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# Necessary and Sufficient Constraint Qualification for Surrogate Duality

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**Abstract** In mathematical programming, constraint qualifications are essential elements for duality theory. Recently, necessary and sufficient constraint qualifications for Lagrange duality results have been investigated. Also, surrogate duality enables one to replace the problem by a simpler one in which the constraint function is a scalar one. However, as far as we know, a necessary and sufficient constraint qualification for surrogate duality has not been proposed yet. In this paper, we propose necessary and sufficient constraint qualifications for surrogate duality and surrogate min-max duality, which are closely related with ones for Lagrange duality.

**Keywords** mathematical programming · quasiconvex functions · surrogate duality · constraint qualification

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## 1 Introduction

In mathematical programming, constraint qualifications are essential elements for duality theory. In convex programming, it is well known that the Slater condition assures the existence of Lagrange multipliers, and Lagrange duality theorems and various constraint qualifications were investigated by many authors. Recently, necessary and sufficient constraint qualifications for Lagrange

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(strong) duality results have been investigated; see [1–5]. To find a necessary and sufficient constraint qualification is one of the destinations of the study of a mathematical programming problem.

Also, surrogate (strong) duality was widely studied by many authors; for example, see [6–11]. These studies showed that surrogate duality is important to consider zero-one integer programming problem, quasiconvex programming problem, and so on. Surrogate duality enables one to replace the problem by a simpler problem, in which the constraint function is a scalar one. In [11], it is shown that the generalized Slater condition assures the existence of surrogate multipliers. However, as far as we know, a necessary and sufficient constraint qualification for surrogate duality has not been proposed yet.

In this paper, we investigate necessary and sufficient constraint qualifications for surrogate duality and surrogate min-max duality. We propose necessary and sufficient constraint qualifications for surrogate duality, which are closely related with ones for Lagrange duality, and prove the main theorems by using Jeyakumar's set containment characterization in [12].

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and notations. In Section 3, we investigate a necessary and sufficient constraint qualification for the surrogate duality. In Section 4, we investigate a necessary and sufficient constraint qualification for the surrogate min-max duality. In Section 5, we compare constraint qualifications in this paper with previous ones.

## 2 Preliminaries

Let $X$ be a locally convex Hausdorff topological vector space, let $X^*$ be the continuous dual space of $X$, and let $\langle x^*, x\rangle$ denote the value of a functional $x^* \in X^*$ at $x \in X$. Given a set $A^* \subset X^*$, we denote the $w^*$-closure, and the conical hull generated by $A^*$, by $\mathrm{cl}\, A^*$, and $\mathrm{cone}\, A^*$, respectively. The indicator function $\delta_A$ of $A \subset X$ is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & \textit{otherwise}. \end{cases}$$

Let $f$ be a function from $X$ to $\overline{\mathbb{R}}$. Here, $f$ is said to be proper iff for all $x \in X$, $f(x) > -\infty$ and there exists $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of $f$ by $\mathrm{dom} f := \{x \in X \mid f(x) < \infty\}$. The epigraph of $f$ is $\mathrm{epi} f := \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\}$, and $f$ is said to be convex iff $\mathrm{epi} f$ is convex. When $f(x) \in \mathbb{R}$, the subdifferential of $f$ at $x$ is defined as $\partial f(x) := \{x^* \in X^* \mid \forall y \in X, f(y) \geq f(x) + \langle x^*, y - x\rangle\}$. Also, the normal cone of $A$ at $x \in A$ is $N_A(x) := \{x^* \in X^* \mid \forall y \in A, \langle x^*, y - x\rangle \leq 0\}$. Clearly, $N_A(x) = \partial \delta_A(x)$. In addition, the Fenchel conjugate of $f$, $f^* : X^* \to \overline{\mathbb{R}}$, is defined as $f^*(u) := \sup_{x \in \mathrm{dom} f}\{\langle u, x\rangle - f(x)\}$. $f$ is said to be quasiconvex iff for all $x_1$, $x_2 \in X$ and $\alpha \in [0, 1]$, $f((1 - \alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}$. Define level sets of $f$ with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \beta) := \{x \in X \mid f(x) \diamond \beta\}$$

for any $\beta \in \mathbb{R}$. Then, $f$ is quasiconvex if and only if for any $\beta \in \mathbb{R}$, $L(f, \leq, \beta)$ is a convex set or, equivalently, for any $\beta \in \mathbb{R}$, $L(f, <, \beta)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true.

