Asymptotic stability of a pendulum with quadratic damping

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Abstract. The equation considered in this paper is

$$x'' + h(t)x'|x'| + \omega^2 \sin x = 0,$$

where $h(t)$ is continuous and nonnegative for $t \geq 0$ and ω is a positive real number. This may be regarded as an equation of motion of an underwater pendulum. The damping force is proportional to the square of the velocity. The primary purpose is to establish necessary and sufficient conditions on the time-varying coefficient $h(t)$ for the origin to be asymptotically stable. The phase plane analysis concerning the positive orbits of an equivalent planar system to the above-mentioned equation is used to obtain the main results. In addition, solutions of the system are compared with a particular solution of the first-order nonlinear differential equation

$$u' + h(t)u|u| + 1 = 0.$$

Some examples are also included to illustrate our results. Finally, our results are extended to be applied to an equation with a nonnegative real-power damping force.

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1. Introduction

As known well, an object under water receives the resistance proportional to the square of the velocity. For example, let us think about the movement of an underwater simple pendulum. Then, all moment of inertia I to act on the pendulum is the sum total of the moment of inertia by the sinker, a thread and the added mass. On the other hand, all torque T is the sum total of the torque by gravity, buoyancy and the drag which act on the sinker and a thread. The torque for gravity or buoyancy is proportional to $\sin\theta$, where θ is the angle of swing. The torque for drag is proportional to the square of the angular velocity θ'. This is the so-called inertial resistance. Generally, since $I\theta'' = T$, the movement of an underwater simple pendulum is described by the equation

$$\theta'' + c\,\theta'|\theta'| + \omega^2 \sin\theta = 0,$$

where c and ω are positive numbers. These numbers are often called the damping (or drag) coefficient and the restoring coefficient per unit of the moment of inertia, respectively. As to specific models of the pendulum, see [5, 10] for example.

The above-mentioned equation is well approximated as follows:

$$\theta'' + c\,\theta'|\theta'| + \omega^2 \theta = 0.$$

Such equations appear frequently in many phenomena, for instance, free rolling motion of a small fishing vessel and damping oscillation by the air resistance. A pendulum and an oscillator with quadratic damping have been researched from various angles in a wide range of fields and there is a lot of literature about them (for example, see [1, 3, 6, 8, 9, 17, 18, 21, 22, 23, 26, 34, 35]).

In applied science and technology, the damping coefficient c is presumed from experimental data by using the least squares method. For this reason, the damping coefficient must be always dealt with as a fixed positive number. Here, a simple doubt is caused. May we really assume that the damping coefficient is a constant? It is a well-known fact that the inertial resistance changes depending on the density of fluid and the form of the object. The density of fluid is influenced by temperature and pressure. From this point of view, it would be reasonable to consider that the damping coefficient changes with time.

We consider the damped pendulum equation

$$x'' + h(t)x'|x'| + \omega^2 \sin x = 0, \tag{P}$$

where the damping coefficient $h(t)$ is continuous and nonnegative for $t \geq 0$. The origin $(x, x') = (0, 0)$ is an equilibrium of (P).

Let $\mathbf{x}(t) = (x(t), x'(t))$ and $\mathbf{x}_0 \in \mathbb{R}^2$, and let $\|\cdot\|$ be any suitable norm. We denote the solution of (P) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. The global existence and uniqueness of solutions of (P) is guaranteed for the initial value problem.

The origin is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The origin is said to be *attractive* if, for any $t_0 \geq 0$, there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \to 0$ as $t \to \infty$. The origin is *asymptotically stable* if it is stable and attractive. With respect to the various definitions of stability, the reader may refer to the books [2, 11, 12, 19, 24, 36] for example.

The purpose of this paper is to establish a criterion for judging whether the origin of (P) is asymptotically stable or not. Recently, the present author [29] has reported the following necessary and sufficient condition for the origin of the damped pendulum

$$x'' + h(t)x' + \omega^2 \sin x = 0, \tag{1.1}$$

to be asymptotically stable.

Theorem A. *Suppose that there exists a* γ_0 *with* $0 < \gamma_0 < \pi/\omega$ *such that*

$$\liminf_{t \to \infty} \int_t^{t+\gamma_0} h(s)ds > 0. \tag{1.2}$$

Then the origin of (1.1) *is asymptotically stable if and only if*

$$\int_0^\infty \frac{\int_0^t e^{H(s)} ds}{e^{H(t)}}\, dt = \infty, \tag{1.3}$$

where

$$H(t) = \int_0^t h(s)ds.$$

The damping force is proportional to the velocity in equation (1.1). This is a big difference point with equation (P). Equation (1.1) is a model on which not the inertial resistance but the viscous resistance acts predominantly.

Smith [27] gave condition (1.3) for the first time as a criterion which judges whether the origin of the linear oscillator

$$x'' + h(t)x' + \omega^2 x = 0$$

is asymptotically stable under the assumption that there exists an $\underline{h} > 0$ such that $h(t) \geq \underline{h}$ for $t \geq 0$. Afterwards, Smith's assumption was weakened to assumption (1.2) by Hatvani and Totik [16]. Even if intervals where $h(t)$ becomes zero are infinitely many, assumption (1.2) may be satisfied if the lengths of intervals are less than π. This is a good point of assumption (1.2).

