Journal of Global Optimization manuscript No.
(will be inserted by the editor)

Some constraint qualifications for quasiconvex vector-valued systems

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Abstract In this paper, we consider minimization problems with a quasiconvex vector-valued inequality constraint. We propose two constraint qualifications, the closed cone constraint qualification for vector-valued quasiconvex programming (the VQ-CCCQ) and the basic constraint qualification for vector-valued quasiconvex programming (the VQ-BCQ). Based on previous results by Benoist, Borwein, and Popovici (Proc. Amer. Math. Soc. 13: 1109–1113, 2002), and the authors (J. Optim. Theory Appl. 149: 554–563, 2011 and Nonlinear Anal. 74: 1279–1285, 2011), we show that the VQ-CCCQ (resp. the VQ-BCQ) is the weakest constraint qualification for Lagrangian-type strong (resp. min-max) duality. As consequences of the main results, we study semi-definite quasiconvex programming problems in our scheme, and we observe the weakest constraint qualifications for Lagrangian-type strong and min-max dualities. Finally, we summarize the characterizations of the weakest constraint qualifications for convex and quasiconvex programming.

Keywords quasiconvex programming · quasiaffine functions · vector-valued · constraint qualification

Mathematics Subject Classification (2000) 90C26 · 26B25

1 Introduction

In mathematical programming, constraint qualifications are essential elements for duality theory. The best-known constraint qualifications are the interior point conditions, also known as the Slater-type constraint qualifications. Often, however, such constraint qualifications are not satisfied for problems that arise in applications. The lack of a constraint qualification can cause theoretical and numerical

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difficulties in applications. In convex programming, research on the weakest constraint qualifications for Lagrangian strong and min-max dualities has been carried out in many studies. Jeyakumar, Dinh and Lee developed the closed cone constraint qualification (the CCCQ) involving epigraphs and extending the Slater-type conditions in [1], and Jeyakumar [2] demonstrated that the CCCQ is the weakest constraint qualification for Lagrangian (strong) duality (in [2], the CCCQ is called [CQ1]). Also, Li, Ng and Pong established the basic constraint qualification (the BCQ) as the weakest constraint qualification for the Lagrangian min-max duality in [3]: let I be an arbitrary set, f and g_i be proper lsc convex functions from X to $\overline{\mathbb{R}}$, and $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$; then $x_0 \in A$ is a global minimizer of f in A if and only if there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that $0 \in \partial f(x_0) + \sum_{i \in I} \lambda_i \partial g_i(x_0)$. Moreover, for a vector-valued convex inequality system, Jeyakumar established the BCQ, which is called [CQ2], for Lagrangian min-max duality as the weakest constraint qualification (see [2]).

In recent research [4, 5] on the weakest constraint qualifications for Lagrangian-type strong and min-max dualities in quasiconvex programming, we established constraint qualifications for real-valued quasiconvex inequality systems (the Q-CCCQ and the Q-BCQ) and proved that these constraint qualifications were the weakest ones for certain Lagrangian-type dualities. To define these constraint qualifications, we introduced the notion of a generator of quasiconvex functions, based on Penot and Volle's interesting result that each lower semi-continuous quasiconvex function consists of a supremum of some family of lower semi-continuous quasiaffine functions [6].

In the present paper, we consider the weakest constraint qualifications for Lagrangian-type dualities of the following minimization programming problem with a quasiconvex vector-valued inequality constraint:

$$\begin{cases} \text{minimize } f(x), \\ \text{subject to } g(x) \in -K, \end{cases}$$

where X and Y are Banach spaces, K is a closed convex cone in Y, f is a function from X to $\overline{\mathbb{R}} = [-\infty, +\infty]$, and g is a K-quasiconvex function from X to Y. We propose and investigate constraint qualifications for a vector-valued quasiconvex inequality system as generalizations of the results in [4] and [5], and show that these qualifications are the weakest ones for certain Lagrangian-type dualities.

The remainder of the present paper is organized as follows. In Section 2, we give some preliminaries and notation. In Section 3, we propose two constraint qualifications, the closed cone constraint qualification for vector-valued quasiconvex programming (the VQ-CCCQ) and the basic constraint qualification for vector-valued quasiconvex programming (the VQ-BCQ). Based on previous results by Benoist, Borwein, and Popovici [7], and the authors [4, 5], we show that the VQ-CCCQ (resp. the VQ-BCQ) is the weakest constraint qualification for Lagrangian-type strong (resp. min-max) duality. As consequences of the main results, in Section 4, we study semi-definite quasiconvex programming problems in our scheme, and we observe the weakest constraint qualifications for Lagrangian-type strong and min-max dualities. Finally, in Section 5, we summarize the characterizations of the weakest constraint qualifications for convex and quasiconvex programming.