Let $Y$ be a locally convex Hausdorff topological vector space, partially ordered by a nonempty, closed and convex cone $K \subset Y$; that is, for $y, z \in Y$, the notation $y \leq_K z$ will mean $z - y \in K$; let $Y^*$ be the continuous dual space of $Y$, and let $g$ be a function from $X$ to $Y$. Also, the positive polar cone of $K$ is $K^+ := \{\lambda \in Y^* \mid \forall y \in K, \langle \lambda, y \rangle \geq 0\}$. A function $g$ is said to be $K$-convex iff for all $x_1, x_2 \in X$, and $\alpha \in [0,1]$, $(1-\alpha)g(x_1) + \alpha g(x_2) \in g((1-\alpha)x_1 + \alpha x_2) + K$. It is well known that $g$ is $K$-convex if and only if $\lambda \circ g$ is convex for all $\lambda \in K^+$.

In [1–5], the following condition was investigated as a necessary and sufficient constraint qualification for the Lagrange duality: $\{g(x) \in -K \mid x \in C\}$ satisfies the closed cone constraint qualification (CCCQ) iff

$$\bigcup_{\lambda \in K^+} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*$$

is $w^*$-closed. Also, $\{g(x) \in -K \mid x \in C\}$ satisfies CCCQ if and only if the alternative form of CCCQ,

$$\operatorname{epi} \delta_A^* \subset \bigcup_{\lambda \in K^+} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*.$$

In [1], Boţ investigated another constraint qualification, which is weaker than CCCQ. Also, Boţ proved that the constraint qualification is equivalent to CCCQ when $g$ is continuous $K$-convex. In this paper, we assume that $g$ is continuous $K$-convex and investigate CCCQ.

In [5], the following constraint qualification was investigated as necessary and sufficient constraint qualification for the Lagrangian min-max duality; $\{g(x) \in -K \mid x \in C\}$ satisfies [CQ2] iff

$$N_A(x_0) = N_C(x_0) + \left\{ x^* \in X^* \,\middle|\, (x^*, \langle x^*, x_0 \rangle) \in \bigcup_{\lambda \in K^+} \operatorname{epi} (\lambda \circ g)^* \right\}$$

for all $x_0 \in A$.

In the research of surrogate duality, many results were investigated with the Slater condition and generalized Slater conditions. Recently, Penot and Volle investigated the following result.

**Theorem 2.1** [11] *Let $C$ be a convex subset of a Banach space $X$, $K$ be a closed and convex cone of a Banach space $Y$, $g$ be a $K$-convex function from $X$ to $Y$, $A = C \cap g^{-1}(-K)$, $f$ be a directionally usc quasiconvex function from $C$ to $\mathbb{R}$, and* $\operatorname{epi} g := \{(x, y) \in X \times Y \mid y \in g(x) + K\}$ *be closed. If $\mathbb{R}_+(g(C) + K)$ is a closed subspace of $Y$, then there exists a surrogate multiplier $\bar{\lambda} \in K^+$.*

In Theorem 2.1, the condition "$\mathbb{R}_+(g(C) + K)$ is a closed subspace of $Y$" is a constraint qualification for surrogate duality. When $X$ and $Y$ are Banach spaces (or Fréchet spaces), this constraint qualification is weaker than the Slater condition; for details see Section 5.

## 3 Necessary and Sufficient Constraint Qualification for Surrogate Duality

In the rest of this paper, let $X$ and $Y$ be locally convex Hausdorff topological vector spaces, $K \subset Y$ be a nonempty, closed and convex cone, $Y$ be partially ordered by $K$, $g$ be a continuous and $K$-convex function from $X$ to $Y$, and $A := C \cap g^{-1}(-K)$ be nonempty.

In this section, we show a necessary and sufficient constraint qualification for surrogate duality. At first, we introduce the following constraint qualification.