It is known that

(i) if $h(t)$ is bounded for $t \geq 0$ or $h(t) = t$, then condition (1.3) is satisfied;
(ii) if $h(t) = t^2$, then condition (1.3) is not satisfied.

(for details, see [15]). However, generally it is not so easy to check condition (1.3). In most cases, it is impossible to confirm whether condition (1.3) is satisfied, by using human's hand calculation. It is hard to verify condition (1.3) even if we perform numerical analysis carried out by a personal computer. We need much patience and time even if possible.

Let

$$u(t) = -\frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}.$$

Then, the function $u(t)$ satisfies the scalar differential equation

$$u' + h(t)u + 1 = 0 \tag{1.4}$$

with $u(0) = 0$. From this relation and Theorem A, in order to determine whether the origin of (1.1) is asymptotically stable or not, we have only to examine whether the integral from 0 to ∞ of the solution $u(t)$ of (1.4) satisfying the initial condition $u(0) = 0$ diverges or not. Based on this fact, we call equation (1.4) a *characteristic equation* for the pendulum (1.1). Turning attention to the particular solution of the characteristic equation (1.4), we can easily obtain its integration value by numerical analysis. This is an advantage in consideration of the characteristic equation.

The first main theorem is as follows:

Theorem 1.1. *Under the assumption* (1.2), *the origin of* (P) *is asymptotically stable if and only if*

$$\int_0^\infty u(t)dt = -\infty, \tag{1.5}$$

where $u(t)$ *is the solution of*

$$u' + h(t)u|u| + 1 = 0 \tag{1.6}$$

satisfying $u(0) = 0$.

As already mentioned, assumption (1.2) is a generalization of Smith's assumption that $h(t) \geq \underline{h} > 0$ for $t \geq 0$. However, assumption (1.2) is not satisfied if

$$\lim_{t\to\infty} h(t) = 0.$$

For example, Theorem 1.1 is inapplicable if

$$h(t) = 1/(1+t) \quad \text{or} \quad h(t) = 1/((2+t)\log(2+t)).$$

To apply even to these cases, we replace the major premise, namely, assumption (1.2).

As preparations to state the second main theorem, we define a family of functions. We say that a nonnegative function $\psi(t)$ is said to be *weakly integrally positive* if

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} \psi(t)dt = \infty$$

for every pairs of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n + \lambda < \sigma_n \leq \tau_{n+1} \leq \sigma_n + \Lambda$ for some $\lambda > 0$ and $\Lambda > 0$. The typical example of the weakly integrally positive function is $1/(1+t)$ or $\sin^2 t/(1+t)$ (for example, see [13, 14, 28, 30, 31]).

Theorem 1.2. *Suppose that* $h(t)$ *is uniformly continuous for* $t \geq 0$ *and weakly integrally positive. Let* $u(t)$ *is the solution of* (1.6) *satisfying* $u(0) = 0$. *Then the origin of* (P) *is asymptotically stable if and only if condition* (1.5) *holds.*

Even if $h(t)$ has infinitely many isolated zeros, it may be weakly integrally positive. However, $h(t)$ is not weakly integrally positive any longer if intervals where $h(t)$ becomes zero appear regularly and frequently. For example, if $h(t) = |\sin 2t| + \sin 2t$, then it is not weakly integrally positive. On the other hand, assumption (1.2) is satisfied in this example. Therefore, Theorems 1.1 and 1.2 supplement each other to expand the adaptation range.

In Theorems 1.1 and 1.2, assumption (1.2) and the weak integral positivity prohibit too fast decline of the damping coefficient $h(t)$, respectively. Conversely, condition (1.5) prohibits too fast growth of the damping coefficient $h(t)$.

2. Preliminary arrangements

In this section, we prepare several lemmas and one proposition, in order to prove our main theorems. To begin with, we consider the scalar differential equation

$$u' = f(t,u), \tag{2.1}$$

where $f(t,u)$ is continuous on $[0,\infty)\times\mathbb{R}$ and satisfies locally a Lipschitz condition with respect to u. As is well known, the following comparison results hold (for example, see [36, p. 5]).

Lemma 2.1. *Let $u(t)$ be a solution of (2.1) on an interval $[a,b]$. Suppose that $\eta(t)$ is continuous on $[a,b]$ and satisfies the inequality*

$$\eta'(t) \ge f(t,\eta(t)) \quad \text{for } a<t<b.$$

If $\eta(a)\ge u(a)$, then $\eta(t)\ge u(t)$ for $a\le t\le b$.

Lemma 2.2. *Let $u(t)$ be a solution of (2.1) on an interval $[a,b]$. Suppose that $\eta(t)$ is continuous on $[a,b]$ and satisfies the inequality*

$$\eta'(t) \le f(t,\eta(t)) \quad \text{for } a<t<b.$$

If $\eta(a)\le u(a)$, then $\eta(t)\le u(t)$ for $a\le t\le b$.

