Surrogate duality for robust optimization

Satoshi Suzuki[a,*], Daishi Kuroiwa[a], Gue Myung Lee[b,1]

[a] *Department of Mathematics, Shimane University, 1060 Nishikawatsu, Matsue, Shimane, Japan*
[b] *Department of Applied Mathematics, Pukyong National University, Busan 608-737, Korea*

Abstract

Robust optimization problems, which have uncertain data, are considered. We prove surrogate duality theorems for robust quasiconvex optimization problems and surrogate min-max duality theorems for robust convex optimization problems. We give necessary and sufficient constraint qualifications for surrogate duality and surrogate min-max duality, and show some examples at which such duality results are used effectively. Moreover, we obtain a surrogate duality theorem and a surrogate min-max duality theorem for semi-definite optimization problems in the face of data uncertainty.

Keywords: Nonlinear programming, quasiconvex programming, robust optimization

1. Introduction

Mathematical programming problems with data uncertainty are becoming important in optimization due to the reality of uncertainty in many real-world optimization problems. Robust optimization, which has emerged as a powerful deterministic approach for studying mathematical programming with data uncertainty, associates an uncertain mathematical program with its

*corresponding author

Email addresses: suzuki@math.shimane-u.ac.jp (Satoshi Suzuki), kuroiwa@math.shimane-u.ac.jp (Daishi Kuroiwa), gmlee@pknu.ac.kr (Gue Myung Lee)

[1] This author was supported by the Korea Science and Engineering Foundation (KOSEF) NRL program grant funded by the Korea government (MEST)(No. ROA-2008-000-20010-0).

robust counterpart. Many researchers ([1, 8, 9, 10, 11, 13]) have investigated duality theory for linear or convex programming problems under uncertainty with the worst-case approach(the robust approach). They used mainly the duality theorem for linear programming, the Lagrange duality theorem, and Sion's min-max theorem. This research gives elegant, powerful, and completely characterized results for robust convex optimization.

On the other hand, recently, many authors ([4, 5, 6, 15, 16, 17, 20]) investigated surrogate duality for quasiconvex programming. Surrogate duality is used in not only quasiconvex programming but also integer programming and the knapsack problem ([3, 4, 5, 6, 15, 16, 17]). Surrogate duality is also closely related to Lagrange duality. In [20], we investigated a necessary and sufficient constraint qualification for surrogate duality. Also, we investigated that the constraint qualification is weaker than the CCCQ, which is a necessary and sufficient constraint qualification for Lagrange duality.

In the present paper, we investigate surrogate duality theorems for quasiconvex programming under data uncertainty via robust optimization. We also propose new constraint qualifications and compare these with previous ones. The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we investigate surrogate strong duality for robust optimization. In Section 4, we investigate surrogate min-max duality for robust optimization, showing some examples. Finally, in Section 5, we obtain a surrogate duality theorem and a surrogate min-max duality theorem for semi-definite optimization problems in the face of data uncertainty.

2. Preliminaries

Let $\langle v, x\rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Given a set $A \subset \mathbb{R}^n$, we denote the closure, the convex hull, and the conical hull generated by A, by $\mathrm{cl}A$, $\mathrm{co}A$, and $\mathrm{cone}A$, respectively. The indicator function δ_A is defined by

$$\delta_A(x) := \begin{cases} 0 & x \in A, \\ \infty & \textit{otherwise}. \end{cases}$$

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Here, f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\mathrm{dom}f$, that is, $\mathrm{dom}f =$

$\{x \in \mathbb{R}^n \mid f(x) < \infty\}$. The epigraph of f, epi$f$, is defined as epi$f = \{(x,r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq r\}$, and f is said to be convex if epi$f$ is convex. In addition, the Fenchel conjugate of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as $f^*(u) = \sup_{x \in \mathrm{dom} f}\{\langle u, x\rangle - f(x)\}$. The subdifferential of f at x is defined as $\partial f(x) = \{x^* \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, f(y) \geq f(x) + \langle x^*, y - x\rangle\}$. Also, the normal cone of A at $x \in A$ is defined as $N_A(x) = \{x^* \in \mathbb{R}^n \mid \forall y \in A, \langle x^*, y - x\rangle \leq 0\}$. It is clear that $N_A(x) = \partial\delta_A(x)$. Recall that f is said to be quasiconvex if for all x_1, $x_2 \in \mathbb{R}^n$ and $\lambda \in (0,1)$, $f((1-\lambda)x_1 + \lambda x_2) \leq \max\{f(x_1), f(x_2)\}$. Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as $L(f, \diamond, \beta) = \{x \in \mathbb{R}^n \mid f(x) \diamond \beta\}$ for any $\beta \in \mathbb{R}$. Then, f is quasiconvex if and only if for any $\beta \in \mathbb{R}$, $L(f, \leq, \beta)$ is a convex set, or equivalently, for any $\beta \in \mathbb{R}$, $L(f, <, \beta)$ is a convex set. Any convex function is quasiconvex, but the converse is not true.

In [7], Jeyakumar investigated the following set containment characterization. This result is very important and useful in the research of necessary and sufficient constraint qualifications for the Lagrange duality theorem in convex programming.

Theorem 1. *[7] Let I be an arbitrary set, and for each $i \in I$, let g_i be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$. In addition, let $\{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\}$ be nonempty, $x^* \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Then, (i) and (ii) given below are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I, g_i(x) \leq 0\} \subset \{x \in \mathbb{R}^n \mid \langle x^*, x\rangle \leq \alpha\}$,

(ii) $(x^*, \alpha) \in \mathrm{cl}\,\mathrm{cone}\,\mathrm{co} \bigcup_{i \in I} \mathrm{epi} g_i^*$.

In [8], the following result was investigated. This result is one of set containment characterization with data uncertainty and is similar to Theorem 1.