**Definition 3.1** $\{g(x) \in -K \mid x \in C\}$ is said to satisfy the closed cone constraint qualification for surrogate duality (S-CCCQ) iff

$$\bigcup_{\lambda \in K^+} \mathrm{cl}\Big[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*\Big]$$

is $w^*$-closed.

S-CCCQ is truly weaker than CCCQ; for details, see Section 5. The following proposition is very important.

**Proposition 3.1** *The following conditions hold:*

*(i) for all* $\lambda \in K^+$, $\mathrm{epi}\,\delta_{B_\lambda}^* = \mathrm{cl}[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*]$,

*(ii)* $\mathrm{epi}\,\delta_A^* = \mathrm{cl}\ \bigcup_{\lambda \in K^+} \mathrm{cl}[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*]$,

*where* $B_\lambda := C \cap L(\lambda \circ g, \leq, 0)$.

*Proof* For all $\lambda \in K^+$, clearly $\mathrm{epi}\,\delta_{B_\lambda}^* = \mathrm{epi}\,(\delta_{L(\lambda \circ g, \leq, 0)} + \delta_C)^*$. Also, we can show that $\mathrm{epi}\,(\delta_{L(\lambda \circ g, \leq, 0)} + \delta_C)^* = \mathrm{cl}[\mathrm{epi}\,\delta_{L(\lambda \circ g, \leq, 0)}^* + \mathrm{epi}\,\delta_C^*]$, see [13,14]. By using Jeyakumar's set containment characterization in [12],

$$\mathrm{cl}[\mathrm{epi}\,\delta_{L(\lambda \circ g, \leq, 0)}^* + \mathrm{epi}\,\delta_C^*] = \mathrm{cl}[\mathrm{cl}\ \mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*].$$

We can check easily that

$$\begin{aligned}\mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^* &\subset \mathrm{cl}[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*] \\ &\subset \mathrm{cl}[\mathrm{cl}\ \mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*] \\ &= \mathrm{epi}\,\delta_{B_\lambda}^* \subset \mathrm{epi}\,\delta_A^*.\end{aligned}$$

Hence,

$$\bigcup_{\lambda \in K^+} \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^* \subset \bigcup_{\lambda \in K^+} \mathrm{cl}[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*] \subset \mathrm{epi}\,\delta_A^*.$$

Because of Lemma 3.1 in [4],

$$\mathrm{cl}\left[\bigcup_{\lambda \in K^+} \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*\right] = \mathrm{cl} \bigcup_{\lambda \in K^+} \mathrm{cl}\,[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*] = \mathrm{epi}\,\delta_A^*.$$

This completes the proof. □

By using Proposition 3.1, we can prove that $\{g(x) \in -K \mid x \in C\}$ satisfies S-CCCQ, if and only if the alternative form of S-CCCQ,

$$\operatorname{epi} \delta_A^* \subset \bigcup_{\lambda \in K^+} \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*].$$

The following theorem shows that S-CCCQ is a necessary and sufficient constraint qualification for surrogate duality.

**Theorem 3.1** *The following conditions are equivalent:*

*(i)* $\{g(x) \in -K \mid x \in C\}$ *satisfies S-CCCQ,*
*(ii) for all usc quasiconvex function* $f$ *from* $X$ *to* $\overline{\mathbb{R}}$*, there exists* $\bar{\lambda} \in K^+$ *such that*

$$\inf\{f(x) \mid x \in A\} = \inf\{f(x) \mid x \in C, \bar{\lambda} \circ g(x) \leq 0\}.$$

*(iii) for all* $v \in X^*$*, there exists* $\bar{\lambda} \in K^+$ *such that*

$$\inf\{v(x) \mid x \in A\} = \inf\{v(x) \mid x \in C, \bar{\lambda} \circ g(x) \leq 0\}.$$

*Proof* At first, we show that (i) implies (ii). Let $f$ be a usc quasiconvex function, and $m = \inf_{x\in A} f(x)$. If $L(f, <, m) \cap C = \emptyset$, then put $\bar{\lambda} = 0$ and this completes the proof. If $L(f, <, m) \cap C \neq \emptyset$, there exists $(x^*, \alpha) \in X^* \times \mathbb{R}$ such that for all $x \in A$ and $y \in L(f, <, m)$,

$$\langle x^*, x\rangle \leq \alpha < \langle x^*, y\rangle\,,$$

since $L(f, <, m) \cap A = \emptyset$ and $L(f, <, m)$ is a nonempty, open and convex set. Because of the condition (i) and Proposition 3.1,

$$(x^*, \alpha) \in \operatorname{epi} \delta_A^* \subset \bigcup_{\lambda \in K^+} \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*] = \bigcup_{\lambda \in K^+} \operatorname{epi} \delta_{B_\lambda}^*.$$