As a special case of equation (2.1), we consider the characteristic equation (1.6). Let T be a nonnegative number. We denote the solution $u(t)$ of (1.6) satisfying $u(T)=0$ by $u(t;T)$. Then, using Lemma 2.1, we obtain the following equivalence relation between $u(t;T)$ and $u(t;0)$. We omit the details (for the proof, see Lemma 2.1 in [32]).

Lemma 2.3. *For any $T\ge 0$,*

$$\int_T^\infty u(t;T)dt = -\infty$$

if and only if

$$\int_0^\infty u(t;0)dt = -\infty.$$

Equation (P) is equivalent to the planar system

$$\begin{aligned} x' &= \omega y, \\ y' &= -\omega\sin x - \omega h(t)y|y|. \end{aligned} \tag{2.2}$$

The origin of (P) corresponds to the zero solution of (2.2), namely, $(x(t),y(t))\equiv(0,0)$. According to custom, we divide $\mathbb{R}^2\setminus\{(0,0)\}$ into four quadrants:

$$Q_1 = \{(x,y)\colon x\ge 0 \text{ and } y>0\}; \qquad Q_2 = \{(x,y)\colon x<0 \text{ and } y\ge 0\};$$

$$Q_3 = \{(x,y)\colon x\le 0 \text{ and } y<0\}; \qquad Q_4 = \{(x,y)\colon x>0 \text{ and } y\le 0\}.$$

Consider the solution $\mathbf{x}(t;t_0,\mathbf{x}_0)$ of (P). The set

$$\Gamma^+_{(2.2)}(t_0,\mathbf{x}_0) \stackrel{\text{def}}{=} \bigcup_{t\ge t_0} \mathbf{x}(t;t_0,\mathbf{x}_0)$$

is called the *positive orbit* of (2.2) starting from a point $\mathbf{x}_0=(x_0,y_0)\in\mathbb{R}^2$ at a time $t_0\ge 0$. Since system (2.2) is nonautonomous, even if $\Gamma^+_{(2.2)}(t_0,\mathbf{x}_0)$ starts from the same point $\mathbf{x}_0$, the shape is different according to the initial time t_0. We call the position of $\mathbf{x}(t;t_0,\mathbf{x}_0)$ on the (x,y)-plane a *phase point* for each time $t\ge 0$. Needless to say, the phase point moves along the positive orbit $\Gamma^+_{(2.2)}(t_0,\mathbf{x}_0)$.

It is natural to choose the total energy

$$V(x,y) = 1 - \cos x + \frac{1}{2}y^2$$

as a suitable Lyapunov function for system (2.2). As a matter of fact, we obtain

$$\dot{V}_{(2.2)}(t,x,y) = (\sin x)x' + y\,y' = -\,\omega h(t) y^2 |y| \le 0$$

on $[0,\infty) \times \mathbb{R}^2$, by differentiating $V(x,y)$ along any solution of (2.2). Let

$$D = \big\{(x,y) \in \mathbb{R}^2 : |x| < \pi/2 \ \text{ and } \ V(x,y) < 1\big\}.$$

Then, it turns out that D is a domain containing the origin and it is a positive invariant set of (2.2), namely, for any $t_0 \ge 0$ and $\mathbf{x}_0 \in D$, the positive orbit $\Gamma^+_{(2.2)}(t_0,\mathbf{x}_0)$ is included in D. Since $V(x,y)$ is positive definite and $\dot{V}_{(2.2)}(t,x,y)$ is nonpositive, it follows from a basic Lyapunov's direct method that the zero solution of (2.2) is stable. Hence, we obtain the following result.

Proposition 2.4. *The origin of* (P) *is stable.*

Proposition 2.4 can be led only under the assumption that $h(t)$ is nonnegative for $t \ge 0$. Unfortunately, the derivative $\dot{V}_{(2.2)}(t,x,y)$ is not negative definite, and hence it is not so easy to demonstrate the global attractivity of the origin of (P). We have to examine the characteristics of positive orbits of (2.2) in detail. For this purpose, we transform system (2.2) into polar coordinates by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Then, we have

$$\begin{aligned} r' &= \omega\, r \sin\theta\cos\theta - \omega\sin\theta\sin(r\cos\theta) - \omega\, h(t)\, r^2 \sin^2\theta\,|\sin\theta|, \\ \theta' &= -\frac{\omega}{r}\sin(r\cos\theta)\cos\theta - \omega\sin^2\theta - \omega\, h(t)\, r\sin\theta\,|\sin\theta|\cos\theta. \end{aligned} \tag{2.3}$$

Since

$$r^2\theta' = -\,\omega\, x\sin x - y^2 - \omega h(t) x y |y| < 0$$

if $(x,y) \in (Q_1 \cup Q_3) \cap D$, the phase point on $\Gamma^+_{(2.2)}(t_0,\mathbf{x}_0)$ turns clockwise around the origin as long as it moves through $Q_1 \cap D$ or $Q_3 \cap D$. Afterwards, how does the phase point move? The following result answers this question.