2 Preliminaries

Let X be a Banach space, let X^* be the continuous dual space of X, and let $\langle x^*, x\rangle$ denote the value of a functional $x^* \in X^*$ at $x \in X$. Given a set $A^* \subset X^*$, we denote the w^*-closure, the convex hull and the conical hull generated by A^*, by $\mathrm{cl}\, A^*$, $\mathrm{co}\, A^*$ and $\mathrm{cone}\, A^*$, respectively. The normal cone of $A \subset X$ at $z_0 \in A$ is denoted by $N_A(z_0) = \{x^* \in X^* \mid \forall y \in A, \langle x^*, y - z_0\rangle \leq 0\}$. The indicator function δ_A of A is defined by

$$\delta_A(x) := \begin{cases} 0 & x \in A, \\ \infty & otherwise. \end{cases}$$

Throughout the present paper, let f be a function from X to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Here, f is said to be proper if for all $x \in X$, $f(x) > -\infty$ and there exists $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\mathrm{dom} f$, that is, $\mathrm{dom} f = \{x \in X \mid f(x) < \infty\}$. The epigraph of f, $\mathrm{epi} f$, is defined as $\mathrm{epi} f = \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\}$, and f is said to be convex if $\mathrm{epi} f$ is convex. In addition, the Fenchel conjugate of f, $f^* : X^* \to \overline{\mathbb{R}}$, is defined as $f^*(u) = \sup_{x \in \mathrm{dom} f}\{\langle u, x\rangle - f(x)\}$. Recall that f is said to be quasiconvex if for all x_1, $x_2 \in X$ and $\alpha \in (0, 1)$,

$$f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \alpha) = \{x \in X \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. Then, f is quasiconvex if and only if for any $\alpha \in \mathbb{R}$, $L(f, \leq, \alpha)$ is a convex set, or equivalently, for any $\alpha \in \mathbb{R}$, $L(f, <, \alpha)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true.

Recall $\mathit{\Gamma}_0(X)$, the set of all proper lsc convex functions. It is well known that a function in $\mathit{\Gamma}_0(X)$ consists of a supremum of some family of affine functions. In the case of quasiconvex functions, a similar result was also proved by Penot and Volle [6]. A function f is said to be quasiaffine if quasiconvex and quasiconcave. It is important that f is lsc quasiaffine if and only if there exists $k \in Q$ and $w \in X^*$ such that $f = k \circ w$, where $Q = \{k : \mathbb{R} \to \overline{\mathbb{R}} \mid k \text{ is lsc and non-decreasing}\}$. By using the notion of quasiaffine function, Penot and Volle proved that f is lsc quasiconvex if and only if there exists $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ such that $f = \sup_{i \in I} k_i \circ w_i$. This result indicates that a lsc quasiconvex function f consists of a supremum of some family of lsc quasiaffine functions. Based on this result, we define a notion of generator for quasiconvex functions in [4], that is, $G = \{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ is said to be a generator of f if $f = \sup_{i \in I} k_i \circ w_i$. From Penot and Volle's result, all lsc quasiconvex functions have at least one generator. Also, it is clear that all proper lsc convex functions have a generator which consists of continuous affine functions.

Moreover, we introduce a generalized notion of inverse function of $k \in Q$. The following function k^{-1} is said to be the hypo-epi-inverse of k:

$$k^{-1}(a) = \inf\{b \in \mathbb{R} \mid a < k(b)\} = \sup\{b \in \mathbb{R} \mid k(b) \leq a\}.$$

If k has the inverse function, then the inverse and the hypo-epi-inverse of k are the same, in detail see [6]. In this paper, we denote the hypo-epi-inverse of k by k^{-1}.

Also, we denote the lower left-hand Dini derivative of $k \in Q$ at t by $D_{-}k(t)$, that is $D_{-}k(t) = \liminf_{\alpha \to 0-} \frac{k(t+\alpha)-k(t)}{\alpha}$. A function k is said to be lower left-hand Dini differentiable if $D_{-}k(t)$ is finite for all $t \in \mathbb{R}$.

