \mathrm{dom} f}\{\langle u, x\rangle - f(x)\}$. Remember that f is said to be quasiconvex if for all x_1, $x_2 \in X$ and $\alpha \in (0, 1)$,

$$f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \alpha) = \{x \in X \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. Then, f is quasiconvex if and only if for any $\alpha \in \mathbb{R}$, $L(f, \leq, \alpha)$ is a convex set, or equivalently, for any $\alpha \in \mathbb{R}$, $L(f, <, \alpha)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true.

It is well known that a proper lsc convex function consists of a supremum of some family of affine functions. In the case of quasiconvex functions, a similar result was also proved by Penot and Volle [7]. First, we introduce a notion of quasiaffine. A function f is said to be quasiaffine if quasiconvex and quasiconcave. It is important that f is lsc quasiaffine if and only if there exists $k \in Q$ and $w \in X^*$ such that $f = k \circ w$, where $Q = \{h : \mathbb{R} \to \overline{\mathbb{R}} \mid h \text{ is lsc and non-decreasing}\}$. By using a notion of quasiaffine, Penot and Volle proved that f is lsc quasiconvex if and only if there exists $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ such that $f = \sup_{i \in I} k_i \circ w_i$. This result indicates that an lsc quasiconvex function f consists of a supremum of some family of lsc quasiaffine functions. Based on this result, in [8], we define a notion of generator for quasiconvex functions, that is, $G = \{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ is said to be a generator of f if $f = \sup_{i \in I} k_i \circ w_i$. Because Penot and Volle's result, all lsc quasiconvex functions have at least one generator. Also, when f is a proper lsc convex function, $B_f = \{(k_v, v) \mid v \in \mathrm{dom} f^*, k_v(t) = t - f^*(v), \forall t \in \mathbb{R}\} \subset Q \times X^*$ is a generator of f. Actually, for all $x \in X$,

$$f(x) = f^{**}(x) = \sup\{\langle v, x\rangle - f^*(v) \mid v \in \mathrm{dom} f^*\} = \sup_{v \in \mathrm{dom} f^*} k_v(\langle v, x\rangle).$$

We call the generator B_f "the basic generator" of a convex function f. The basic generator is very important to the comparison of convex and quasiconvex programming.

Moreover, we introduce a generalized notion of inverse function of $h \in Q$. The following function h^{-1} is said to be the hypo-epi-inverse of h:

$$h^{-1}(a) = \inf\{b \in \mathbb{R} \mid a < h(b)\} = \sup\{b \in \mathbb{R} \mid h(b) \leq a\}.$$

It is known that if h has an inverse function, then the inverse and the hypo-epi-inverse of h are the same, in detail see [7]. In the present paper, we denote the hypo-epi-inverse of h by h^{-1}.

In mathematical programming, research on constraint qualification is very important. In convex programming, the closed cone constraint qualification (the CCCQ) has been investigated extensively, see [2, 3, 5]. Also, in [8], we investigated the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ). In this paper, we redefine the Q-CCCQ for infinitely constraints quasiconvex programming.

Definition 1. *[8] Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i for each $i \in I$, and $T = \{t = (i,j) \mid i \in I, j \in J_i\}$. Assume that $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$ is non-empty set. Then, the quasiconvex inequality system $\{g_i(x) \leq 0 \mid i \in I\}$ satisfies the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ) w.r.t. $\{(k_t, w_t) \mid t \in T\}$ if*

$$\text{cone co} \bigcup_{t \in T} \{(w_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty)$$

is w^*-closed.

Also, $\{g_i(x) \leq 0 \mid i \in I\}$ satisfies the Q-CCCQ if and only if the alternative form of Q-CCCQ,

$$\text{epi}\delta_A^* \subset \text{cone co} \bigcup_{t \in T} \{(w_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty)$$

holds. The Q-CCCQ is the weakest constraint qualification for the following duality. Recall $\Gamma_0(X)$, the set of all proper lsc convex functions.