Theorem 2. *[8] Let $I = \{1, \cdots, m\}$ and let g_i be continuous functions from $\mathbb{R}^n \times \mathbb{R}^q$ to $\mathbb{R}$ such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is a convex function. Let $\mathcal{V}_i$, $i = 1, \cdots, m$, be subsets of $\mathbb{R}^q$, $\mathcal{V} = \prod_{i=1}^m \mathcal{V}_i$, and $F = \{x \mid \forall v_i \in \mathcal{V}_i, g_i(x, v_i) \leq 0, \forall i = 1, \cdots, m\} \neq \emptyset$. Then,*

$$\mathrm{epi}\delta_F^* = \mathrm{cl}\,\mathrm{co} \bigcup_{v \in \mathcal{V}, \lambda \in \mathbb{R}_+^m} \mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i)\right)^* .$$

3. Surrogate duality

Throughout this paper, let $I = \{1, \cdots, m\}$, g_i continuous functions from $\mathbb{R}^n \times \mathbb{R}^q$ to $\mathbb{R}$ such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ a convex function, $\mathcal{V}_i$, $i = 1, \cdots, m$, nonempty subsets of $\mathbb{R}^q$, $\mathcal{V} = \prod_{i=1}^m \mathcal{V}_i$, $F = \{x \in \mathbb{R}^n \mid \forall i \in \{1, \cdots, m\}, \forall v_i \in \mathcal{V}_i, g_i(x, v_i) \leq 0\} \neq \emptyset$, and $F_{(v,\lambda)} = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^m \lambda_i g_i(x, v_i) \leq 0 \right\}$.

In this section, we investigate surrogate duality for robust quasiconvex optimization problem. First, we show a set containment characterization with data uncertainty.

Theorem 3. *The following condition hold:*

$$\mathrm{epi}\delta_F^* = \mathrm{cl}\,\mathrm{co} \bigcup_{v \in \mathcal{V}, \lambda \in \mathbb{R}_+^m} \mathrm{cl}\,\mathrm{cone}\,\mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^* .$$

PROOF. For all $v \in \mathcal{V} = \displaystyle\prod_{i=1}^m \mathcal{V}_i$ and $\lambda \in \mathbb{R}_+^m$, it is clear that $F \subset F_{(v,\lambda)}$ and it is easy to verify that

$$\mathrm{epi}\delta_F^* \supset \mathrm{epi}\delta_{F_{(v,\lambda)}}^* = \mathrm{cl}\,\mathrm{cone}\,\mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^* \supset \mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^* .$$

Then,

$$\mathrm{epi}\delta_F^* \supset \bigcup_{v \in \mathcal{V}, \lambda \in \mathbb{R}_+^m} \mathrm{cl}\,\mathrm{cone}\,\mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^* \supset \bigcup_{v \in \mathcal{V}, \lambda \in \mathbb{R}_+^m} \mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^* .$$

By Theorem 2,

$$\mathrm{epi}\delta_F^* = \mathrm{cl}\,\mathrm{co} \bigcup_{v \in \mathcal{V}, \lambda \in \mathbb{R}_+^m} \mathrm{cl}\,\mathrm{cone}\,\mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^* = \mathrm{cl}\,\mathrm{co} \bigcup_{v \in \mathcal{V}, \lambda \in \mathbb{R}_+^m} \mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^* .$$

This completes the proof.

Remark 1. *Assume that $F \neq \emptyset$. From the proof of Theorem 3,*

$$\mathrm{cl}\,\mathrm{co} \bigcup_{v \in \mathcal{V}, \lambda \in \mathbb{R}_+^m} \mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^* = \mathrm{cl}\,\mathrm{co} \bigcup_{v \in \mathcal{V}, \lambda \in \mathbb{R}_+^m} \mathrm{cl}\,\mathrm{cone}\,\mathrm{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^*$$

and if $\bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}^m_+} \operatorname{epi}\left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i)\right)^*$ *is closed and convex, then*

$$\bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}^m_+} \operatorname{cl}\operatorname{cone}\operatorname{epi}\left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i)\right)^*$$

is closed and convex. But the converse does not hold as shown in the following example.

Example 1. *Let* $\mathcal{V} = [1, 2]$*, for each* $v \in \mathcal{V}$*,*

$$g(x, v) = \begin{cases} \frac{v}{2}(x-v)^2 & x \geq v, \\ 0 & -v \leq x \leq v, \\ \frac{v}{2}(x+v)^2 & x \leq -v. \end{cases}$$

We can calculate Fenchel conjugate of $g(\cdot, v)$ *as follows:*

$$(g(\cdot, v))^*(w) = \begin{cases} \frac{w^2}{2v} - vw & w \geq 0, \\ \frac{w^2}{2v} + vw & w \leq 0. \end{cases}$$

Then,

$$\bigcup_{v\in\mathcal{V},\lambda\geq 0} \operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot, v))^* = \{(x, \alpha) \in \mathbb{R}^2 \mid |x| \leq \alpha\},$$

and hence the set is closed and convex. However,

$$\bigcup_{v\in\mathcal{V},\lambda\geq 0} \operatorname{epi}(\lambda g(\cdot, v))^* = \{(x, \alpha) \in \mathbb{R}^2 \mid |x| < \alpha\} \cup \{(0, 0)\},$$

and hence the set is not closed.

In the following theorem, we show a necessary and sufficient constraint qualification of surrogate duality for robust quasiconvex optimization problem.

Theorem 4. *The following conditions are equivalent:*

(i)

$$\bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}^m_+} \mathrm{cl\,cone\,epi}\left(\sum_{i=1}^m \lambda_i g_i(\cdot,v_i)\right)^*$$

is closed and convex,

(ii) for all upper semicontinuous (usc) quasiconvex function f from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ with $\mathrm{dom} f\cap F\neq\emptyset$, there exist $\bar{v}\in\mathcal{V}$ and $\bar{\lambda}\in\mathbb{R}^m_+$ such that

$$\inf\{f(x)\mid x\in F\}=\inf\{f(x)\mid \sum_{i=1}^m \bar{\lambda}_i g_i(x,\bar{v}_i)\leq 0\}.$$

(iii) for all continuous linear function f from $\mathbb{R}^n$ to $\mathbb{R}$, there exist $\bar{v}\in\mathcal{V}$ and $\bar{\lambda}\in\mathbb{R}^m_+$ such that

$$\inf\{f(x)\mid x\in F\}=\inf\{f(x)\mid \sum_{i=1}^m \bar{\lambda}_i g_i(x,\bar{v}_i)\leq 0\}.$$