Let Y be a Banach space, and K a nonempty closed convex cone of Y. By introducing the binary relation $\leq_K$ on Y by $y \leq_K z$ if and only if $z - y \in K$, $(Y, \leq_K)$ becomes a partially ordered set. Also, $(Y, \leq_K)$ is said to be directed if for all y_1, $y_2 \in Y$, there exists $z \in Y$ such that $y_1 \leq_K z$ and $y_2 \leq_K z$. Let Y^* be the continuous dual space of Y, K^+ the positive polar cone of K, and $\operatorname{extd} K^+$ the set of all extreme directions of K^+. Recall that $K^+ = \{y^* \in Y^* \mid \forall y \in K, \langle y^*, y\rangle \geq 0\}$ and $y^* \in \operatorname{extd} K^+$ if and only if $y^* \in K^+ \setminus \{0\}$ and for all y_1^*, $y_2^* \in K^+$ with $y^* = y_1^* + y_2^*$, y_1^*, $y_2^* \in \mathbb{R}_+\{y^*\}$. A function g is said to be K-convex if for all x_1, $x_2 \in X$, and $\alpha \in (0,1)$, $(1-\alpha)g(x_1) + \alpha g(x_2) \in g((1-\alpha)x_1 + \alpha x_2) + K$. It is well known that g is K-convex if and only if $y^* \circ g$ is convex for all $y^* \in K^+$. Also, a function g is said to be K-quasiconvex if for all $y \in Y$, x_1, $x_2 \in X$, and $\alpha \in (0,1)$ with $y \in (g(x_1)+K) \cap (g(x_2)+K)$, $y \in g((1-\alpha)x_1 + \alpha x_2) + K$. In [7], K-quasiconvexity is characterized as follows:

Theorem 2.1 *[7] Let D be a nonempty convex subset of a vector space Z, and K be a closed convex cone in Banach space Y satisfying $(Y, \leq_K)$ is directed and $K^+ = \operatorname{cl} \operatorname{co} \operatorname{extd} K^+$. For a function g from D to Y, the following conditions are equivalent:*

(i) g is K-quasiconvex,
(ii) $y^* \circ g$ is quasiconvex for all $y^* \in \operatorname{extd} K^+$.

Next we mention about previous results concerned with the weakest constraint qualifications. In convex optimization, the closed cone constraint qualification (the CCCQ) have been investigated extensively as the weakest one for the Lagrangian strong duality see [1,3,8,9], and the basic constraint qualification (the BCQ) for the Lagrangian min-max duality, see [2,3]. In quasiconvex optimization, the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ) has been established in [4], as the weakest one for the Lagrangian-type strong duality:

Definition 2.1 [4] Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i for each $i \in I$ and $T = \{t = (i,j) \mid i \in I, j \in J_i\}$. Assume that $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$ is non-empty set. Then, the quasiconvex system $\{g_i(x) \leq 0 \mid i \in I\}$ satisfies the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ) w.r.t. $\{(k_t, w_t) \mid t \in T\}$ if

$$\operatorname{cone} \operatorname{co} \bigcup_{t \in T} \{(w_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty)$$

is w^*-closed.

Theorem 2.2 *[4] Let f be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$. Assume that $A = \{x \in X \mid f(x) \leq 0\}$ is non-empty set. Then, the following statements are equivalent:*

(i) $\{f(x) \leq 0\}$ satisfies the Q-CCCQ w.r.t. $\{(k_i, w_i) \mid i \in I\}$,

(ii) for all $h \in \Gamma_0(X)$ *with* $\mathrm{dom} h \cap A \neq \emptyset$ *and* $\mathrm{epi} h^* + \mathrm{epi}\delta_A^*$ *is* w^**-closed,*

$$\inf_{x\in A} h(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x\in X} \left\{ h(x) + \sum_{i\in I} \lambda_i (w_i(x) - k_i^{-1}(0)) \right\}.$$

Also the basic constraint qualification for quasiconvex programming (the Q-BCQ) has been established as the weakest one for Lagrangian-type min-max duality by the authors, see [5]:

Definition 2.2 [5] Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i for each $i \in I$, $T = \{t = (i,j) \mid i \in I, j \in J_i\}$, $T(x) = \{t \in T \mid k_t(\langle w_t, x\rangle) = 0,\ k_t^{-1}(0) = \langle w_t, x\rangle\}$, and $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$.