Theorem 1. *[8] Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i for each $i \in I$, and $T = \{t = (i,j) \mid i \in I, j \in J_i\}$. Assume that $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$ is non-empty set. Then, the following statements are equivalent:*

(i) $\{g_i(x) \le 0 \mid i \in I\}$ *satisfies the Q-CCCQ w.r.t.* $\{(k_t, w_t) \mid t \in T\}$*,*

(ii) for all $f \in \Gamma_0(X)$ *with* $\mathrm{epi} f^* + \mathrm{epi} \delta_A^*$ *is* w^**-closed,*

$$\inf_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in X} \left\{ f(x) + \sum_{i \in I} \lambda_i (w_i - k_i^{-1}(0)) \right\}.$$

We introduce the following constraint qualification in [9].

Definition 2. *[9] Let* $\{g_i \mid i \in I\}$ *be a family of lsc quasiconvex functions from* X *to* $\overline{\mathbb{R}}$*,* $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ *be a generator of* g_i *for each* $i \in I$*,* $T = \{t = (i, j) \mid i \in I, j \in J_i\}$*,* $T(x) = \{t \in T \mid k_t(\langle w_t, x \rangle) = 0,$ $k_t^{-1}(0) = \langle w_t, x \rangle\}$*, and* $A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\}$*.*

The family $\{g_i \mid i \in I\}$ *is said to satisfy the basic constraint qualification for quasiconvex programming (the Q-BCQ) with respect to* $\{(k_t, w_t) \mid t \in T\}$ *at* $x \in A$ *if*

$$N_A(x) = \mathrm{coneco} \bigcup_{t \in T(x)} \{w_t\}.$$

Also, the Q-BCQ is equivalent to the following inclusion

$$N_A(x) \subset \mathrm{coneco} \bigcup_{t \in T(x)} \{w_t\}.$$

The Q-BCQ is the weakest constraint qualification for the following optimality condition. Let $Q_F(X)$ be the set of all quasiconvex functions which have a finite and lower left-hand Dini differentiable generator, that is,

$$Q_F(X) = \left\{ \sup_{s \in S} k_s \circ w_s \;\middle|\; \begin{array}{l} \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*, \ S : \text{finite}, \\ \forall s \in S, k_s : \text{continuous and lower left-hand Dini diff.} \end{array} \right\}.$$

Theorem 2. *[9] Let* $\{g_i \mid i \in I\}$ *be a family of lsc quasiconvex functions from* X *to* $\overline{\mathbb{R}}$*,* $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ *be a generator of* g_i *for each* $i \in I$*,* $T = \{t = (i, j) \mid i \in I, j \in J_i\}$*,* $T(x) = \{t \in T \mid k_t(\langle w_t, x \rangle) = 0,$ $k_t^{-1}(0) = \langle w_t, x \rangle\}$*,* $A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\}$ *and* $x_0 \in A$*. Then, the following statements (i), (ii) and (iii) are equivalent:*

(i) $\{g_i \mid i \in I\}$ *satisfies the Q-BCQ w.r.t.* $\{(k_t, w_t) \mid t \in T\}$ *at* x_0*,*

(ii) *for each* $f \in \Gamma_0(X)$ *with* $\mathrm{dom} f \cap A \neq \emptyset$ *and* $\mathrm{epi} f^* + \mathrm{epi}\delta_A^*$ *is* w^**-closed,* x_0 *is a minimizer of* f *in* A *if and only if there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\lambda_t = 0$ *for all* $t \in T \setminus T(x_0)$*, and*

$$0 \in \partial f(x_0) + \sum_{t \in T} \lambda_t w_t,$$

(iii) *for all* $f \in Q_F(X)$ *with a generator* $G = \{(k_s, w_s) \mid s \in S\}$*, if* x_0 *is a local minimizer of* f *in* A*, then, there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\lambda_t = 0$ *for all* $t \in T \setminus T(x_0)$*, and*

$$0 \in \partial_G f(x_0) + \sum_{t \in T} \lambda_t w_t,$$

where $\partial_G f(x_0) = \{D_- k_s(\langle w_s, x_0\rangle) w_s \mid f(x_0) = k_s \circ w_s(x_0)\}$.

3. Equivalent condition and Sufficient conditions of the Q-CCCQ

In convex programming, the Slater condition is one of the well known constraint qualification as a sufficient condition for FM and the BCQ. However, the Slater condition is not a sufficient condition for Q-CCCQ, see the following Example.