Therefore, there exists $\bar{\lambda} \in K^+$ such that $(x^*, \alpha) \in \operatorname{epi} \delta_{B_{\bar{\lambda}}}^*$. By using the above separation inequality, we can prove that for all $x \in C$,

$$\begin{aligned} \bar{\lambda} \circ g(x) \leq 0 &\iff x \in B_{\bar{\lambda}} \\ &\Longrightarrow \langle x^*, x\rangle \leq \alpha \\ &\Longrightarrow x \notin L(f, <, m) \\ &\iff f(x) \geq m, \end{aligned}$$

that is, $\inf\{f(x) \mid x \in C, \bar{\lambda} \circ g(x) \leq 0\} \geq m$, this shows (ii) holds.

Clearly, (ii) implies (iii).

Next, we show that (iii) implies (i). Because of the alternative form of S-CCCQ, we show that $\operatorname{epi} \delta_A^* \subset \bigcup_{\lambda \in K^+} \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*]$. Let $(x^*, \alpha) \in \operatorname{epi} \delta_A^*$, then $\delta_A^*(x^*) \in \mathbb{R}$ and $\delta_A^*(x^*) = -\inf_{x\in A} \langle -x^*, x\rangle$. By the condition (iii), there exists $\bar{\lambda} \in K^+$ such that

$$\inf_{x\in A} \langle -x^*, x\rangle = \inf\{\langle -x^*, x\rangle \mid x \in C, \bar{\lambda} \circ g(x) \leq 0\}.$$

Hence, for all $x \in C$,

$$\bar{\lambda} \circ g(x) \leq 0 \Longrightarrow \langle -x^*, x \rangle \geq -\delta_A^*(x^*) \iff \langle x^*, x \rangle \leq \delta_A^*(x^*).$$

This implies $\delta^*_{B_{\bar{\lambda}}}(x^*) \leq \delta_A^*(x^*) \leq \alpha$, that is,

$$(x^*, \alpha) \in \operatorname{epi} \delta^*_{B_{\bar{\lambda}}} = \operatorname{cl}\left[\operatorname{cone} \operatorname{epi} (\bar{\lambda} \circ g)^* + \operatorname{epi} \delta_C^*\right]$$

because of Proposition 3.1. This completes the proof. □

**Remark 3.1** *Because of the weak duality, we show that $\{g(x) \in -K \mid x \in C\}$ satisfies S-CCCQ, if and only if for all usc quasiconvex function $f$,*

$$\inf\{f(x) \mid x \in A\} = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\}.$$

## 4 Necessary and Sufficient Constraint Qualification for Surrogate Min-Max Duality

In this section, we show a necessary and sufficient constraint qualification for surrogate min-max duality. At first, we introduce the following constraint qualification.

**Definition 4.1** $\{g(x) \in -K \mid x \in C\}$ is said to satisfy the basic constraint qualification for surrogate duality (S-BCQ) at $x_0 \in A$ iff

$$N_A(x_0) = \bigcup_{\lambda \in K^+} \left\{ x^* \in X^* \,\middle|\, (x^*, \langle x^*, x_0 \rangle) \in \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*] \right\}.$$

Furthermore, $\{g(x) \in -K \mid x \in C\}$ is said to satisfy S-BCQ iff for all $y \in A$, $\{g(x) \in -K \mid x \in C\}$ satisfies S-BCQ at $y$.

S-BCQ is closely related to S-CCCQ and [CQ2]. For details, see Section 5. The following proposition is very important.