The family $\{g_i \mid i \in I\}$ is said to satisfy the basic constraint qualification for quasiconvex programming (the Q-BCQ) with respect to $\{(k_t, w_t) \mid t \in T\}$ at $x \in A$ if

$$N_A(x) = \mathrm{cone}\,\mathrm{co} \bigcup_{t\in T(x)} \{w_t\}.$$

Theorem 2.3 *[5] Let* $\{g_i \mid i \in I\}$ *be a family of lsc quasiconvex functions from* X *to* $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ *be a generator of* g_i *for each* $i \in I$, $T = \{t = (i,j) \mid i \in I, j \in J_i\}$, $T(x) = \{t \in T \mid k_t(\langle w_t, x\rangle) = 0,\ k_t^{-1}(0) = \langle w_t, x\rangle\}$, $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$ *and* $x_0 \in A$. *Then, the following statements (i), (ii) and (iii) are equivalent:*

(i) $\{g_i(x) \leq 0 \mid i \in I\}$ *satisfies the Q-BCQ w.r.t.* $\{(k_t, w_t) \mid t \in T\}$ *at* x_0,

(ii) for each $f \in \Gamma_0(X)$ *with* $\mathrm{dom} f \cap A \neq \emptyset$ *and* $\mathrm{epi} f^* + \mathrm{epi}\delta_A^*$ *is* w^**-closed,* x_0 *is a minimizer of* f *in* A *if and only if there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\lambda_t = 0$ *for all* $t \in T \setminus T(x_0)$, *and*

$$0 \in \partial f(x_0) + \sum_{t\in T} \lambda_t w_t,$$

(iii) for all $f \in Q_F(X) \cup Q_C(X)$ *with a generator* G, *if* x_0 *is a local minimizer of* f *in* A, *then, there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\lambda_t = 0$ *for all* $t \in T \setminus T(x_0)$, *and*

$$0 \in \partial_G f(x_0) + \sum_{t\in T} \lambda_t w_t,$$

where $\partial_G f(x_0)$ *is the subdifferential in [5] of* f *at* x_0, *that is,* $\partial_G f(x_0) = \mathrm{cl}\,\mathrm{co}\,\{D_- k_s(\langle w_s, x_0\rangle) w_s \mid s \in S(x_0)\}$, *and* $Q_F(X)$ *and* $Q_C(X)$ *are the following families of quasiconvex functions:*

$$Q_F(X) = \left\{ \sup_{s\in S} k_s \circ w_s \;\middle|\; \begin{array}{l} \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*,\ S : \textit{finite}, \\ \forall s \in S, k_s : \textit{continuous and lower left-hand Dini diff.} \end{array} \right\},$$

$$Q_C(X) = \left\{ \sup_{s\in S} k_s \circ w_s \;\middle|\; \begin{array}{l} \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*,\ S : \textit{compact}, \\ s \mapsto w_s : \textit{continuous}, (s,t) \mapsto k_s(t) : \textit{usc}, \\ D_- k_s(t) \in \mathbb{R} \textit{ and } (s,t) \mapsto D_- k_s(t) : \textit{continuous.} \end{array} \right\}.$$

3 Constraint qualifications for vector-valued quasiconvex functions

As mention in Section 1, we consider the weakest constraint qualifications for Lagrangian-type dualities of the following minimization programming problem with a quasiconvex vector-valued inequality constraint:

$$\begin{cases} \text{minimize } f(x), \\ \text{subject to } g(x) \in -K, \end{cases}$$

where X and Y are Banach spaces, K is a closed convex cone in Y, f is a function from X to $\overline{\mathbb{R}} = [-\infty, +\infty]$, g is a K-quasiconvex function from X to Y, and $A = \{x \in X \mid g(x) \in -K\}$. Also, assume that $(Y, \leq_K)$ is directed and $K^+ = \operatorname{cl}\operatorname{co}\operatorname{extd} K^+$.

Definition 3.1 Let g be a continuous K-quasiconvex function, $\{(k_{(y^*,j)}, w_{(y^*,j)}) \mid j \in J_{y^*}\} \subset Q \times X^*$ be a generator of $y^* \circ g$ for each $y^* \in \operatorname{extd} K^+$, and $T = \{t = (y^*, j) \mid y^* \in \operatorname{extd} K^+, j \in J_{y^*}\}$. Assume that A is non-empty set. Then, the vector-valued quasiconvex system $\{g(x) \in -K\}$ satisfies the closed cone constraint qualification for vector-valued quasiconvex programming (the VQ-CCCQ) with respect to $\{(k_t, w_t) \mid t \in T\}$ if

$$\operatorname{cone}\operatorname{co} \bigcup_{t \in T} \{(w_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty)$$

is w^*-closed.

By the similar way in [4], we can check that $\{g(x) \in -K\}$ satisfies the VQ-CCCQ if and only if the alternative form of the VQ-CCCQ,

$$\operatorname{epi}\delta_A^* \subset \operatorname{cone}\operatorname{co} \bigcup_{t \in T} \{(w_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty)$$

holds. In the next theorem, we can see that the VQ-CCCQ is the weakest constraint qualification of a vector-valued quasiconvex inequality system for Lagrangian-type duality.