Example 1. *Let* $X = \mathbb{R}^2$*,* $I = (0, 1]$*,* $w_i = (-i, i-1)$*,* k_i *be a function as follows:*

$$k_i(t) = \begin{cases} it & t > 0, \\ -1 & t \leq 0, \end{cases}$$

and $g = \sup_{i \in I} k_i \circ w_i$*. Then,* $A = \{x \in X \mid g(x) \leq 0\} = \mathbb{R}_+^2$*, and* $f(1,1) < 0$*, that is, the Slater condition holds. However, it is clear that* $\{g(x) \leq 0\}$ *is not satisfies the Q-CCCQ w.r.t.* $\{(k_i, w_i) \mid i \in I\}$*. Also, even if* $G = \{(k, w) \mid k \in Q, w \in \mathbb{R}^2, k \circ w \leq g\}$*, it is the biggest generator of* f*, then* g *does not satisfy the Q-CCCQ w.r.t.* G*. Indeed,* $\mathrm{epi}\sigma_A = \{(x, \alpha) \mid x \in -\mathbb{R}_+^2, \alpha \geq 0\}$*, and* $((0,-1), 0) \notin \mathrm{cone\ co}\ \{(w, \delta) \mid k^{-1}(0) \leq \delta, k \circ w \leq f\} + \{0\} \times [0, \infty)$ *since if* $w = (0, -1)$ *and* $k \circ w \leq f$*, then* $k \leq 0$*, that is* $k^{-1}(0) = \infty$*.*

Hence, in this section, we consider an equivalent condition and some sufficient conditions of the Q-CCCQ. From now on we consider the problem with only one constraint function, and we fix the generator of a constraint function as follows: $G = \{(k, w) \mid k \in Q, w \in X^*, k \circ w \leq f\}$ that is, G is the set of all lsc quasiaffine functions which is smaller than f. Since this G is uniquely defined to f, we can consider the Q-CCCQ without taking care with how to take a generator. Also, in convex programming, the basic generator is the set of all affine functions which is smaller than f, hence the above generator G is a natural notion.

At first, we show the following lemma which concerns non-decreasing functions.

Lemma 1. *Let k be a function from $\mathbb{R}$ to $\overline{\mathbb{R}}$, and $\mathrm{cl}k$ be the lsc hull of h, that is, $\mathrm{epicl}k = \mathrm{clepi}k$. Then, the following (i) and (ii) hold:*

(i) If k is non-decreasing, then, $\mathrm{cl}k \in Q$,

(ii) If k is non-decreasing, then $k^{-1}(0) = (\mathrm{cl}k)^{-1}(0)$.

PROOF. We prove the statement (i). We only show that $\mathrm{cl}k$ is non-decreasing. If there exist t_1 and t_2 such that $t_1 < t_2$ and $\mathrm{cl}k(t_1) > \mathrm{cl}k(t_2)$. Then, for all $t \in [t_1, \infty)$, $\mathrm{cl}k(t_2) < \mathrm{cl}k(t_1) \leq k(t_1) \leq k(t)$. Thus, $(t_2, \mathrm{cl}k(t_2)) \notin \mathrm{clepi}k$, this is a contradiction.

Next, we prove the statement (ii). Since $k \geq \mathrm{cl}k$, $k^{-1}(0) \leq (\mathrm{cl}k)^{-1}(0)$. If $k^{-1}(0) < (\mathrm{cl}k)^{-1}(0)$, then, there exists $t_0 \in \mathbb{R}$ such that $k^{-1}(0) < t_0$ and $\mathrm{cl}k(t_0) \leq 0$. Also there exists $t' \in \mathbb{R}$ such that $k^{-1}(0) < t' < t_0$. Then, for all $t \in [t', \infty)$, $\mathrm{cl}k(t_0) \leq 0 < k(t') \leq k(t)$. Thus, $(t_0, \mathrm{cl}k(t_0)) \notin \mathrm{clepi}k$, this is a contradiction.

By using Lemma 1, we rewrite the Q-CCCQ.

Lemma 2. *Let f be a lsc quasiconvex function. Then,*

(i) $K = \{(w, \delta) \mid \exists k \in Q \text{ s.t. } k \circ w \leq f, k^{-1}(0) \leq \delta\}$ is a convex cone,

(ii) $\{f(x) \leq 0\}$ satisfies the Q-CCCQ if and only if K is w^*-closed.