Lemma 2.5. *There is no positive orbit of* (2.2) *which is included in* $(Q_1 \cup Q_3) \cap D$ *ultimately.*

Proof. Let $t_0 \ge 0$ and $\mathbf{x}_0 \in Q_1 \cap D$ (resp., $Q_3 \cap D$). Suppose that $\Gamma^+_{(2.2)}(t_0,\mathbf{x}_0)$ is contained in $Q_1 \cap D$ (resp., $Q_3 \cap D$). Let $(r(t),\theta(t))$ be the solution of (2.3) corresponding to $\Gamma^+_{(2.2)}(t_0,\mathbf{x}_0)$. Then, $0 < r^2(t) < \pi^2/4 + 2$ for $t \ge t_0$. It follows from the assumptions that $h(t) \ge 0$ for $t \ge 0$ and $\sin\theta(t)\cos\theta(t) \ge 0$ for $t \ge t_0$ that

$$\begin{aligned} r^2(t)\,\theta'(t) &= -\,\omega\, r(t)\cos\theta(t)\sin(r(t)\cos\theta(t)) - r^2(t)\sin^2\theta(t) \\ &\quad - \omega\, h(t)\, r^3(t)\cos\theta(t)\sin\theta(t)|\sin\theta(t)| \\ &\le -\,\omega\, r(t)\cos\theta(t)\sin(r(t)\cos\theta(t)) \end{aligned}$$

for $t \ge t_0$. Since $x\sin x$ is decreasing for $-\pi/2 \le x \le 0$ and increasing for $0 \le x \le \pi/2$, we see that

$$\begin{aligned} \theta'(t) &\le -\frac{\omega\, r(t_0)\cos\theta(t_0)\sin(r(t_0)\cos\theta(t_0))}{r^2(t)} \\ &\le -\frac{\omega\, r(t_0)\cos\theta(t_0)\sin(r(t_0)\cos\theta(t_0))}{\pi^2/4 + 2} \end{aligned}$$

for $t \ge t_0$. Integrating this inequality from t_0 to t, we obtain

$$\theta(t) < \theta(t_0) - M(t - t_0) \to -\infty \quad \text{as } \ t \to \infty,$$

where $M = \omega\, r(t_0)\cos\theta(t_0)\sin(r(t_0)\cos\theta(t_0))/(\pi^2/4 + 2) > 0$. This is a contradiction. Thus, such a positive orbit does not exist. □

From Lemma 2.5, we conclude that system (2.2) has three types of positive orbits: (i) a positive orbit coils itself around the origin; (ii) a positive orbit is contained in Q_4 (resp., Q_2) ultimately and the phase point that runs on the positive orbit approaches the origin through Q_4 (resp., Q_2); (iii) a positive orbit is includes in Q_4 (resp., Q_2) ultimately and the phase point that runs on the positive orbit approaches an interior point in Q_4 (resp., Q_2). From the vector field of (2.2), we see that any phase point moves to the left in Q_4, and moves to the right in Q_2. However, it does not always rotate around the origin $(0,0)$, and may go up and down in Q_4 and Q_2.

3. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. By virtue of Proposition 2.4, we have only to discuss the attractivity of the origin of (P).

3.1. Necessity

We will prove that if condition (1.5) does not hold, then the origin of (P) is not attractive. Let ε be an arbitrary positive number and let $L = \max\{1, \omega^2\}$. Then, there exists a $T > 0$ such that

$$\int_T^\infty u(t)dt > -\frac{\varepsilon}{2L}.$$

Recall that $u(t;T)$ is the solution of (1.6) satisfying $u(T;T) = 0$. From the uniqueness of solutions of (1.6) for the initial value problem, we see that

$$u(t) = u(t;0) \le u(t;T) < 0 \quad \text{for } \; t > T.$$

Hence, we have

$$\int_T^\infty u(t;T)dt > -\frac{\varepsilon}{2L}. \tag{3.1}$$

Consider the positive orbit $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$, where $t_0 = T$ and $\mathbf{x}_0 = (\varepsilon, 0)$. Taking the vector field of (2.2) into account, we see that the phase point on $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ goes into Q_4 afterwards and it does not enter Q_1 passing through the positive x-axis. Let $(x(t), y(t))$ be the solution of (2.2) satisfying $x(T) = \varepsilon$ and $y(T) = 0$. If

$$x(t) > \frac{\varepsilon}{2} \quad \text{for } \; t \ge T, \tag{3.2}$$

then naturally the origin of (P) is not attractive. This completes the proof of 'only if'-part.