PROOF. First, we show that (i) implies (ii). Let f be a usc quasiconvex function and $m=\inf_{x\in F} f(x)$. If $m=-\infty$, then (ii) holds trivially. So, assume that m is finite. If $L(f,<,m)$ is empty, then putting $\lambda=0$ and taking any $v\in\mathcal{V}$, the equality holds. If $L(f,<,m)$ is not empty, then there exists $(x^*,\alpha)\in\mathbb{R}^n\times\mathbb{R}$ such that for all $x\in F$ and $y\in L(f,<,m)$,

$$\langle x^*,x\rangle\leq\alpha<\langle x^*,y\rangle\,,$$

since $L(f,<,m)\cap F=\emptyset$ and $L(f,<,m)$ is a nonempty open convex set. By condition (i) and Theorem 3,

$$(x^*,\alpha)\in\mathrm{epi}\delta^*_F=\bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}^m_+} \mathrm{cl\,cone\,epi}\left(\sum_{i=1}^m \lambda_i g_i(\cdot,v_i)\right)^*.$$

Hence, there exist $\bar{v}\in\mathcal{V}$ and $\bar{\lambda}\in\mathbb{R}^m_+$ such that

$$(x^*,\alpha)\in\mathrm{cl\,cone\,epi}\left(\sum_{i=1}^m \bar{\lambda}_i g_i(\cdot,\bar{v}_i)\right)^*.$$

Also, by Theorem 1,

$$\begin{aligned} \text{epi}\delta^*_{F_{(\bar v,\bar\lambda)}} &= \text{epi}\left(\sup_{\alpha\geq 0}\left(\alpha\sum_{i=1}^m \bar\lambda_i g_i(\cdot,\bar v_i)\right)\right)^* \\ &= \text{cl co}\bigcup_{\alpha\geq 0}\text{epi}\left(\alpha\sum_{i=1}^m \bar\lambda_i g_i(\cdot,\bar v_i)\right)^* \\ &= \text{cl}\,\text{cone}\,\text{epi}\left(\sum_{i=1}^m \bar\lambda_i g_i(\cdot,\bar v_i)\right)^*, \end{aligned}$$

in detail, see [2, 7, 14]. Hence, $(x^*,\alpha)\in \text{epi}\delta^*_{F_{(\bar v,\bar\lambda)}}$. By using the above separation inequality, we can prove that for all $x\in\mathbb{R}^n$,

$$\begin{aligned} \sum_{i=1}^m \bar\lambda_i g_i(x,\bar v_i)\leq 0 \quad &\Longleftrightarrow \quad x\in F_{(\bar v,\bar\lambda)} \\ &\Longrightarrow \quad \langle x^*,x\rangle\leq\alpha \\ &\Longrightarrow \quad x\notin L(f,<,m) \\ &\Longleftrightarrow \quad f(x)\geq m, \end{aligned}$$

that is, $\inf\{f(x)\mid \sum_{i=1}^m \bar\lambda_i g_i(x,\bar v_i)\leq 0\}\geq m$, which shows that (ii) holds.

It is clear that (ii) implies (iii).

Finally, we show that (iii) implies (i). Because of Theorem 3, we only show that $\text{epi}\delta^*_F\subset\bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}^m_+}\text{cl}\,\text{cone}\,\text{epi}\,(\sum_{i=1}^m\lambda_i g_i(\cdot,v_i))^*$. Let $(x^*,\alpha)\in\text{epi}\delta^*_F$. Then, $\delta^*_F(x^*)\in\mathbb{R}$ and $\delta^*_F(x^*)=-\inf_{x\in F}\langle -x^*,x\rangle$. Since $-x^*$ is a continuous linear function, by condition (iii), there exist $\bar v\in\mathcal{V}$ and $\bar\lambda\in\mathbb{R}^m_+$ such that

$$\inf_{x\in F}\langle -x^*,x\rangle=\inf\{\langle -x^*,x\rangle\mid\sum_{i=1}^m\bar\lambda_i g_i(x,\bar v_i)\leq 0\}.$$

Hence, for all $x\in\mathbb{R}^n$,

$$\begin{aligned} \sum_{i=1}^m \bar\lambda_i g_i(x,\bar v_i)\leq 0 \quad &\Longrightarrow \quad \langle -x^*,x\rangle\geq -\delta^*_F(x^*) \\ &\Longleftrightarrow \quad \langle x^*,x\rangle\leq \delta^*_F(x^*). \end{aligned}$$

This implies $\delta^*_{F_{(\bar v,\bar\lambda)}}(x^*)\leq\delta^*_F(x^*)\leq\alpha$, and hence by Theorem 1,

$$(x^*,\alpha)\in\text{epi}\delta^*_{F_{(\bar v,\bar\lambda)}}=\text{cl}\,\text{cone}\,\text{epi}\left(\sum_{i=1}^m\bar\lambda_i g_i(\cdot,\bar v_i)\right)^*.$$

This completes the proof.

Now we give an example illustrating Theorem 4.

Example 2. *Consider the following optimization problem (UP) with an uncertainty parameter v:*

$$(UP)\ \textit{minimize}\ f(x) := x^3\ \textit{subject to}\ g(x,v) \leq 0, v \in \mathcal{V} := [0,1],$$

where g is a function as follows:

$$g(x,v) = \begin{cases} v(x-2+v)^2 & x \geq 2-v, \\ 0 & -1-v \leq x \leq 2-v, \\ (1-v)(x+1+v)^2 & x \leq -1-v. \end{cases}$$

Then, f is continuous quasiconvex and $F = [-1,1]$. Also, we can check that

$$\begin{aligned} \mathrm{epi}\delta_F^* &= \{(x,\alpha) \in \mathbb{R}^2 \mid |x| \leq \alpha\} \\ &= \{(x,\alpha) \in \mathbb{R}^2 \mid 0 \leq x \leq \alpha\} \bigcup \{(x,\alpha) \in \mathbb{R}^2 \mid 0 \leq -x \leq \alpha\} \\ &= \mathrm{cl\,cone\,epi}(g(\cdot,1))^* \bigcup \mathrm{cl\,cone\,epi}(g(\cdot,0))^* \\ &\subset \bigcup_{v\in\mathcal{V},\lambda\geq 0} \mathrm{cl\,cone\,epi}(\lambda g(\cdot,v))^*. \end{aligned}$$

Hence by Theorem 3, $\bigcup_{v\in\mathcal{V},\lambda\geq 0} \mathrm{cl\,cone\,epi}(\lambda g(\cdot,v))^*$ is closed and convex. Moreover, let $(\bar{v},\bar{\lambda}) = (0,1)$, then

$$\inf\{f(x) \mid \forall v \in \mathcal{V}, g(x,v) \leq 0\} = -1 = \inf\{f(x) \mid \bar{\lambda} g(x,\bar{v}) \leq 0\}.$$

We give examples showing that without the closed cone constraint qualification (in Theorem 4), Theorem 4 may not hold.