Theorem 3.1 *Let g be a continuous K-quasiconvex function, $\{(k_{(y^*,j)}, w_{(y^*,j)}) \mid j \in J_{y^*}\} \subset Q \times X^*$ be a generator of $y^* \circ g$ for each $y^* \in \operatorname{extd} K^+$, and $T = \{t = (y^*, j) \mid y^* \in \operatorname{extd} K^+, j \in J_{y^*}\}$. Assume that A is non-empty set. Then, the following statements are equivalent:*

(i) $\{g(x) \in -K\}$ *satisfies the VQ-CCCQ w.r.t.* $\{(k_t, w_t) \mid t \in T\}$,

(ii) for all $f \in \Gamma_0(X)$ with $\operatorname{epi} f^* + \operatorname{epi}\delta_A^*$ is w^*-closed, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$\inf_{x \in A} f(x) = \inf_{x \in X} \{f(x) + \sum_{t \in T} \lambda_t (w_t(x) - k_t^{-1}(0))\}.$$

Proof We can verify that

$$A = \{x \in X \mid g(x) \in -K\} = \{x \in X \mid \forall y^* \in \operatorname{extd} K^+, y^* \circ g(x) \leq 0\}$$

since $K^+ = \operatorname{cl}\operatorname{co}\operatorname{extd} K^+$ and $K = \{y \in Y \mid \forall y^* \in K^+, \langle y^*, y\rangle \geq 0\}$, and we have

$$A = \{x \in X \mid \forall t \in T, k_t \circ w_t(x) \leq 0\}$$

from the definition of T. Put $h = \sup_{t \in T} k_t \circ w_t$, then h is a lsc quasiconvex function with generator $\{(k_t, w_t) \mid t \in T\}$. Hence, by using Theorem 2.2, we can prove this theorem.

Next, we introduce the following constraint qualification for Lagrangian-type min-max duality:

Definition 3.2 Let g be a continuous K-quasiconvex function, $\{(k_{(y^*,j)}, w_{(y^*,j)}) \mid j \in J_{y^*}\} \subset Q \times X^*$ be a generator of $y^* \circ g$ for each $y^* \in \operatorname{extd} K^+$, $T = \{t = (y^*, j) \mid y^* \in \operatorname{extd} K^+, j \in J_{y^*}\}$, and $T(x) = \{t \in T \mid k_t(\langle w_t, x\rangle) = 0,\ k_t^{-1}(0) = \langle w_t, x\rangle\}$. Assume that A is non-empty set. Then, the vector-valued quasiconvex system $\{g(x) \in -K\}$ is said to satisfy the basic constraint qualification for vector-valued quasiconvex programming (the VQ-BCQ) with respect to $\{(k_t, w_t) \mid t \in T\}$ at $x \in A$ if

$$N_A(x) = \operatorname{cone} \operatorname{co} \bigcup_{t \in T(x)} \{w_t\}.$$

Also, we can check that the VQ-BCQ is equivalent to the following inclusion

$$N_A(x) \subset \operatorname{cone} \operatorname{co} \bigcup_{t \in T(x)} \{w_t\}.$$

The following result shows us the VQ-BCQ is the weakest constraint qualification of a vector-valued quasiconvex inequality system for Lagrangian-type min-max duality:

Theorem 3.2 *Let g be a continuous K-quasiconvex function, $\{(k_{(y^*,j)}, w_{(y^*,j)}) \mid j \in J_{y^*}\} \subset Q \times X^*$ be a generator of $y^* \circ g$ for each $y^* \in \operatorname{extd} K^+$, $T = \{t = (y^*, j) \mid y^* \in \operatorname{extd} K^+, j \in J_{y^*}\}$, and $T(x) = \{t \in T \mid k_t(\langle w_t, x\rangle) = 0,\ k_t^{-1}(0) = \langle w_t, x\rangle\}$. and $x_0 \in A$. Then, the following statements (i), (ii) and (iii) are equivalent:*

(i) $\{g(x) \in -K\}$ *satisfies the VQ-BCQ w.r.t.* $\{(k_t, w_t) \mid t \in T\}$ *at* x_0,

(ii) for each $f \in \Gamma_0(X)$ *with* $\operatorname{dom} f \cap A \neq \emptyset$ *and* $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ *is* w^**-closed,* x_0 *is a minimizer of* f *in* A *if and only if there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\lambda_t = 0$ *for all* $t \in T \setminus T(x_0)$*, and*

$$0 \in \partial f(x_0) + \sum_{t \in T} \lambda_t w_t,$$

(iii) for all $f \in Q_F(X) \cup Q_C(X)$ *with a generator* G*, if* x_0 *is a local minimizer of* f *in* A*, then, there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\lambda_t = 0$ *for all* $t \in T \setminus T(x_0)$*, and*

$$0 \in \partial_G f(x_0) + \sum_{t \in T} \lambda_t w_t,$$

Proof In the similar way of the proof of Theorem 3.1, we have $A = \{x \in X \mid \forall t \in T, k_t \circ w_t(x) \leq 0\}$, and $h = \sup_{t \in T} k_t \circ w_t$ is a lsc quasiconvex function with generator $\{(k_t, w_t) \mid t \in T\}$. By using Theorem 2.3, we can prove this theorem.