By way of contradiction, suppose that (3.2) is not true. Then, we can find a $T_1 > T$ such that $x(T_1) = \varepsilon/2$ and $\varepsilon/2 < x(t) \le \varepsilon$ for $T \le t < T_1$. Since $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ does not intersect the positive x-axis, we see that

$$y(t) < 0 \quad \text{for } \; T < t \le T_1.$$

Let $\eta(t) = \omega y(t)/L$. Then, from the second equation of (2.2) it follows that

$$\begin{aligned}\eta'(t) &= -\frac{\omega^2}{L}\sin x(t) - \frac{\omega^2}{L}h(t)y(t)|y(t)| \\ &\ge -1 - Lh(t)\eta(t)|\eta(t)| \ge -1 - h(t)\eta(t)|\eta(t)|\end{aligned}$$

for $T \le t \le T_1$. Let $f(t,u) = -1 - h(t)u|u|$. Then, we have

$$\eta'(t) \ge f(t, \eta(t)) \quad \text{for } \; T \le t \le T_1.$$

We compare $\eta(t)$ with $u(t;T)$. Since $u'(t;T) = f(t, u(t;T))$ for $t \ge T$ and $\eta(T) = \omega y(T)/L = 0$, it follows from Lemma 2.1 that

$$x'(t) = \omega y(t) = L\eta(t) \ge Lu(t;T)$$

for $T \le t \le T_1$. Hence, using (3.1), we obtain

$$x(T_1) \ge x(T) + L\int_T^{T_1} u(t;T)dt > \varepsilon + L\int_T^\infty u(t;T)dt > \frac{\varepsilon}{2} = x(T_1).$$

This is a contradiction.

3.2. Sufficiency

Let $x(t)$ be any solution of (P) with the initial time $t_0 \geq 0$ and let $y(t) = x'(t)$ for $t \geq t_0$. Then, $(x(t), y(t))$ is a solution of (2.2), which corresponds to $x(t)$. Let $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ be the positive orbit of (2.2) corresponding to the solution $(x(t), y(t))$. To prove 'if'-part of the theorem, we have only to show that if $\mathbf{x}_0 = (x(t_0), y(t_0)) \in D$, then $(x(t), y(t))$ tends to $(0,0)$ as $t \to \infty$.

Define

$$v(t) = V(x(t), y(t))$$

for $t \geq t_0$. Then, $v'(t) = -\omega h(t) y^2(t)|y(t)| \leq 0$ for $t \geq t_0$. Hence, $v(t)$ is decreasing and has the limiting value $v_0 \geq 0$. If v_0 is zero, then the proof of 'if'-part is complete. We will show that the case of $v_0 > 0$ does not occur provided assumptions (1.2) and (1.5) hold.

Suppose that v_0 is positive. Then, $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ is contained in the annulus

$$A = \left\{(x,y) \in \mathbb{R}^2 : |x| < \pi/2 \text{ and } v_0 < V(x,y) < 1\right\} \subset D$$

for all future time. Consider the closed curve given by $V(x,y) = v_0 > 0$. It is clear that this curve is a symmetric oval. Hence, it intersects with the x-axis only at two points $(\mu, 0)$ and $(-\mu, 0)$, where $0 < \mu = \arccos(1 - v_0) < \pi/2$.

As already mentioned, it turns out from Lemma 2.5 that $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ must belong to either of three types. However, $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ does not belong to the second type, namely, it is contained in Q_4 (resp., Q_2) and the phase point that runs on $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ approaches the origin through Q_4 (resp., Q_2), because it stays in the annulus A that does not contain the origin. Under the assumptions (1.2) and (1.5), $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ does not belong to either of the first type and the third type. Hereafter, we will confirm this fact by dividing into two steps.

Step (i). Suppose that $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ belongs to the first type, namely, it coils itself around the origin while remaining in the annulus A. Let $(r(t), \theta(t))$ be the solution of (2.3) corresponding to $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$. Then, there exist divergent sequences $\{\tau_n\}$ and $\{\sigma_n\}$ with $t_0 \leq \tau_n < \sigma_n$ such that $\theta(\tau_n) = 3\pi/2$ and $\theta(\sigma_n) = \pi/2 \pmod{2\pi}$. In other words, $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ intersects the negative y-axis at $t = \tau_n$, and it intersects the positive y-axis at $t = \sigma_n$ for $n \in \mathbb{N}$. Let ε be so small that

$$0 < \varepsilon < \frac{\pi - \omega\gamma_0}{2}, \tag{3.3}$$

where γ_0 is the number given in assumption (1.2). Then, $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ crosses the straight lines $y = (\tan\varepsilon)x$ and $y = (\tan(\pi - \varepsilon))x$ infinitely many times. Recall that the phase point on $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ moves clockwise in $(Q_1 \cup Q_3) \cap A$. However, in $(Q_2 \cup Q_4) \cap A$, it does not always rotate clockwise and may go up and down. The shape of $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ may be so simple in $(Q_2 \cup Q_4) \cap A$. For this reason, the point in the set $\{t \in (\sigma_n, \tau_{n+1}) : \theta(t) = \varepsilon\}$ is unique, but the point in the set $\{t \in (\tau_n, \sigma_n) : \theta(t) = \pi - \varepsilon\}$ might not be only one. For $n \in \mathbb{N}$, let s_n be the unique point satisfying $\sigma_n < s_n < \tau_{n+1}$ and $\theta(s_n) = \varepsilon$, and let t_n be the supremum of all $t \in (\tau_n, \sigma_n)$ for which $\theta(t) \geq \pi - \varepsilon$. Then, $t_0 \leq \tau_n < t_n < \sigma_n < s_n$, $\theta(t_n) = \pi - \varepsilon$, $\theta(s_n) = \varepsilon$ and