PROOF. We prove the statement (i). Put k as follows:

$$k(t) := \begin{cases} -\infty & t \leq 0, \\ \infty & \mathit{otherwise}, \end{cases}$$

then, $k \in Q$, $k^{-1}(0) = 0$ and $k \circ 0 \leq f$, hence $(0, 0) \in K$. Let $(w, \delta) \in K$ and $\lambda > 0$, then there exists $k \in Q$ such that $k \circ w \leq f$ and $k^{-1}(0) \leq \delta$. Put

$k_\lambda(t) = k(\frac{t}{\lambda})$, we can check that $k_\lambda \in Q$, $k_\lambda \circ \lambda w \leq f$ and $k_\lambda^{-1}(0) \leq \lambda\delta$, hence K is a cone.

Let (w_1, δ_1), $(w_2, \delta_2) \in K$, then there exist k_1 and $k_2 \in Q$ such that $k_1 \circ w_1 \leq f$, $k_1^{-1}(0) \leq \delta_1$, $k_2 \circ w_2 \leq f$ and $k_2^{-1}(0) \leq \delta_2$. Put $\bar{k}$ as follows:

$$\bar{k}(t) := \begin{cases} -\infty & t \leq k_1^{-1}(0) + k_2^{-1}(0), \\ \inf\{f(x) \mid t \leq \langle w_1 + w_2, x\rangle\} & \textit{otherwise}, \end{cases}$$

then it is clear that $\bar{k}$ is non-decreasing. Also, $\bar{k}^{-1}(0) = k_1^{-1}(0) + k_2^{-1}(0)$. Actually, for all $t > k_1^{-1}(0) + k_2^{-1}(0)$, there exists $\varepsilon > 0$ such that $t > k_1^{-1}(0) + k_2^{-1}(0) + 2\varepsilon$. Then, for all $x \in X$ with $\langle w_1 + w_2, x\rangle \geq t$, $\langle w_1, x\rangle > k_1^{-1}(0) + \varepsilon$ or $\langle w_2, x\rangle > k_2^{-1}(0) + \varepsilon$. If $\langle w_1, x\rangle > k_1^{-1}(0) + \varepsilon$, then $f(x) \geq k_1 \circ w_1(x) \geq k_1(k_1^{-1}(0) + \varepsilon) \geq \min\{k_1(k_1^{-1}(0) + \varepsilon), k_2(k_2^{-1}(0) + \varepsilon)\}$. It is similar that if $\langle w_2, x\rangle > k_2^{-1}(0)+\varepsilon$, $f(x) \geq \min\{k_1(k_1^{-1}(0)+\varepsilon), k_2(k_2^{-1}(0)+\varepsilon)\}$. Hence, $\bar{k}(t) = \inf\{f(x) \mid t \leq \langle w_1 + w_2, x\rangle\} \geq \min\{k_1(k_1^{-1}(0)+\varepsilon), k_2(k_2^{-1}(0)+\varepsilon)\} > 0$, that is, $\bar{k}^{-1}(0) = k_1^{-1}(0) + k_2^{-1}(0)$. By using Lemma 1, $\mathrm{cl}\bar{k} \in Q$ and $(\mathrm{cl}\bar{k})^{-1}(0) = \bar{k}^{-1}(0) = k_1^{-1}(0) + k_2^{-1}(0)$. Also, $\mathrm{cl}\bar{k}(\langle w_1 + w_2, x\rangle) \leq \bar{k}(\langle w_1 + w_2, x\rangle) \leq f(x)$, therefore K is convex.

The statement (ii) is clear because of the definition of the Q-CCCQ and the statement (i).

Next, we show a sufficient and necessary condition of the Q-CCCQ. This condition is very important when we consider some sufficient conditions for the Q-CCCQ.

Theorem 3. *The following (i) and (ii) are equivalent,*

(i) $\{f(x) \leq 0\}$ *satisfies the Q-CCCQ,*

(ii) for all $v \in X^* \setminus \{0\}$ *and* $t > \sigma_A(v)$, $\inf\{f(x) \mid \langle v, x\rangle \geq t\} > 0$.