Example 3. *Let $g(x,v) := \sqrt{v}|x| - v$, for all $x \in \mathbb{R}$ and $v \in \mathcal{V} := [0,1]$. Then, $g(\cdot,v)$ is convex and for each $v \in \mathcal{V}$ and $\lambda \geq 0$,*

$$(\lambda g(\cdot,v))^*(x^*) = \begin{cases} \lambda v, & x^* \in [-\lambda\sqrt{v}, \lambda\sqrt{v}], \\ \infty, & \textit{otherwise}. \end{cases}$$

Then, for each $\lambda \geq 0$,

$$\bigcup_{v\in\mathcal{V}} \mathrm{cl\,cone\,epi}(\lambda g(\cdot,v))^* = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \cup \{(0,0)\}.$$

Hence, $\bigcup_{v\in\mathcal{V},\lambda\geq 0} \operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*$ *is not closed, that is, the closed cone constraint qualification in Theorem 4 does not hold.*

Let f *be a function from* $\mathbb{R}$ *to* $\mathbb{R}$ *as follows:*

$$f(x) = \begin{cases} 0 & x \geq 0, \\ -1 & x < 0. \end{cases}$$

Then, f *is a usc quasiconvex function and* $\inf\{f(x) \mid \forall v \in \mathcal{V}, g(x,v) \leq 0\} = 0$*. However, for all* $v \in \mathcal{V}$ *and* $\lambda \geq 0$*,* $\inf\{f(x) \mid \lambda g(x,v) \leq 0\} = -1$*, that is, surrogate duality does not hold.*

Furthermore, if $f(x) = x^3$*,*

$$\inf\{f(x) \mid \forall v \in \mathcal{V}, g(x,v) \leq 0\} = \sup_{\lambda\geq 0, v\in\mathcal{V}} \inf\{f(x) \mid \lambda g(x,v) \leq 0\}.$$

However, the maximum does not attained. Actually, we can check that $\inf\{f(x) \mid \forall v \in \mathcal{V}, g(x,v) \leq 0\} = 0$*. Let* $v \in \mathcal{V}$ *and* $\lambda \geq 0$*. If* $v = 0$ *or* $\lambda = 0$*, then* $\inf\{f(x) \mid \lambda g(x,v) \leq 0\} = -\infty$*. If* $v > 0$ *and* $\lambda > 0$*, then* $\inf\{f(x) \mid \lambda g(x,v) \leq 0\} = -v\sqrt{v} < 0$*.*

Example 4. *Let* $g(x,v) = v^2|x_1| + \max\{x_2, 0\} - 2v$ *for all* $x = (x_1,x_2) \in \mathbb{R}^2$ *and* $v \in \mathcal{V} = [0,1]$*. Then,* $F = \{x \in \mathbb{R}^2 \mid \forall v \in \mathcal{V}, g(x,v) \leq 0\} = \{x \in \mathbb{R}^2 \mid |x_1| \leq 2, x_2 \leq 0\}$*,*

$$(\lambda_1 g_1(\cdot,v_1))^*(x^*) = \begin{cases} 2\lambda_1 v_1, & x_1^* \in [-\lambda_1 v_1^2, \lambda_1 v_1^2], x_2^* \in [0,\lambda_1], \\ \infty, & \textit{otherwise}, \end{cases}$$

for each $v \in \mathcal{V}$ *and* $\lambda \in \mathbb{R}_+$*, and* $\bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}_+} \operatorname{epi}(\lambda g(\cdot,v))^*$ *is not convex (see Example 2.1 in [8]). Hence, we can not apply Theorem 3.1 in [8] to this function* g*. Furthermore,* $\bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}_+} \operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*$ *is not convex. Actually, we can check that* $(2g((0,2),0))^* = 0$ *and* $(2g((2,2),1))^* = 4$*, that is,*

$$\begin{aligned} ((0,2),0) &\in \operatorname{epi}(2g(\cdot,0))^* \subset \bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}_+} \operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*, \\ ((2,2),4) &\in \operatorname{epi}(2g(\cdot,1))^* \subset \bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}_+} \operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*. \end{aligned}$$

However,

$$\frac{1}{2}((0,2),0) + \frac{1}{2}((2,2),4) = ((1,2),2) \notin \bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}_+} \operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*.$$

If $((1,2),2) \in \bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}_+} \operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*$*, then there exist* $v \in \mathcal{V}$ *and* $\lambda \in \mathbb{R}_+$ *such that* $((1,2),2) \in \operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*$*. Since* $\operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*$ *is closed,* $((1,2),2) \in \operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*$ *and hence there exists* $\gamma > 0$ *such that* $\gamma((1,2),2) \in \operatorname{epi}(\lambda g(\cdot,v))^*$*. This implies that*

$$-\lambda v^2 \le \gamma \le \lambda v^2,\ 0 \le 2\gamma \le \lambda,\ \textit{and } 2\lambda v \le 2\gamma.$$

Then, we can see that $\frac{\gamma}{\lambda} \le \frac{1}{2}$ *and* $\frac{\gamma}{\lambda} \ge 1$*, which is a contradiction. Hence,* $\bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}_+} \operatorname{cl}\operatorname{cone}\operatorname{epi}(\lambda g(\cdot,v))^*$ *is not convex. So we can not apply Theorem 4 to this function* g*.*

Let f *be a function from* $\mathbb{R}$ *to* $\mathbb{R}$ *as follows:*

$$f(x_1,x_2) = \begin{cases} -1 & x_2 > 0, \\ 0 & x_2 \ge 0. \end{cases}$$

Then, we can check that surrogate duality does not hold by the similar way of Example 3.