$$\varepsilon < \theta(t) < \pi - \varepsilon \quad \text{for } t_n < t < s_n.$$

Since the curve $V(x,y) = v_0$ is an oval, it intersects the straight line $y = (\tan\varepsilon)x$ at only one point in Q_1. Let $\delta(\varepsilon)$ be the y-component of the intersections. Since $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ does not enter the region $\left\{(x,y) \in \mathbb{R}^2 : V(x,y) \leq v_0\right\}$, it follows that $y(t) > \delta$ for $t_n \leq t \leq s_n$. Hence,

$$v'(t) = -\omega h(t) y^2(t)|y(t)| \leq -\omega h(t)\delta^3 \tag{3.4}$$

for $t_n \leq t \leq s_n$.

Suppose that there exists an $N \in \mathbb{N}$ such that $s_n - t_n \geq \gamma_0$ for $n \geq N$. Then, it turns out from (3.4) that

$$v(s_n) - v(t_n) \leq -\omega\delta^3 \int_{t_n}^{s_n} h(t)dt \leq -\omega\delta^3 \int_{t_n}^{t_n+\gamma_0} h(t)dt$$

for $n \geq N$. Since $v'(t) = -\omega h(t) y^2(t)|y(t)| \leq 0$, it is clear that

$$v(t_{n+1}) - v(s_n) \leq 0 \quad \text{for} \ \ n \in \mathbb{N}.$$

Hence, we obtain

$$v(t_{n+1}) - v(t_n) \leq -\omega\delta^3 \int_{t_n}^{t_n+\gamma_0} h(t)dt \quad \text{for} \ \ n \geq N,$$

and therefore,

$$v_0 - v(t_N) \leq v(t_{n+1}) - v(t_N) \leq -\omega\delta^3 \sum_{i=N}^{n} \int_{t_i}^{t_i+\gamma_0} h(t)dt. \tag{3.5}$$

From assumption (1.2), we see that

$$\sum_{i=N}^{\infty} \int_{t_i}^{t_i+\gamma_0} h(t)dt = \infty,$$

which contradicts (3.5). Thus, there exists a sequence $\{n_k\}$ with $n_k \in \mathbb{N}$ and $n_k \to \infty$ as $k \to \infty$ such that

$$s_{n_k} - t_{n_k} < \gamma_0. \tag{3.6}$$

The annulus A is included in a circle. Let $\overline{r}$ be the radius of the circle. Since $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ is contained in A, we see that $r(t) \leq \overline{r}$ for $t \geq t_0$. Hence, we can estimate that

$$\begin{aligned}\theta'(t) \geq &-\frac{\omega}{r(t)}|\sin(r(t)\cos\theta(t))||\cos\theta(t)| - \omega\sin^2\theta(t) \\ &-\omega h(t) r(t)\sin^2\theta(t)|\cos\theta(t)| \\ \geq &-\omega\cos^2\theta(t) - \omega\sin^2\theta(t) - h(t)r(t) \geq -\omega - h(t)\overline{r}\end{aligned}$$

for $t \geq t_0$. It turns out from (3.6) that

$$\begin{aligned}\varepsilon - (\pi - \varepsilon) &= \theta(s_{n_k}) - \theta(t_{n_k}) \\ &\geq -\omega(s_{n_k} - t_{n_k}) - \overline{r}\int_{t_{n_k}}^{s_{n_k}} h(t)dt > -\omega\gamma_0 - \overline{r}\int_{t_{n_k}}^{s_{n_k}} h(t)dt\end{aligned}$$

for each $k \in \mathbb{N}$, namely,

$$\overline{r}\int_{t_{n_k}}^{s_{n_k}} h(t)dt > \pi - \omega\gamma_0 - 2\varepsilon \quad \text{for} \ \ k \in \mathbb{N}.$$

Using this inequality and (3.4), we obtain

$$v(s_{n_k}) - v(t_{n_k}) \leq -\omega\delta^3 \int_{t_{n_k}}^{s_{n_k}} h(t)dt < -\frac{\omega\delta^3}{\overline{r}}(\pi - \omega\gamma_0 - 2\varepsilon)$$

for $k \in \mathbb{N}$. Since $v(t_{n_{k+1}}) - v(s_{n_k}) \leq 0$ for $k \in \mathbb{N}$, we see that

$$v(t_{n_{k+1}}) - v(t_{n_k}) < -\frac{\omega\delta^3}{\overline{r}}(\pi - \omega\gamma_0 - 2\varepsilon) \quad \text{for} \ \ k \in \mathbb{N}.$$

From (3.3), we can conclude that

$$v_0 - v(t_0) \leq \sum_{k=1}^{\infty} \left(v(t_{n_{k+1}}) - v(t_{n_k})\right) = -\infty,$$

which is a contradiction. Thus, $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ does not belong to the first type.