**Proposition 4.1** *Let $x_0 \in A$. The following conditions hold:*

*(i) for all $\lambda \in K^+$,*

$$N_{B_\lambda}(x_0) = \{x^* \in X^* \mid (x^*, \langle x^*, x_0 \rangle) \in \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*]\},$$

*(ii)* $$N_A(x_0) \supset \bigcup_{\lambda \in K^+} \left\{ x^* \in X^* \,\middle|\, (x^*, \langle x^*, x_0 \rangle) \in \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*] \right\}.$$

*Proof* By using Proposition 3.1 (i), we can prove that for all $\lambda \in K^+$,

$$\begin{aligned} x^* \in N_{B_\lambda}(x_0) &\iff \delta^*_{B_\lambda}(x^*) \leq \langle x^*, x_0 \rangle \\ &\iff (x^*, \langle x^*, x_0 \rangle) \in \operatorname{epi} \delta^*_{B_\lambda} \\ &\iff (x^*, \langle x^*, x_0 \rangle) \in \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*], \end{aligned}$$

and then $\{x^* \in X^* \mid (x^*, \langle x^*, x_0\rangle) \in \mathrm{cl}[\mathrm{cone\ epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*]\} = N_{B_\lambda}(x_0)$. Since $A \subset B_\lambda$, $N_{B_\lambda}(x_0) \subset N_A(x_0)$, hence,

$$\bigcup_{\lambda \in K^+} \left\{ x^* \in X^* \,\middle|\, (x^*, \langle x^*, x_0\rangle) \in \mathrm{cl}[\mathrm{cone\ epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*] \right\} \subset N_A(x_0),$$

this completes the proof. □

By using Proposition 4.1, we can prove that $\{g(x) \in -K \mid x \in C\}$ satisfies S-BCQ, if and only if

$$N_A(x_0) \subset \bigcup_{\lambda \in K^+} \left\{ x^* \in X^* \,\middle|\, (x^*, \langle x^*, x_0\rangle) \in \mathrm{cl}[\mathrm{cone\ epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*] \right\}.$$

The following theorem shows that S-BCQ is a necessary and sufficient constraint qualification for the surrogate min-max duality. Recall that $\Gamma_0(X)$ is the set of all proper lsc convex function from $X$ to $\overline{\mathbb{R}}$.

**Theorem 4.1** *Let $x_0 \in A$. The following conditions are equivalent:*

*(i)* $\{g(x) \in -K \mid x \in C\}$ *satisfies S-BCQ at* $x_0$,

*(ii) for all* $f \in \Gamma_0(X)$ *with* $\mathrm{dom} f \cap A \neq \emptyset$ *and* $\mathrm{epi}\, f^* + \mathrm{epi}\,\delta_A^*$ *is* $w^*$*-closed,* $x_0$ *is a global minimizer of* $f$ *in* $A$ *if and only if there exists* $\bar\lambda \in K^+$ *such that*

$$f(x_0) = \min\{f(x) \mid x \in C, \bar\lambda \circ g(x) \leq 0\}.$$

*(iii) for all* $v \in X^*$ $x_0$ *is a global minimizer of* $v$ *in* $A$ *if and only if there exists* $\bar\lambda \in K^+$ *such that*

$$v(x_0) = \min\{v(x) \mid x \in C, \bar\lambda \circ g(x) \leq 0\}.$$

*Proof* At first, we show that (i) implies (ii). Because of assumptions of $f$, $x_0$ is a global minimizer of $f$ in $A$ if and only if $0 \in \partial f(x_0) + N_A(x_0)$. Because of S-BCQ and Proposition 4.1 (i),

$$0 \in \partial f(x_0) + N_A(x_0) \iff 0 \in \partial f(x_0) + \bigcup_{\lambda \in K^+} N_{B_\lambda}(x_0),$$

that is, there exists $\bar\lambda \in K^+$ such that $0 \in \partial f(x_0) + N_{B_{\bar\lambda}}(x_0)$. This means that $x_0$ is a global minimizer of $f$ in $B_{\bar\lambda} = C \cap L(\bar\lambda \circ g, \leq, 0)$, this completes the first part of the proof.

It is clear that (ii) implies (iii).