PROOF. Assume that $\{f(x) \leq 0\}$ satisfies the Q-CCCQ. Then, for all $v \in \mathrm{dom}\sigma_A \setminus \{0\}$, $(v, \sigma_A(v)) \in \mathrm{epi}\sigma_A$. Because of the alternative form of the Q-CCCQ, $\mathrm{epi}\sigma_A = K$, and there exists $k \in Q$ such that $k \circ v \leq f$ and $k^{-1}(0) \leq \sigma_A(v)$. Also, we can check easily that for all $t \in \mathbb{R}$, $\inf\{f(x) \mid \langle v, x\rangle \geq t\} \geq k(t)$. Hence, for all $t > \sigma_A(v)$, $\inf\{f(x) \mid \langle v, x\rangle \geq t\} \geq k(t) > 0$.

Next, we want to prove that if (ii) holds, then K is w^*-closed. Let $\{(w_\alpha, \delta_\alpha) \mid \alpha \in D\} \subset K$ be a net which converges to (w_0, δ_0) in w^*-topology. Then, for each $\alpha \in D$, there exists $k_\alpha \in Q$ such that $k_\alpha \circ w_\alpha \leq f$ and $k_\alpha^{-1}(0) \leq$

δ_α. Beause of the alternative form of the Q-CCCQ, $(w_\alpha, k_\alpha^{-1}(0)) \in \text{epi}\sigma_A$, that is, $\sigma_A(w_\alpha) \leq k_\alpha^{-1}(0) \leq \delta_\alpha$. Since σ_A is w^*-lsc, $\sigma_A(w_0) \leq \liminf_\alpha \sigma_A(w_\alpha) \leq \liminf_\alpha \delta_\alpha = \delta_0$. Put k_0 as follows:

$$k_0(t) = \begin{cases} \inf\{f(x) \mid \langle w_0, x\rangle \geq t\} & t > \sigma_A(w_0), \\ -\infty & otherwise, \end{cases}$$

It is clear that k_0 is non-decreasing and $k_0 \circ w_0 \leq f$. Since (ii) holds, $k_0^{-1}(0) = \sigma_A(w_0)$. By using Lemma 1, $\text{cl}k_0 \in Q$ and $\text{cl}k_0^{-1}(0) = k_0^{-1}(0) = \sigma_A(w_0)$. Also, we can check $\text{cl}k_0 \circ w_0 \leq k_0 \circ w_0 \leq f$ and $\text{cl}k_0^{-1}(0) \leq \delta_0$, that is, K is w^*-closed.

By using Theorem 3, we show some sufficient conditions for the Q-CCCQ. At first we show the following result in finite dimensional Euclidean space.

Corollary 1. *Let $X = \mathbb{R}^n$ and A be compact, then $\{f(x) \leq 0\}$ satisfies the Q-CCCQ.*

PROOF. Because of Theorem 3, we only prove that for all $v \in X^* \setminus \{0\}$ and $t > \sigma_A(v)$, $\inf\{f(x) \mid \langle v, x\rangle \geq t\} > 0$. Assume that there exist $v_0 \in X^* \setminus \{0\}$ and $t_0 > \sigma_A(v_0)$ such that $\inf\{f(x) \mid \langle v_0, x\rangle \geq t_0\} \leq 0$. If $\inf\{f(x) \mid \langle v_0, x\rangle \geq t_0\} < 0$, then there exists $x_0 \in \mathbb{R}^n$ such that $\langle v_0, x_0\rangle \geq t_0 > \sigma_A(v_0)$ and $f(x) < 0$. This is a contradiction since $x \in A$.

We assume that $\inf\{f(x) \mid \langle v_0, x\rangle \geq t_0\} = 0$. Since A is compact, there exists $x_0 \in A$ such that $\langle v_0, x_0\rangle = \sup_{x\in A} \langle v_0, x\rangle = \sigma_A(v_0)$. Also, $M \geq 0$ such that $A \subset \{x \in \mathbb{R}^n \mid \|x_0 - x\| \leq M - 1\}$. Because $\inf\{f(x) \mid \langle v_0, x\rangle \geq t_0\} = 0$, there exists $\{x_k\} \subset \mathbb{R}^n$ such that $\langle v_0, x_k\rangle \geq t_0$ and $f(x_k) \leq \frac{1}{k}$. If $\{x_k\}$ is bounded, there exist $\{x_{k_i}\} \subset \{x_k\}$ such that x_{k_i} converges to some $\bar{x}$. Then, $\langle v_0, \bar{x}\rangle \geq t_0$ and $f(\bar{x}) \leq \liminf f(x_{k_i}) \leq 0$, that is, $\bar{x} \in A$. This is a contradiction. If $\{x_k\}$ is not bounded, for large enough $k \in \mathbb{N}$, $x_k \notin \{x \in \mathbb{R}^n \mid \|x_0 - x\| \leq M\}$. Put $y_k = \frac{M}{\|x_k - x_0\|}(x_k - x_0) + x_0$, then $\|x_0 - y_k\| = M$ and $y_k \in [x_0, x_k]$. Since f is quasiconvex, $f(y_k) \leq \max\{f(x_0), f(x_k)\} = \frac{1}{k}$. Because $\{y_k\} \subset \{x \in \mathbb{R}^n \mid \|x_0 - x\| \leq M\}$, there exist $\{y_{k_i}\} \subset \{y_k\}$ and $y_0 \in \mathbb{R}^n$ such that y_{k_i} converges to y_0. However, $\|x_0 - y_0\| = M$ and $f(y_0) \leq \liminf f(y_{k_i}) \leq 0$, this is a contradiction.