Next, we investigate uncertainty in the objective function. The following theorem indicates that the constraint qualification also characterizes completely surrogate duality for uncertainty in the objective function.

Theorem 5. *The following conditions are equivalent:*

(i)

$$\bigcup_{v\in\mathcal{V},\lambda\in\mathbb{R}^m_+} \operatorname{cl}\operatorname{cone}\operatorname{epi}\left(\sum_{i=1}^m \lambda_i g_i(\cdot,v_i)\right)^*$$

is closed and convex,

(ii) for all continuous function f *from* $\mathbb{R}^n \times \mathbb{R}^p$ *to* $\mathbb{R}$ *such that* $f(\cdot,u)$ *is quasiconvex for each* $u \in \mathbb{R}^p$ *and* $f(x,\cdot)$ *is quasiconcave for each* $x \in \mathbb{R}^n$*, and any compact convex subset* $\mathcal{U}$ *of* $\mathbb{R}^p$*,*

$$\inf_{x\in F}\max_{u\in\mathcal{U}} f(x,u) = \max_{u\in\mathcal{U},v\in\mathcal{V},\lambda\in\mathbb{R}^m_+} \inf\{f(x,u) \mid \sum_{i=1}^m \lambda_i g_i(x,v_i) \le 0\}.$$

PROOF. We first prove that (i) implies (ii). Let f be a continuous function $\mathbb{R}^n \times \mathbb{R}^q$ to $\mathbb{R}$ such that $f(\cdot, u)$ is quasiconvex for each $u \in \mathbb{R}^p$ and $f(x, \cdot)$ is quasiconcave for each $x \in \mathbb{R}^n$, and let $\mathcal{U}$ be a compact convex subset of $\mathbb{R}^p$. Then, $\max_{u\in\mathcal{U}} f(\cdot, u)$ is continuous and quasiconvex. So, by Theorem 4, there exist $\bar{v} \in \mathcal{V}$ and $\bar{\lambda} \in \mathbb{R}^m_+$ such that

$$\inf_{x\in F} \max_{u\in\mathcal{U}} f(x, u) = \inf_{x\in F_{(\bar{v},\bar{\lambda})}} \max_{u\in\mathcal{U}} f(x, u).$$

We know that $F_{(\bar{v},\bar{\lambda})}$ is convex since $g_i(\cdot, v_i)$ is convex for all i and v_i. Hence, by Sion's min-max theorem ([12]),

$$\inf_{x\in F_{(\bar{v},\bar{\lambda})}} \max_{u\in\mathcal{U}} f(x, u) = \max_{u\in\mathcal{U}} \inf_{x\in F_{(\bar{v},\bar{\lambda})}} f(x, u).$$

Thus there exist $\bar{u} \in \mathcal{U}$ such that

$$\inf_{x\in F} \max_{u\in\mathcal{U}} f(x, u) = \inf_{x\in F_{(\bar{v},\bar{\lambda})}} f(x, \bar{u}).$$

And also, for all $\lambda_0 \in \mathbb{R}^m_+$, $v_0 \in \mathcal{V}$ and $u_0 \in \mathcal{U}$,

$$\inf_{x\in F} \max_{u\in\mathcal{U}} f(x, u) \geq \inf_{x\in F_{(v_0,\lambda_0)}} \max_{u\in\mathcal{U}} f(x, u) \geq \max_{u\in\mathcal{U}} \inf_{x\in F_{(v_0,\lambda_0)}} f(x, u) \geq \inf_{x\in F_{(v_0,\lambda_0)}} f(x, u_0).$$

Hence (ii) holds.

The converse implication is obtained by choosing for f a continuous linear form.

4. Surrogate min-max duality

In this section, we consider a surrogate min-max duality theorem for robust convex optimization problem.

For all $(v, \lambda) \in \mathcal{V} \times \mathbb{R}^m_+$,

$$N_{F_{(v,\lambda)}}(\bar{x}) = \left\{ x^* \in \mathbb{R}^n \;\middle|\; (x^*, \langle x^*, \bar{x}\rangle) \in \operatorname{cl} \operatorname{cone} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* \right\}. \tag{1}$$

Actually, by using Theorem 1, we can prove that for each $(v, \lambda) \in \mathcal{V} \times \mathbb{R}^m_+$,

$$\begin{aligned} x^* \in N_{F_{(v,\lambda)}}(\bar{x}) &\iff \delta^*_{F_{(v,\lambda)}}(x^*) \leq \langle x^*, \bar{x}\rangle \\ &\iff (x^*, \langle x^*, \bar{x}\rangle) \in \operatorname{epi}\delta^*_{F_{(v,\lambda)}} \\ &\iff (x^*, \langle x^*, \bar{x}\rangle) \in \operatorname{cl} \operatorname{cone} \operatorname{epi}(\sum_{i\in I(\bar{x})} \lambda_i g_i(\cdot, v_i))^*, \end{aligned}$$

and then we have $\{x^* \in \mathbb{R}^n \mid (x^*, \langle x^*, \bar{x}\rangle) \in \mathrm{cl\,cone\,epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^*\} = N_{F_{(v,\lambda)}}(\bar{x})$. Also, since $F \subset F_{(v,\lambda)}$, we can check that $N_{F_{(v,\lambda)}}(\bar{x}) \subset N_F(\bar{x})$.

Now, we define the following constraint qualification:

$$N_F(\bar{x}) = \bigcup_{(v,\lambda)\in J(\bar{x})} \left\{ x^* \in \mathbb{R}^n \,\middle|\, (x^*, \langle x^*, \bar{x}\rangle) \in \mathrm{cl\,cone\,epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^* \right\},$$

where $J(\bar{x}) = \{(v,\lambda) \in \mathcal{V} \times \mathbb{R}^m_+ \mid \forall i \in I, \lambda_i g_i(\bar{x}, v_i) = 0\}$. We show that this constraint qualification is a necessary and sufficient constraint qualification for surrogate min-max duality for robust optimization.