4 Applications

In this section, we investigate some applications of results in this paper. Let S^n be the vector space of $(n \times n)$ symmetric matrices with the trace inner product which is partially ordered by $\succeq$, that is, for M, $N \in S^n$, $M \succeq N$ if and only if $(M-N)$ is positive semidefinite. Consider the following minimization programming problem:

$$\begin{cases} \text{minimize } f(x), \\ \text{subject to } G(x) \preceq 0, \end{cases}$$

where X is a Banach space, f is a function from X to $\overline{\mathbb{R}}$, and G is a function from X to S^n. Let $S^n_+ = \{M \in S^n \mid M \succeq 0\}$. Then S^n_+ is a closed convex cone, $(S^n_+)^+ = S^n_+$ and $\operatorname{int} S^n_+ = \{M \in S^n \mid M \text{ is positive definite}\}$, in detail, see [2]. Since $\operatorname{int} S^n_+$ is nonempty, $\operatorname{cl}\operatorname{co}\operatorname{extd} S^n_+ = S^n_+$ and $(S^n, \leq_{S^n_+})$ is directed, in detail, see [7]. Hence, Theorem 3.1 and 3.2 apply to any S^n_+-quasiconvex functions.

In the rest of this section, we consider the case when $n = 2$. Let $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in S^2$. Then, $M \in S^2_+$ if and only if

$$\Big(a = 0,\ b = 0 \text{ and } c \geq 0\Big) \text{ or } \Big(a > 0 \text{ and } c \geq \frac{b^2}{a}\Big).$$

Now we show that

$$\operatorname{extd} S^2_+ = \operatorname{cone}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \bigcup \left\{\begin{pmatrix} 1 & b \\ b & b^2 \end{pmatrix} \,\middle|\, b \in \mathbb{R}\right\}\right) \setminus \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B_b = \begin{pmatrix} 1 & b \\ b & b^2 \end{pmatrix}$ for all $b \in \mathbb{R}$, $M = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix}$, and $N = \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} \in S^2_+$. First we show that $A \in \operatorname{extd} S^2_+$. If $M + N = A$, then $a_1 + a_2 = 0$ and a_1, $a_2 \geq 0$ since M and $N \in S^2_+$. This implies that $a_1 = a_2 = 0$. Similarly, $b_1 = b_2 = 0$. Also, clearly c_1, $c_2 \geq 0$, this implies that M, $N \in \mathbb{R}_+\{A\}$, that is, $A \in \operatorname{extd} S^2_+$. Next we show that $B_b \in \operatorname{extd} S^2_+$ for each $b \in \mathbb{R}$. Assume that $M + N = B_b$. If $a_2 = 0$, then $1 = a_1$, $b = b_1$, and $b^2 = c_1 + c_2$. Since $a_1 = 1$ and $c_2 \geq 0$,

$$b^2 = c_1 + c_2 \geq c_1 \geq \frac{b_1^2}{a_1} = b^2,$$

that is $c_1 = b^2$ and $c_2 = 0$. Hence, $M = \begin{pmatrix} 1 & b \\ b & b^2 \end{pmatrix}$ and $N = 0 \in \mathbb{R}\{B\}$. If $a_1 \neq 0$ and $a_2 \neq 0$, then $1 = a_1 + a_2$, $b = b_1 + b_2$, $b^2 = c_1 + c_2$, $c_1 \geq \frac{b_1^2}{a_1}$ and $c_2 \geq \frac{b_2^2}{a_2}$. Clearly, $1 > a_1 > 0$. Then,

$$\frac{b_1^2}{a_1} + \frac{b_2^2}{a_2} - b^2 = \frac{b_1^2}{a_1} + \frac{(b - b_1)^2}{1 - a_1} - b^2 = \frac{(b_1 - ba_1)^2}{a_1(1 - a_1)} \geq 0.$$