Step (ii). Suppose that $\Gamma^+(t_0, \mathbf{x}_0)$ belongs to the third type, namely, it is included in $Q_4 \cap A$ (resp., $Q_2 \cap A$) ultimately and the phase point on $\Gamma^+(t_0, \mathbf{x}_0)$ approaches to an interior point in $Q_4 \cap A$ (resp., $Q_2 \cap A$). Then, there exist a point $\mathbf{x}_1 \in Q_4 \cap A$ (resp., $Q_2 \cap A$) and a time $T \geq t_0$ such that $\mathbf{x}_1$ is the phase point on $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ for $t = T$ and $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ is contained in $Q_4 \cap A$ (resp., $Q_2 \cap A$) afterwards. From the uniqueness of solutions of (P) to initial value problems, we see that $\Gamma^+_{(2.2)}(T, \mathbf{x}_1)$ is a part of $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$. There are two cases that we should consider: (a) $\Gamma^+_{(2.2)}(T, \mathbf{x}_1)$ is included in $Q_4 \cap A$ for all the future; (b) $\Gamma^+_{(2.2)}(T, \mathbf{x}_1)$ is included in $Q_2 \cap A$ for all the future. We consider only the former, because the latter is carried out in the same way.

Since $(x(t), y(t)) \in Q_4$ for $t \geq T$, we see that $x'(t) = y(t) \leq 0$ for $t \geq T$. Hence, there exists a number $c \in \mathbb{R}$ with $0 \leq c < \pi/2$ such that $x(t) \searrow c$ as $t \to \infty$. Recall that $v(t) = 1 - \cos x(t) + y^2(t)/2 \searrow v_0$ as $t \to \infty$. Hence, it turns out that

$$\frac{1}{2} y^2(t) \to \rho \quad \text{as } t \to \infty,$$

where $\rho = v_0 - 1 + \cos c$. Naturally, $\rho \geq 0$. If $\rho > 0$, then we can choose a $T_1 \geq T$ so large that

$$y^2(t) > \rho \quad \text{for } t \geq T_1.$$

Hence, we have

$$v'(t) = -\omega h(t) y^2(t)|y(t)| \leq -\omega \rho \sqrt{\rho}\, h(t)$$

for $t \geq T_1$. Integrate this inequality to obtain

$$-\infty < v_0 - v(T_1) < v(t) - v(T_1) \leq -\omega \rho \sqrt{\rho} \int_{T_1}^{t} h(s) ds.$$

This is a contradiction, because it follows from assumption (1.2) that

$$\int_{T_1}^{\infty} h(t) dt = \infty.$$

Thus, it turns out that $\rho = 0$, namely, $c = \arccos(1 - v_0) = \mu$. We therefore conclude that the phase point on $\Gamma^+_{(2.2)}(T, \mathbf{x}_1)$ approaches an interior point $(\mu, 0) \in Q_4 \cap A$, which is a intersection of the closed curve $V(x, y) = v_0$ and the x-axis.

From the above-mentioned argument, we see that

$$0 < \mu < x(t) \leq x(T) < \frac{\pi}{2} \quad \text{and} \quad y(t) < 0$$

for $t \geq T$. Note that $\sin x(t) > \sin \mu > 0$ for $t \geq T$. Let $\varepsilon_0 = \min\{1, \omega^2 \sin \mu\}$. Then, we can estimate that

$$\begin{aligned} \left(\frac{\omega y(t)}{\varepsilon_0} \right)' &= -\frac{\omega^2}{\varepsilon_0} \sin x(t) - \frac{\omega^2 h(t)}{\varepsilon_0} y(t)|y(t)| \\ &\leq -1 - h(t) \frac{\omega y(t)}{\varepsilon_0} \left| \frac{\omega y(t)}{\varepsilon_0} \right| \end{aligned}$$

for $t \geq T$. Let $\eta(t) = \omega y(t)/\varepsilon_0$ for $t \geq t_0$ and let $f(t, u) = -1 - h(t) u|u|$. Then, $\eta'(t) \leq f(t, \eta(t))$ for $t \geq T$. We compare $\eta(t)$ with the solution $u(t; T)$ of (1.6) satisfying $u(T; T) = 0$. Since $\eta(T) = \omega y(T)/\varepsilon_0 < 0$, it follows from Lemma 2.2 that

$$\frac{\omega y(t)}{\varepsilon_0} = \eta(t) \leq u(t; T) \leq 0$$

for $t \geq T$. Hence, we have

$$x'(t) = \omega y(t) \leq \varepsilon_0 u(t; T) \quad \text{for } t \geq T.$$

Integrating both sides of this inequality from T to t, we obtain

$$-\frac{\pi}{2} < \mu - x(T) < x(t) - x(T) \leq \varepsilon_0 \int_T^t u(s; T) ds.$$

However, by means of Lemma 2.3 and condition (1.5), we conclude that

$$\int_T^t u(s;T)ds \to -\infty \quad \text{as} \quad t \to \infty.$$

This is a contradiction. Thus, $\Gamma^+(t_0, \mathbf{x}_0)$ does not belong to the third type.