Next, we show a sufficient condition for the Q-CCCQ in Hilbert space by using the Gâteaux differential.

Corollary 2. *Let X be a Hilbert space, f be a Gâteaux differentiable quasionvex function, $\inf_{x\in\partial A} \|f'(x)\| > 0$ and $f'(\cdot)$ is uniformly continuous on ∂A, then $\{f(x) \leq 0\}$ satisfies the Q-CCCQ.*

PROOF. Assume that there exist $v_0 \in X \setminus \{0\}$ and $t_0 > \sigma_A(v_0)$ such that $\inf\{f(x) \mid \langle v_0, x\rangle \geq t\} \leq 0$. If $\inf\{f(x) \mid \langle v_0, x\rangle \geq t\} < 0$, there exists $x_0 \in X$ such that $f(x_0) < 0$ and $\langle v_0, x\rangle > \sigma_A(v_0)$, this is a contradiction. If $\inf\{f(x) \mid \langle v_0, x\rangle \geq t\} = 0$, there exists $\{x_k\} \subset X$ such that for all $k \in \mathbb{N}$, $\langle v_0, x_k\rangle \geq t$ and $0 < f(x_k) \leq \frac{1}{k}$. Since X is a Hilbert space and A is a closed convex set, there exists the metric projection of x_k onto A, denoted by $P_A(x_k) \in \partial A$. Because of the continuity of f, we can check that $f(P_A(x_k)) = 0$. Since $\langle v_0, x_k\rangle \geq t > \sigma_A(v_0)$ and $P_A(x_k) \in A$, there exists $M > 0$ such that $\|x_k - P_A(x_k)\| \geq M$ for all $k \in \mathbb{N}$. Since $P_A(x_k)$ is the metric projection and f is differentiable quasiconvex, there exists $\lambda_k > 0$ such that $\lambda_k f'(P_A(x_k)) = x_k - P_A(x_k)$ for all $k \in \mathbb{N}$. Put $\varepsilon = \frac{1}{2}\inf_{x\in\partial A}\|f'(x)\| > 0$, then there exists $\delta > 0$ such that for all $x \in \partial A$ and $y \in X$ with $\|x - y\| < \delta$, $\|f'(x) - f'(y)\| < \varepsilon$ because of the uniformly continuity of the Gâteaux differenrial of f. Then, for all $k \in \mathbb{N}$ and $y \in X$ with $\|P_A(x_k) - y\| < \delta$,

$$\begin{aligned}
f'\left(y, \frac{x_k - P_A(x_k)}{\|x_k - P_A(x_k)\|}\right) &= f'\left(y, \frac{\lambda_k f'(P_A(x_k))}{\|x_k - P_A(x_k)\|}\right) \\
&= \left\langle f'(y), \frac{\lambda_k f'(P_A(x_k))}{\|x_k - P_A(x_k)\|}\right\rangle \\
&= \frac{\lambda_k}{\|x_k - P_A(x_k)\|}\langle f'(y), f'(P_A(x_k))\rangle \\
&\geq \frac{\lambda_k}{\|x_k - P_A(x_k)\|}(\|f'(P_A(x_k))\| - \varepsilon)\|f'(P_A(x_k))\| \\
&\geq \frac{\lambda_k\|f'(P_A(x_k))\|}{\|x_k - P_A(x_k)\|}\varepsilon \\
&= \varepsilon.
\end{aligned}$$

Put $\bar{M} = \min\{\delta, M\} > 0$, then $f(x_k) \geq \varepsilon\bar{M}$ for all $k \in \mathbb{N}$ since f is quasiconvex and directional derivative is positive. This is a contradiction since $f(x_k)$ converges to 0.