Next, we show that (iii) implies (i). Because of Proposition 4.1, we only show that

$$N_A(x_0) \subset \bigcup_{\lambda \in K^+} \{x^* \in X^* \mid (x^*, \langle x^*, x_0\rangle) \in \mathrm{cl}[\mathrm{cone\ epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*]\}.$$

Let $x^* \in N_A(x_0)$, then $x_0$ is a global minimizer of $-x^*$ in $A$. From (iii), there exists $\bar\lambda \in K^+$ such that

$$\langle -x^*, x_0\rangle = \min\{\langle -x^*, x\rangle \mid x \in C, \bar\lambda \circ g(x) \leq 0\}.$$

For all $x \in C$,

$$\bar{\lambda} \circ g(x) \leq 0 \Longrightarrow \langle -x^*, x \rangle \geq \langle -x^*, x_0 \rangle \iff \langle x^*, x - x_0 \rangle \leq 0,$$

that is $x^* \in N_{B_{\bar{\lambda}}}(x_0)$. This completes the proof. □

**Remark 4.1** *Because of the weak duality, we show that* $\{g(x) \in -K \mid x \in C\}$ *satisfies S-BCQ, if and only if for all* $f \in \Gamma_0(X)$ *with* $\operatorname{dom} f \cap A \neq \emptyset$ *and* $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ *is* $w^*$*-closed,*

$$\min_{x \in A} f(x) = \max_{\lambda \in K^+} \inf\{f(x) \mid x \in C, \lambda \circ g(x) \leq 0\}.$$

**Remark 4.2** *By using Theorem 1 in* [15], *we can prove that the following conditions are equivalent:*

*(i)* $\{g(x) \in -K \mid x \in C\}$ *satisfies S-BCQ at* $x_0$,
*(ii) for all quasiconvex functions* $f \in Q_F(X)$ *with a generator* $G$, *if* $x_0$ *is a local minimizer of* $f$ *in* $A$, *then there exists* $\bar{\lambda} \in K^+$ *such that*

$$0 \in \partial_G f(x_0) + \left\{x^* \in X^* \mid (x^*, \langle x^*, x_0 \rangle) \in \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\bar{\lambda} \circ g)^* + \operatorname{epi} \delta_C^*]\right\}.$$

*For details, see* [15].

## 5 Comparisons of Constraint Qualifications

In this section, we compare S-CCCQ, S-BCQ and known constraint qualifications in [1–5, 11, 16]. For a given set $S \subset Y$, we denote the interior, the affine hull, and the core of $S$, by int$S$, aff$S$, and core$S$, respectively. The core of $S$ relative to aff$S$ is called intrinsic core of $S$ and denoted by icr$S$. If $S$ is convex, the strong quasi-relative interior of $S$, sqri$S$, is the set of those $y \in S$, for which cone$(S - y)$ is a closed subspace.

**Proposition 5.1** *The following statements hold:*

*(i) if* $\{g(x) \in -K \mid x \in C\}$ *satisfies CCCQ, then* $\{g(x) \in -K \mid x \in C\}$ *satisfies S-CCCQ,*
*(ii) if* $\{g(x) \in -K \mid x \in C\}$ *satisfies [CQ2], then* $\{g(x) \in -K \mid x \in C\}$ *satisfies S-BCQ,*
*(iii) if* $\{g(x) \in -K \mid x \in C\}$ *satisfies S-CCCQ, then* $\{g(x) \in -K \mid x \in C\}$ *satisfies S-BCQ.*

*Proof* (i) In the proof of Proposition 3.1, we show the following inclusion,

$$\operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^* \subset \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*] = \operatorname{epi} \delta_{B_\lambda}^* \subset \operatorname{epi} \delta_A^*,$$

for all $\lambda \in K^+$, that is,

$$\bigcup_{\lambda \in K^+} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^* \subset \bigcup_{\lambda \in K^+} \operatorname{cl}[\operatorname{cone} \operatorname{epi} (\lambda \circ g)^* + \operatorname{epi} \delta_C^*] \subset \operatorname{epi} \delta_A^*.$$

If $\{g(x) \in -K \mid x \in C\}$ satisfies CCCQ, then

$$\bigcup_{\lambda \in K^+} \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^* = \bigcup_{\lambda \in K^+} \mathrm{cl}[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*] = \mathrm{epi}\,\delta_A^*,$$

this means that $\{g(x) \in -K \mid x \in C\}$ satisfies S-CCCQ.