Theorem 6. *The following conditions are equivalent:*

(i) $N_F(\bar{x}) = \bigcup_{(v,\lambda)\in J(\bar{x})} \left\{ x^* \in \mathbb{R}^n \,\middle|\, (x^*, \langle x^*, \bar{x}\rangle) \in \mathrm{cl\,cone\,epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^* \right\}$,

(ii) for any real-valued convex function f on $\mathbb{R}^n$, $\bar{x}$ is a minimizer of f over F if and only if there exist $v \in \mathcal{V}$ and $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i g_i(\bar{x}, v_i) = 0$ and

$$f(\bar{x}) = \min\left\{ f(x) \,\middle|\, \sum_{i=1}^m \lambda_i g_i(x, v_i) \leq 0 \right\}.$$

PROOF. First, we show that (i) implies (ii). Let f be a convex function. Then, $\bar{x}$ is a minimizer of f over F if and only if $0 \in \partial f(\bar{x}) + N_F(\bar{x})$. By condition (i), there exist $(v,\lambda) \in J(\bar{x})$ such that

$$0 \in \partial f(\bar{x}) + \left\{ x^* \in \mathbb{R}^n \,\middle|\, (x^*, \langle x^*, \bar{x}\rangle) \in \mathrm{cl\,cone\,epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^* \right\}.$$

By the equation (1), we can prove that $\bar{x}$ is a minimizer of f over $F_{(v,\lambda)}$, that is, (ii) holds.

Conversely, let $v \in N_F(\bar{x})$; then $\bar{x}$ is a global minimizer of $-v$ in F. From (ii), there exist $(v,\lambda) \in J(\bar{x})$ such that

$$\langle -v, \bar{x}\rangle = \min\left\{ \langle -v, x\rangle \,\middle|\, \sum_{i=1}^m \lambda_i g_i(x, v_i) \leq 0 \right\}.$$

For all $x \in \mathbb{R}^n$,

$$\sum_{i=1}^{m} \lambda_i g_i(x, v_i) \leq 0 \quad \Longrightarrow \quad \langle -v, x \rangle \geq \langle -v, \bar{x} \rangle$$

$$\Longleftrightarrow \quad \langle v, x - \bar{x} \rangle \leq 0,$$

that is $v \in N_{F_{(v,\lambda)}}(\bar{x})$. So, by the equation (1), (i) holds.

Example 5. *Let $g(x, v) = v^2|x_1| + \max\{x_2, 0\} - 2v$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $v \in \mathcal{V} = [0, 1]$. Then, by Example 4, $F = \{x \in \mathbb{R}^2 \mid \forall v \in \mathcal{V}, g(x, v) \leq 0\} = \{x \in \mathbb{R}^2 \mid |x_1| \leq 2, x_2 \leq 0\}$,*

$$(\lambda g(\cdot, v))^*(x^*) = \begin{cases} 2\lambda v, & x_1^* \in [-\lambda v^2, \lambda v^2], x_2^* \in [0, \lambda], \\ \infty, & \textit{otherwise}, \end{cases}$$

for all $v \in \mathcal{V}$ and $\lambda \in \mathbb{R}_+$, and $\bigcup_{v \in \mathcal{V}, \lambda \in \mathbb{R}_+} \operatorname{cl} \operatorname{cone} \operatorname{epi}(\lambda g(\cdot, v))^*$ is not convex. So we can not apply Theorem 4 to this function g.

However, we can apply Theorem 6 to this function g. Let $x = (x_1, x_2) = (2, -1)$. Then $x \in F$ and

$$N_F(x) = \bigcup_{(v,\lambda) \in J(x)} \left\{ x^* \in \mathbb{R}^n \,\middle|\, (x^*, \langle x^*, x \rangle) \in \operatorname{cl} \operatorname{cone} \operatorname{epi}(\lambda g(\cdot, v))^* \right\}.$$

We can show that $N_F(x) = \bigcup_{(v,\lambda) \in J(x)} N_{F_{(v,\lambda)}}(x)$. Let $v = 1$ and $\lambda = 1$. Then, $(v, \lambda) \in J(x)$ and $N_F(x) = N_{F_{(v,\lambda)}}(x)$. This implies that the above equation hold. So, we can apply Theorem 6 to this function g at $x = (2, -1)$. Unfortunately, we can not apply Theorem 6 to g at for all $y \in F$. Actually, the above equation does not hold at $(2, 0)$.

Theorem 7. *The following conditions are equivalent:*

(i) $$N_F(\bar{x}) = \bigcup_{(v,\lambda) \in J(\bar{x})} \left\{ x^* \in \mathbb{R}^n \,\middle|\, (x^*, \langle x^*, \bar{x} \rangle) \in \operatorname{cl} \operatorname{cone} \operatorname{epi}\Big(\sum_{i \in I(\bar{x})} \lambda_i g_i(\cdot, v_i)\Big)^* \right\},$$

(ii) for any continuous function f from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}$ such that $f(\cdot, u)$ is convex for each $u \in \mathbb{R}^p$ and $f(x, \cdot)$ is quasiconcave for each $x \in \mathbb{R}^n$, and any compact convex subset $\mathcal{U}$ of $\mathbb{R}^p$, $\bar{x}$ is a minimizer of f over F if and only if there exist $v \in \mathcal{V}$ and $\lambda \in \mathbb{R}_+^m$ such that $\lambda_i g_i(x, v_i) = 0$ and

$$f(\bar{x}) = \max_{u \in \mathcal{U}} \min \left\{ f(x, u) \,\middle|\, \sum_{i=1}^{m} \lambda_i g_i(x, v_i) \leq 0 \right\}.$$

PROOF. We first prove that (i) implies (ii). Let f be a continuous function such that $f(\cdot, u)$ is convex for each $u \in \mathbb{R}^p$ and $f(x, \cdot)$ is quasiconcave for each $x \in \mathbb{R}^n$, and let $\mathcal{U}$ be a compact convex subset of $\mathbb{R}^p$. By assumption, $\max_{u \in \mathcal{U}} f(\cdot, u)$ is continuous and convex. So, by Theorem 6, $\bar{x}$ is a minimizer of $\max_{u \in \mathcal{U}} f(\cdot, u)$ over F if and only if there exist $v \in \mathcal{V}$ and $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i g_i(x, v_i) = 0$ and $f(\bar{x}) = \min\{\max_{u \in \mathcal{U}} f(x, u) \mid \sum_{i=1}^{m} \lambda_i g_i(x, v_i) \leq 0\}$. By Sion's min-max theorem, we can show that

$$f(\bar{x}) = \max_{u \in \mathcal{U}} \min \left\{ f(x, u) \,\middle|\, \sum_{i=1}^{m} \lambda_i g_i(x, v_i) \leq 0 \right\}.$$

The converse implication can be proved as in the proof of Theorem 4.