4. Necessary condition of the Q-CCCQ

In this section, we investigate a necessary condition of the Q-CCCQ. Indeed, the Q-BCQ is a necessary condition of the Q-CCCQ. In convex programming, the BCQ is a necesarry condition of FM, hence we show a similar result in the following theorem.

Theorem 4. *Assume that* $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$ *is nonempty. If the family* $\{g_i \mid i \in I\}$ *satisfies the Q-CCCQ w.r.t.* $\{(k_t, w_t) \mid t \in T\}$*, then it satisfies the Q-BCQ w.r.t.* $\{(k_t, w_t) \mid t \in T\}$ *at for all* $x \in A$*. The converse implication also holds if* $\mathrm{dom}\delta_A^* \subset \bigcup_{x\in A} N_A(x)$.

PROOF. Assume that the family $\{g_i \mid i \in I\}$ satisfies the Q-CCCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ and $x \in A$. It is clear that $N_A(x) \supset \mathrm{coneco} \bigcup_{t\in T(x)} \{w_t\}$, hence we only prove the inverse inclusion. If $x \in \mathrm{int}S$, then $N_A(x) = \{0\}$, and this completes the proof. So we want to show that if $x \in A \setminus \mathrm{int}A$ and $x^* \in N_A(x) \setminus \{0\}$ then $x^* \in \mathrm{coneco} \bigcup_{t\in T(x)} \{w_t\}$. Because $x^* \in N_A(x)$, $\delta_A^*(x^*) = \langle x^*, x\rangle$. By using the assumption, the Q-CCCQ, there exists $\lambda \in \mathbb{R}_+^{(T)}$, $\delta \in \mathbb{R}^{(T)}$, and $r \geq 0$ such that $\delta_t = 0$ for all $t \in T$ with $\lambda_t = 0$, $\delta_t \geq k_t^{-1}(0)$ for all $t \in T$ with $\lambda_t \neq 0$, and

$$(x^*, \langle x^*, x\rangle) = \sum_{t\in T} \lambda_t (w_t, \delta_t) + (0, r).$$

Because $\delta_A^*(x^*) = \langle x^*, x\rangle$, we can prove that for all $t \in T$ with $\lambda_t \neq 0$, $\delta_t = k_t^{-1}(0)$ and $r = 0$, that is,

$$(x^*, \langle x^*, x\rangle) = \sum_{t\in T} \lambda_t (w_t, k_t^{-1}(0)).$$

Since $x \in A$, $0 \geq g_i(x) \geq k_t(\langle w_t, x\rangle)$ for all $t \in T$, this implies that $k_t^{-1}(0) \geq \langle w_t, x\rangle$. Then, for all $t \in T$ with $\lambda_t \neq 0$, $k_t^{-1}(0) = \langle w_t, x\rangle$, because $\sum_{t\in T} \lambda_t \langle w_t, x\rangle = \langle x^*, x\rangle = \sum_{t\in T} \lambda_t k_t^{-1}(0)$. Therefore, for all $t \in T$ with $\lambda_t \neq 0$, $t \in T(x)$ that is, $x^* \in \mathrm{coneco} \bigcup_{t\in T(x)} \{w_t\}$.

Conversely, assume that $\mathrm{dom}\delta_A^* \subset \bigcup_{x\in A} N_A(x)$. We need to show only the alternative form of the Q-CCCQ,

$$\mathrm{epi}\delta_A^* \subset \text{cone co} \bigcup_{t\in T} \{(w_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty)$$

holds. Let $(y^*, \alpha) \in \mathrm{epi}\delta_A^* \setminus \{0\}$. Since $y^* \in \mathrm{dom}\delta_A^* \subset \bigcup_{x\in A} N_A(x)$, there exists $x_0 \in A$ such that $y^* \in N_A(x_0)$. The definition of the Q-BCQ implies that there exists $\lambda \in \mathbb{R}_+^{(T(x_0))}$ such that $y^* = \sum_{t\in T(x_0)} \lambda_t w_t$. By the definition of $T(x_0)$, $k_t(\langle w_t, x_0\rangle) = 0$ and $k_t^{-1}(0) = \langle w_t, x_0\rangle$ for each $t \in T(x_0)$. Also,

$$\alpha \geq \langle y^*, x_0\rangle = \sum_{t\in T(x_0)} \lambda_t \langle w_t, x_0\rangle = \sum_{t\in T(x_0)} \lambda_t k_t^{-1}(0).$$