(ii) Let $x_0 \in A$. In Proposition 4.1, we show the following inclusion,

$$\bigcup_{\lambda \in K^+} \{x^* \in X^* \mid (x^*, \langle x^*, x_0\rangle) \in \mathrm{cl}[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*]\} \subset N_A(x_0).$$

Also, we can check that for all $\lambda \in K^+$,

$$\begin{aligned} N_C(x_0) + \{x^* \in X^* \mid (x^*, \langle x^*, x_0\rangle) \in \mathrm{epi}\,(\lambda \circ g)^*\} \\ \subset \{x^* \in X^* \mid (x^*, \langle x^*, x_0\rangle) \in \mathrm{cl}[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*]\}, \end{aligned}$$

that is,

$$\begin{aligned} N_C(x_0) + \{x^* \in X^* \mid (x^*, \langle x^*, x_0\rangle) \in \bigcup_{\lambda \in K^+} \mathrm{epi}\,(\lambda \circ g)^*\} \\ \subset \bigcup_{\lambda \in K^+} \{x^* \in X^* \mid (x^*, \langle x^*, x_0\rangle) \in \mathrm{cl}[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*]\} \\ \subset N_A(x_0). \end{aligned}$$

Hence, if $\{g(x) \in -K \mid x \in C\}$ satisfies [CQ2], then

$$\begin{aligned} N_C(x_0) + \{x^* \in X^* \mid (x^*, \langle x^*, x_0\rangle) \in \bigcup_{\lambda \in K^+} \mathrm{epi}\,(\lambda \circ g)^*\} \\ = \bigcup_{\lambda \in K^+} \{x^* \in X^* \mid (x^*, \langle x^*, x_0\rangle) \in \mathrm{cl}[\mathrm{cone}\ \mathrm{epi}\,(\lambda \circ g)^* + \mathrm{epi}\,\delta_C^*]\} \\ = N_A(x_0), \end{aligned}$$

this means that $\{g(x) \in -K \mid x \in C\}$ satisfies S-BCQ at $x_0$.

(iii) We assume that $\{g(x) \in -K \mid x \in C\}$ satisfies S-CCCQ. We show the condition (iii) of Theorem 4.1. Let $x_0 \in A$ and $v \in X^*$. Since S-CCCQ is satisfied, there exists $\lambda \in K^+$ such that $\inf_{x \in A} \langle v, x\rangle = \inf_{x \in B_\lambda} \langle v, x\rangle$. If $x_0$ is a minimizer of $v$ in $A$, then it is clear that $\langle v, x_0\rangle = \min_{x \in B_\lambda} \langle v, x\rangle$. Conversely, if $\langle v, x_0\rangle = \min_{x \in B_\lambda} \langle v, x\rangle$, then $x_0$ is a minimizer of $v$ in $A$ since $x_0 \in A \subset B_\lambda$. Hence, "$x_0$ is a minimizer of $v$ in $A$" if and only if "there exists $\lambda \in K^+$ such that $\langle v, x_0\rangle = \min_{x \in B_\lambda} \langle v, x\rangle$", that is, the condition (iii) holds. Because of Theorem 4.1, this completes the proof. □

Based on [4,5,11], we assume that $X$ and $Y$ are Banach spaces. By Proposition 5.1 and [4,5,11], we investigate the following relations among well-known constraint qualifications.