5. Surrogate duality for robust semi-definite optimization problem

In this section, we obtain a surrogate duality theorem and a surrogate min-max duality theorem for semi-definite optimization problem in the face of data uncertainty.

Let S^n be the space of $n \times n$ symmetric matrices. For $A \in S^n$, $A \succeq 0$ mean that A is positive semidefinite. Let $T = \{A \in S^n \mid A \succeq 0\}$ and $I = \{0, 1, \ldots, m\}$. We denote the trace of the matrix A by $\mathrm{Tr}[A]$.

Following the proof of Theorem 4, we can prove the following surrogate duality theorem for semi-definite optimization problems in the face of data uncertainty.

Theorem 8. *Let $\mathcal{V}_i$, $i = 0, 1, \ldots, m$, be a closed and convex subset of S^n, $\mathcal{V} = \prod_{i=0}^{m} \mathcal{V}_i$, and $F = \{x \in \mathbb{R}^m \mid A_0 + \sum_{i=1}^{m} x_i A_i \succeq 0, \forall A_i \in \mathcal{V}_i, \forall i \in I\} \neq \emptyset$. Then the following conditions are equivalent:*

(i)

$$\bigcup_{Z \in T, A \in \mathcal{V}} \left\{ (-\mathrm{Tr}[ZA_1], \ldots, -\mathrm{Tr}[ZA_m], \mathrm{Tr}[ZA_0] + \delta)^T \mid \delta \geq 0 \right\}$$

is closed and convex,

(ii) for all usc quasiconvex function f from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ with $\mathrm{dom} f \cap F \neq \emptyset$, there exist $\bar{Z} \in T$ and $\bar{A} \in \mathcal{V}$ such that

$$\inf\{f(x) \mid x \in F\} = \inf\{f(x) \mid \mathrm{Tr}[\bar{Z}\bar{A}_0] + \sum_{i=1}^{m} x_i \mathrm{Tr}[\bar{Z}\bar{A}_i)] \geq 0\}.$$

PROOF. Let $K_{(Z,A)} = \{(-\mathrm{Tr}[ZA_1], \ldots, -\mathrm{Tr}[ZA_m], \mathrm{Tr}[ZA_0] + \delta)^T \mid \delta \geq 0\}$ for each $Z \in T$ and $A \in \mathcal{V}$. Then, we can prove that

$$\bigcup_{Z \in T, A \in \mathcal{V}} \mathrm{cl\ cone}\ K_{(Z,A)} = \bigcup_{Z \in T, A \in \mathcal{V}} K_{(Z,A)}.$$

Indeed, it is clear that $\bigcup_{Z \in T, A \in \mathcal{V}} \mathrm{cl\ cone}\ K_{(Z,A)} \supset \bigcup_{Z \in T, A \in \mathcal{V}} K_{(Z,A)}$. Let $\bar{Z} \in T$ and $\bar{A} \in \mathcal{V}$. If $(-\mathrm{Tr}[\bar{Z}\bar{A}_1], \ldots, -\mathrm{Tr}[\bar{Z}\bar{A}_m]) = 0$, then

$$\begin{aligned} \mathrm{cl\ cone}\ K_{(\bar{Z},\bar{A})} &= \{0\} \times \mathrm{cone}\{\mathrm{Tr}[\bar{Z}\bar{A}_0] + \delta \mid \delta \geq 0\} \\ &= \{0\} \times [0, \infty) \\ &= K_{(0,\bar{A})} \\ &\subset \bigcup_{Z \in T, A \in \mathcal{V}} K_{(Z,A)} \end{aligned}$$

since $\bar{Z}\bar{A}_0$ is positively semi-definite. Assume that $(-\mathrm{Tr}[\bar{Z}\bar{A}_1], \ldots, -\mathrm{Tr}[\bar{Z}\bar{A}_m]) \neq 0$. Then $(\bar{a}, \bar{\alpha}) \in \mathrm{cl\ cone}\ K_{(\bar{Z},\bar{A})}$. Then, there exist $\{(a_k, \alpha_k)\} \subset \mathbb{R}^m \times \mathbb{R}$ such that $(a_k, \alpha_k) \in \mathrm{cone}\ K_{(\bar{Z},\bar{A})}$ and (a_k, α_k) converges to $(\bar{a}, \bar{\alpha})$. For each $k \in \mathbb{N}$, there exist $\lambda_k \geq 0$ and $\delta_k \geq 0$ such that $(a_k, \alpha_k) = \lambda_k(-\mathrm{Tr}[\bar{Z}\bar{A}_1], \ldots, -\mathrm{Tr}[\bar{Z}\bar{A}_m], \mathrm{Tr}[\bar{Z}\bar{A}_0] + \delta_k)$. Since $\lambda_k(-\mathrm{Tr}[\bar{Z}\bar{A}_1], \ldots, -\mathrm{Tr}[\bar{Z}\bar{A}_m])$ converges to $\bar{a}$, λ_k converges to some $\bar{\lambda} \geq 0$ and $\bar{a} = \bar{\lambda}(-\mathrm{Tr}[\bar{Z}\bar{A}_1], \ldots, -\mathrm{Tr}[\bar{Z}\bar{A}_m])$. Also, $\lambda_k \delta_k$ converges some $\delta \geq 0$ and $\bar{\alpha} = \bar{\lambda}(\mathrm{Tr}[\bar{Z}\bar{A}_0]) + \delta$. Since T is a cone,

$$\begin{aligned} (\bar{a}, \bar{\alpha}) &= \bar{\lambda}(-\mathrm{Tr}[\bar{Z}\bar{A}_1], \ldots, -\mathrm{Tr}[\bar{Z}\bar{A}_m], \mathrm{Tr}[\bar{Z}\bar{A}_0]) + (0, \delta) \\ &= (-\mathrm{Tr}[\bar{\lambda}\bar{Z}\bar{A}_1], \ldots, -\mathrm{Tr}[\bar{\lambda}\bar{Z}\bar{A}_m], \mathrm{Tr}[\bar{\lambda}\bar{Z}\bar{A}_0] + \delta) \\ &\in \bigcup_{Z \in T, A \in \mathcal{V}} K_{(Z,A)}. \end{aligned}$$

Hence, by using Theorem 4, this completes the proof.