Put $r = \alpha - \sum_{t\in T(x_0)} \lambda_t k_t^{-1}(0)$ then $r \geq 0$, and

$$(y^*, \alpha) = \sum_{t\in T(x_0)} \lambda_t(w_t, k_t^{-1}(0)) + (0, r),$$

that is, $(y^*, \alpha) \in \text{cone co} \bigcup_{t\in T}\{(w_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty)$. This completes the proof.

Because of Theorem 4, the Q-CCCQ is a sufficient condition for the Q-BCQ. Hence, some sufficient conditions for the Q-CCCQ in Section 3 are also sufficient conditions for the Q-BCQ, and an optimality condition by using subdifferential in [9] is valid for quaiconvex progrmming problem which satisfies those sufficient conditions.

5. Comparisons of constraint qualifications

In this section, we investigate relations between some constraint qualifications for quasiconvex programming. We investigated the relations between the Q-BCQ, the Q-BCQ relative to a closed convex set C, and the strong conical hull intersection property (the strong CHIP).

Let C be a closed convex set in X. The family $\{g_i \mid i \in I\}$ is said to satisfy the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ relative to C at $x \in C \cap A$ if

$$N_{C\cap A}(x) = N_C(x) + \text{coneco} \bigcup_{t\in T(x)} \{w_t\}.$$

Theorem 5. *The family $\{g_i \mid i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ relative to C at $x \in C \cap A$ if and only if the family $\{\delta_A, g_i \mid i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T'\}$ at x, where $T' = T \cup \{(i_0, w) \mid w \in \text{dom}\delta_C^*\}$.*

PROOF. Take $i_0 \notin I$ and set $g_{i_0} = \delta_C$. Writing $I' := I \cup \{i_0\}$, the family $\{\delta_C, g_i \mid i \in I\}$ becomes $\{g_i \mid i \in I'\}$ such that $C \cap A = \{y \in X \mid \forall i \in I', g_i(y) \leq 0\}$. Also, $\{(k_w, w) \mid w \in \text{dom}\delta_C^*, \forall a \leq \delta_C^*(w), k_w(a) = 0, \forall a > \delta_C^*(w), k_w(a) = \infty\}$ is a generator of δ_C and $T'(x) = T(x) \cup \{(i_0, w) \mid w \in \text{dom}\delta_C^*, \langle w, x\rangle = \delta_C^*(w)\}$, where for $x \in C \cap A$. Then,

$$\begin{aligned} N_C(x) + \text{coneco} \bigcup_{t\in T(x)} \{w_t\} &= \{w \mid (i_0, w) \in T'(x)\} + \text{coneco} \bigcup_{t\in T(x)} \{w_t\} \\ &= \text{coneco} \bigcup_{t\in T'(x)} \{w_t\}. \end{aligned}$$

This completes the proof.

A family of closed convex sets $\{C_i \mid i \in I\}$ is said to have the storong conical hull intersection property (the storong CHIP) at $x \in \cap_{i \in I} C_i$ if

$$N_{\cap_{i \in I} C_i}(x) = \sum_{i \in I} N_{C_i}(x).$$

Theorem 6. *Let $x \in C \cap A$, and suppose that the family $\{g_i \mid i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x. Then $\{C, A\}$ has the strong CHIP at x if and only if the $\{g_i \mid i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ relative to C at x.*

PROOF. Assume that $\{C, A\}$ has the strong CHIP at x The definition of $\{C, A\}$ has the strong CHIP at x is that $N_{C \cap A}(x) = N_C(x) + N_A(x)$. By the assumption, $N_A(x) = \text{coneco} \bigcup_{t \in T(x)} \{w_t\}$. Therefore, $\{g_i \mid i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ relative to C at x. The converse is similar.

Theorem 4, 5, and 6 are similar results in [6] which is concerend with the BCQ and the other constraint qualifications.

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