Hence,

$$b^2 = c_1 + c_2 \geq \frac{b_1^2}{a_1} + \frac{b_2^2}{a_2} \geq b^2,$$

that is $c_1 = \frac{b_1^2}{a_1}$ and $c_2 = \frac{b_2^2}{a_2}$. Therefore, $M = a_1 \begin{pmatrix} 1 & \frac{b_1}{a_1} \\ \frac{b_1}{a_1} & \frac{b_1^2}{a_1^2} \end{pmatrix}$, $N = a_2 \begin{pmatrix} 1 & \frac{b_2}{a_2} \\ \frac{b_2}{a_2} & \frac{b_2^2}{a_2^2} \end{pmatrix} \in \mathbb{R}_+\{B_b\}$. This implies that $B_b \in \operatorname{extd} S_+^2$. Since $\lambda M \in \operatorname{extd} S_+^2$ holds for any $M \in \operatorname{extd} S_+^2$ and $\lambda > 0$, we have

$$\operatorname{extd} S_+^2 \supset \operatorname{cone}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \bigcup \left\{\begin{pmatrix} 1 & b \\ b & b^2 \end{pmatrix} \,\middle|\, b \in \mathbb{R}\right\}\right) \setminus \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

Conversely, let $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \operatorname{extd} S_+^2$. If $a = 0$, then $a = b = 0$ and $c > 0$ since $M \in S_+^2$ and $M \neq 0$, that is, $M = cA$. If $a \neq 0$, then $a > 0$ and $c \geq \frac{b^2}{a}$ since $M \in S_+^2$. Then, $c = \frac{b^2}{a}$ and M is an element of the right-hand side. Actually, if $c > \frac{b^2}{a}$ then

$$M = \begin{pmatrix} a & b \\ b & \frac{b^2}{a} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c - \frac{b^2}{a} \end{pmatrix}.$$

This is a contradiction. Therefore, the equality holds. Let G be a function from X to S^2 and g_1, g_2, and g_3 be functions from X to $\overline{\mathbb{R}}$ as follows:

$$g_1 = \operatorname{Tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} G\right], g_2 = \operatorname{Tr}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} G\right], g_3 = \operatorname{Tr}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} G\right],$$

where $\operatorname{Tr}M$ is the trace of a matrix M. Then, we can verify that $G = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_3 \end{pmatrix}$, and by Theorem 2.1, G is S_+^2-quasiconvex if and only if g_3 and $g_1 + 2bg_2 + b^2 g_3$ ($b \in \mathbb{R}$) are quasiconvex.

Assume that G is S_+^2-quasiconvex. Then, for each $b \in \mathbb{R}$, there exists a generator $\{(k_{(b,j)}, w_{(b,j)}) \mid j \in J_b\} \subset Q \times X^*$ of $g_1 + 2bg_2 + b^2 g_3$ and there exists a generator $\{(k_{(\gamma,j)}, w_{(\gamma,j)}) \mid j \in J_\gamma\} \subset Q \times X^*$ of g_3, where γ is an index. We can reput generators $\{(k_{(b,w)}, w) \mid w \in S_{X^*}\} \subset Q \times X^*$ of $g_1 + 2bg_2 + a^2 g_3$ where $S_{X^*} = \{x^* \in X^* \mid \|x^*\| = 1\}$. Actually, let $J_{(b,0)} = \{j \in J_b \mid w_{(b,j)} = 0\}$, $J_{(b,w)} = \{j \in J_b \mid w = \frac{w_{(b,j)}}{\|w_{(b,j)}\|}\}$ for all $w \in S_{X^*}$ and let $k_{(b,w)}(t) = \max\{\sup_{j \in J_{(b,w)}} k_{(b,j)}(\|w_{(b,j)}\| t), \sup_{j \in J_{(b,0)}} k_{(b,j)}(0)\}$ for all $w \in S_{X^*}$. Then, $k_{(b,w)} \in Q$ and $g_1 + 2bg_2 + b^2 g_3 = \sup_{w \in S_{X^*}} k_{(b,w)} \circ w$. Also, we can reput a generator of g_3 similarly. Let

$$D = \operatorname{cone} \operatorname{co} \bigcup_{w \in S_{X^*}} \left\{(w, \delta) \,\middle|\, \inf_{b \in \mathbb{R} \cup \{\gamma\}} k_{(b,w)}^{-1}(0) \leq \delta\right\} + \{0\} \times [0, \infty).$$

We can show the following corollary.