Following the proof of Theorem 6, we can prove the following surrogate min-max duality theorem for semi-definite optimization problem in the face of data uncertainty. We omit its proof.

Theorem 9. *Let $\mathcal{V}_i$, $i = 0, 1, \ldots, m$, be a closed and convex subset of S^n, $\mathcal{V} = \prod_{i=0}^m \mathcal{V}_i$, and $F = \{x \in \mathbb{R}^m \mid A_0 + \sum_{i=1}^m x_i A_i \succeq 0, \forall A_i \in \mathcal{V}_i, \forall i \in I\} \neq \emptyset$. Let $J(\bar{x}) = \{(Z, A) \in T \times \mathcal{V} \mid \mathrm{Tr}[ZA_0] + \sum_{i=1}^m x_i \mathrm{Tr}[ZA_i] = 0\}$. Then the following conditions are equivalent:*

(i)

$$N_F(\bar{x}) = \bigcup_{(Z,A)\in J(\bar{x})} \{(-\mathrm{Tr}[ZA_1], \ldots, -\mathrm{Tr}[ZA_m])^T\},$$

(ii) for any real-valued convex function f on $\mathbb{R}^n$, $\bar{x}$ is a minimizer of f over F if and only if there exist $\bar{Z} \in T$ and $\bar{A} \in \mathcal{V}$ such that $\mathrm{Tr}[\bar{Z}\bar{A}_0] + \sum_{i=1}^m x_i \mathrm{Tr}[\bar{Z}\bar{A}_i] = 0$ and

$$f(\bar{x}) = \min\{f(x) \mid \mathrm{Tr}[\bar{Z}\bar{A}_0] + \sum_{i=1}^m x_i \mathrm{Tr}[\bar{Z}\bar{A}_i)] \geq 0\}.$$

6. Conclusion

In this paper, we showed surrogate duality and surrogate min-max duality theorems for optimization problems with data uncertainty via robust optimization. We investigated necessary and sufficient constraint qualifications for surrogate and surrogate min-max duality, and showed some examples that these duality theorems are used effectively. Also, we obtained a surrogate duality theorem and a surrogate min-max duality theorem for semi-definite optimization problems in the face of data uncertainty.

Acknowledgements

The authors are grateful to anonymous referees for many comments and suggestions improved the quality of the paper.

References

[1] Beck, A., Ben-Tal, A. (2009). Duality in robust optimization: Primal worst equals dual best. Operations Research Letters, 37, 1–6.

[2] Fang, D. H., Li, C., Ng, K. F., (2009). Constraint qualifications for extended Farkas's lemmas and Lagrangian dualities in convex infinite programming. SIAM Journal on Optimization, 20, 1311–1332.

[3] Glover, F. (1965). A Multiphase-Dual Algorithm for the Zero-One Integer Programming Problem. Operations Research, 13, 879–919.

[4] Greenberg, H. J. (1973). Quasi-conjugate functions and surrogate duality. Operations Research, 21, 162–178.

[5] Greenberg, H. J., Pierskalla, W. P. (1970). Surrogate mathematical programming. Operations Research, 18, 924–939.

[6] Greenberg, H. J., Pierskalla, W. P. (1973). Quasi-conjugate functions and surrogate duality. Cahiers Centre Études Recherche Opér, 15, 437–448.

[7] Jeyakumar, V. (2003). Characterizing set containments involving infinite convex constraints and reverse-convex constraints. SIAM Journal on Optimization, 13, 947–959.

[8] Jeyakumar, V., Li, G. Y. (2010). Strong duality in robust convex programming: complete characterizations. SIAM Journal on Optimization, 20, 3384–3407.

[9] Jeyakumar, V., Li, G. Y. (2010). Characterizing robust set containments and solutions of uncertain linear programs without qualifications. Operations Research Letters, 38, 188–194.

[10] Jeyakumar, V., Li, G. Y. (2011). Robust Farkas' lemma for uncertain linear systems with applications. Positivity, 15, 331–342.

[11] Jeyakumar, V., Li, G. Y. (2012). Strong duality in robust semi-definite linear programming under data uncertainty. Optimization, DOI:10.1080/02331934.2012.690760.

[12] Komiya, H. (1988). Elementary proof for Sion's minimax theorem. Kodai Mathematical Journal, 11, 5–7.

[13] Li, G. Y., Jeyakumar, V., Lee, G. M. (2011). Robust conjugate duality for convex optimization under uncertainty with application to data classification. Nonlinear Analysis, 74, 2327–2341.

[14] Li, G. Y., Ng, K. F., (2008). On extension of Fenchel duality and its application. SIAM Journal on Optimization, 19, 1489–1509.

[15] Luenberger, D. G. (1988). Quasi-convex programming. SIAM Journal on Applied Mathematics, 16, 1090–1095.

[16] Penot, J. P., Volle, M. (1990). On quasi-convex duality. Mathematics of Operations Research, 15, 597–625.

[17] Penot, J. P., Volle, M. (2004) Surrogate programming and multipliers in quasi-convex programming. SIAM Journal on Control and Optimization, 42, 1994–2003.

[18] Suzuki, S., Kuroiwa, D. (2011). On set containment characterization and constraint qualification for quasiconvex programming. Journal of Optimization Theory and Applications, 149, 554–563.

[19] Suzuki, S., Kuroiwa, D. (2011). Optimality conditions and the basic constraint qualification for quasiconvex programming. Nonlinear Analysis, 74, 1279–1285.

[20] Suzuki, S., Kuroiwa, D. (2012) Necessary and sufficient constraint qualification for surrogate duality. Journal of Optimization Theory and Applications, 152, 366–377.