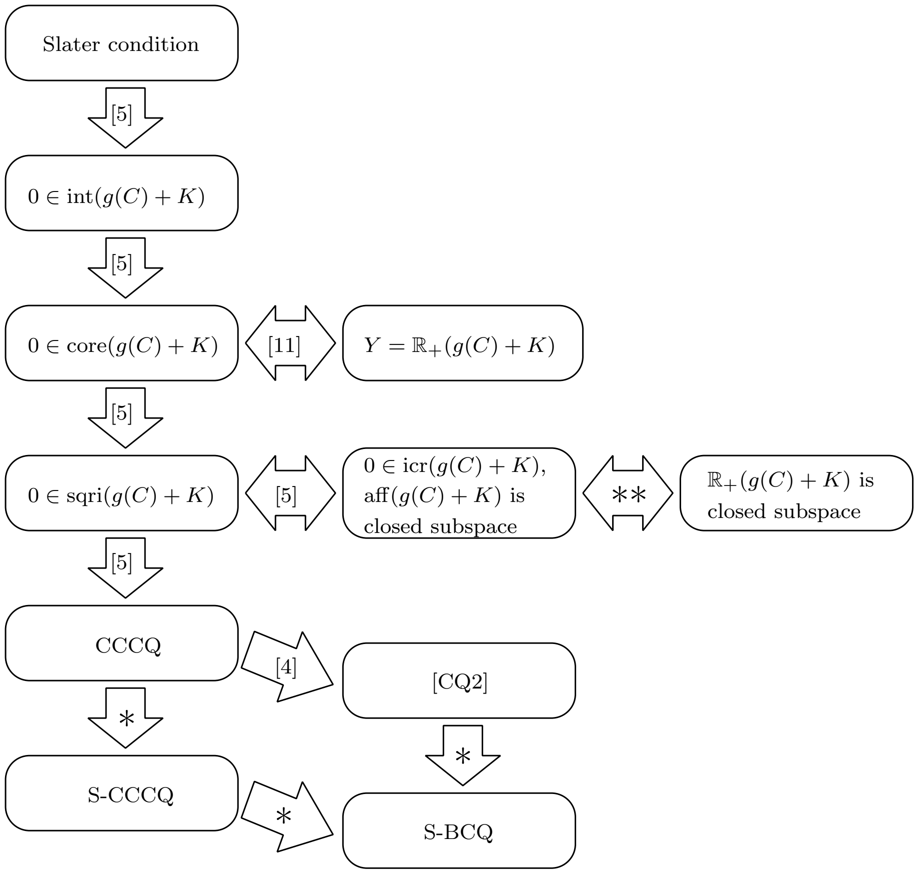


**Fig. 1**

The above asterisk ($*$) indicates that these implications are proved in Proposition 5.1. The double asterisk ($**$), "$0 \in \mathrm{icr}(g(C)+K)$ and $\mathrm{aff}(g(C)+K)$ is a closed subspace" if and only if "$\mathbb{R}_+(g(C)+K)$ is a closed subspace", is proved easily. Also, according to [1], CCCQ holds when $X$ and $Y$ are Fréchet spaces and $0 \in \mathrm{sqri}(g(C)+K)$. However, in locally convex topological vector space, such a result does not hold; for details, see [1].

In Proposition 3.1, we show if CCCQ is satisfied then S-CCCQ is satisfied. However, the opposite is not generally true. Actually, if $X = C = \mathbb{R}$, $K = \mathbb{R}_+$, and $g(x) = x^2$, then $\{g(x) \leq 0\}$ satisfies S-CCCQ but does not satisfy CCCQ. Similarly, even if S-BCQ is satisfied, [CQ2] is not always satisfied.

Recently, regularity conditions are investigated by many researchers; for example, see [1, 17–19]. The papers by Moldovan and Pellegrini [18, 19] aimed at giving a reference scheme for walking in the jungle of regularity conditions and constraint qualifications for various inequality constraints. In general, a condition, which guarantees the strong duality, is called regularity condition or constraint qualification, according to whether the condition does or does not involve the objective function, respectively. In this paper, we study only necessary and sufficient constraint qualifications for convex inequality constraints, but do not consider regularity conditions. The application of regularity conditions to quasiconvex programming problems may constitute a topic for future research.

## 6 Conclusion

In this paper, we investigate necessary and sufficient constraint qualifications for surrogate duality and surrogate min-max duality. Also, we compare these constraint qualifications with previous ones in convex and quasiconvex programming. The following table shows necessary and sufficient constraint qualifications for Lagrange duality and surrogate duality.

| | strong | min-max |
|---|---|---|
| real-valued Lagrange | FM [3] | BCQ [16] |
| vector-valued Lagrange | CCCQ [2–5] | [CQ2] [4] |
| surrogate | S-CCCQ | S-BCQ |

**Table 1**

Also, in quasiconvex programming, we investigated Q-CCCQ and Q-BCQ for necessary and sufficient constraint qualifications for Lagrange-type dualities; for details, see [15,20].

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