Corollary 4.1 *Assume that $A = \{x \in X \mid G(x) \preceq 0\}$ is nonempty. Then, D is closed if and only if for all $f \in \Gamma_0(X)$ with $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ is w^*-closed, there exists $\lambda \in \mathbb{R}_+^{(S_{X^*})}$ such that*

$$\inf_{x \in A} f(x) = \inf_{x \in X} \left\{ f(x) + \sum_{w \in S_{X^*}} \lambda_w (\langle w, x \rangle - \inf_{b \in \mathbb{R} \cup \{\gamma\}} k_{(b,w)}^{-1}(0)) \right\}.$$

Proof We can verify that

$$\begin{aligned} D : \text{closed} &\iff \{g_3(x) \leq 0, (g_1 + bg_2 + b^2 g_3)(x) \leq 0 \mid b \in \mathbb{R}\} : \text{Q-CCCQ} \\ &\iff \{G(x) \preceq 0\} : \text{VQ-CCCQ}. \end{aligned}$$

By using Theorem 3.1, we can proof this corollary.

We compare the condition "D is closed" with previous constraint qualifications. In [9], the following constraint qualification, Farkas-Minkowski (FM), was investigated. Let I be an index set, for each $i \in I$, let $g_i \in \Gamma_0(X)$. The convex system $\{g_i(x) \leq 0 \mid i \in I\}$ is said to be FM if $\operatorname{cone}\operatorname{co} \bigcup_{i \in I} \operatorname{epi} g_i^*$ is w^*-closed. Also, FM is closely related to the closedness of D and the Slater type constraint qualification, "there exists $x \in X$ such that for all $w \in S_{X^*}$ $\langle w, x\rangle < \inf_{b \in \mathbb{R} \cup \{\gamma\}} k^{-1}_{(b,w)}(0)$", assures that D is closed. Actually,

$$\begin{aligned} D : \text{closed} &\iff \{g_3(x) \leq 0, (g_1 + bg_2 + b^2 g_3)(x) \leq 0 \mid b \in \mathbb{R}\} : \text{Q-CCCQ} \\ &\iff \{w(x) - k^{-1}_{(b,w)}(0) \leq 0 \mid b \in \mathbb{R} \cup \{\gamma\}, w \in S_{X^*}\} : \text{FM} \\ &\iff \{w(x) - \inf_{b \in \mathbb{R} \cup \{\gamma\}} k^{-1}_{(b,w)}(0) \leq 0 \mid w \in S_{X^*}\} : \text{FM} \end{aligned}$$

and the Slater constraint qualification implies the FM, in detail see [1,4,9].

Similarly, we can show the following corollary.

Corollary 4.2 *Let $A = \{x \in X \mid G(x) \preceq 0\}$, $x_0 \in A$, and $S_{X^*}(x_0) = \{w \in S_{X^*} \mid \langle w, x\rangle = \inf_{b \in \mathbb{R} \cup \{\gamma\}} k^{-1}_{(b,w)}(0)\}$. Then, the following conditions are equivalent.*

(i) $N_A(x_0) = \operatorname{cone}\operatorname{co} S_{X^*}(x_0)$,

(ii) for each $f \in \Gamma_0(X)$ with $\operatorname{dom} f \cap A \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ is w^*-closed, x_0 is a minimizer of f in A if and only if there exists $\lambda \in \mathbb{R}_+^{(S_{X^*})}$ such that $\lambda_w = 0$ for all $w \in S_{X^*} \setminus S_{X^*}(x_0)$, and

$$0 \in \partial f(x_0) + \sum_{w \in S_{X^*}} \lambda_w w.$$

Proof We can verify that $\{G(x) \preceq 0\}$ satisfies VQ-BCQ if and only if $N_A(x_0) = \operatorname{cone}\operatorname{co} S_{X^*}(x_0)$. By using Theorem 3.2, we can proof this corollary.

5 Conclusion

We define constraint qualifications the VQ-CCCQ and the VQ-BCQ for quasiconvex vector-valued systems, and we prove the VQ-CCCQ is the weakest constraint qualifications for Lagrangian-type strong duality (Theorem 3.1), and the VQ-BCQ is one for Lagrangian-type min-max duality (Theorem 3.2).

The following table shows the weakest constraint qualifications of real/vector-valued convex/quasiconvex inequality systems for Lagrangian strong/min-max dualities.

	strong	min-max
real-valued convex	FM [9]	BCQ [3]
vector-valued convex	CCCQ [1,2]	[CQ2] [2]
real-valued quasiconvex	Q-CCCQ [4]	Q-BCQ [5]
vector-valued quasiconvex	VQ-CCCQ(Theorem 3.1)	VQ-BCQ(Theorem 3.2)

Acknowledgements

The authors are grateful to anonymous referees for many comments and suggestions improved the quality of the paper.

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