The proof of Theorem 1.1 is thus complete. □

4. Proof of Theorem 1.2

Recall that the damped pendulum equation (P) is equivalent to the planar system (2.2). Let $(x(t), y(t))$ be any solution of (2.2) with the initial time $t_0 \geq 0$, and let $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ be the positive orbit of (2.2) corresponding to the solution $(x(t), y(t))$, where $\mathbf{x}_0 = (x(t_0), y(t_0)) \in D$. The proof of Theorem 1.1 was composed of 'only if'-part and 'if'-part, and the proof of 'if'-part was divided into two steps:

(i) $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ coils itself around the origin while remaining in the annulus A;

(ii) $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ is contained in $Q_4 \cap A$ (resp., $Q_2 \cap A$) ultimately and the phase point on $\Gamma^+_{(2.2)}(T, \mathbf{x}_0)$ approaches an interior point in $Q_4 \cap A$ (resp., $Q_2 \cap A$).

Assumption (1.2) was not used in the proof of 'only if'-part, and the proof of the second step of 'if'-part. In the second step, the damping coefficient $h(t)$ had only to satisfy

$$\int_0^\infty h(t)dt = \infty.$$

Therefore, we need to prove only the first step of 'if'-part, when changing assumption (1.2) to the assumption that $h(t)$ is uniformly continuous for $t \geq 0$ and weakly integrally positive.

Suppose that $\Gamma^+_{(2.2)}(t_0, \mathbf{x}_0)$ coils itself around the origin while remaining in the annulus A. Then, we conclude that

$$\liminf_{t\to\infty} |y(t)| = 0 < \sqrt{2v_0} = \limsup_{t\to\infty} |y(t)|. \tag{4.1}$$

Since $h(t)$ is uniformly continuous for $t \geq 0$, we can find numbers $T > 0$ and $\kappa > 0$ so that

$$|h(\alpha) - h(\beta)| < 1 \tag{4.2}$$

whenever $\alpha \geq T$ and $\beta \geq T$ with $|\alpha - \beta| < \kappa$. Note that κ is independent of t. Let ν be so small that

$$2\sqrt{2\nu} \leq \kappa(\omega^2 - 7\nu)\sqrt{v_0}. \tag{4.3}$$

Needless to say, it is possible to find such a positive number ν, which is less than $\omega^2/7$. By (4.1), we can choose three divergent sequences $\{\tau_n\}$, $\{t_n\}$ and $\{\sigma_n\}$ with $T < \tau_n < t_n < \sigma_n \leq \tau_{n+1}$ such that $|y(\tau_n)| = |y(\sigma_n)| = \sqrt{\nu v_0}/\omega$, $|y(t_n)| = \sqrt{2\nu v_0}/\omega$,

$$|y(t)| > \sqrt{\nu v_0}/\omega \quad \text{for} \quad \tau_n < t < \sigma_n, \tag{4.4}$$

$$\sqrt{\nu v_0}/\omega < |y(t)| < \sqrt{2\nu v_0}/\omega \quad \text{for} \quad \tau_n < t < t_n, \tag{4.5}$$

and

$$|y(t)| \leq \sqrt{2\nu v_0}/\omega \quad \text{for} \quad \sigma_n \leq t \leq \tau_{n+1}. \tag{4.6}$$

In fact, it turns out from (4.1) that $|y(t^*)| \leq \sqrt{\nu v_0}$ for some $t^* > T$. Let

$$t_1 = \min\{t > t^* : |y(t)| = \sqrt{2\nu v_0}/\omega\},$$

$$\tau_1 = \max\{t < t_1 : |y(t)| = \sqrt{\nu v_0}/\omega\},$$

and

$$\sigma_1 = \min\{t > t_1 : |y(t)| = \sqrt{\nu v_0}/\omega\}.$$

Such numbers always exist because of (4.1) and the continuity of $|y(t)|$. Using σ_1 instead of t^*, we define t_2, τ_2 and σ_2 similarly to t_1, τ_1 and σ_1, and so on. Then, $t_0 < \tau_n < t_n < \sigma_n \leq \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$. Also, (4.4)–(4.6) are satisfied. To be precise, if $y(\sigma_n)y(\tau_{n+1}) > 0$, then $|y(t)| \leq \sqrt{\nu v_0}/\omega$ for $\sigma_n \leq t \leq \tau_{n+1}$; if $y(\sigma_n)y(\tau_{n+1}) < 0$, then $|y(t)| \leq \sqrt{2\nu v_0}/\omega$ for $\sigma_n \leq t \leq \tau_{n+1}$.

Let us estimate the distance between τ_n and σ_n for each $n \in \mathbb{N}$. Since $|y(\tau_n)| = \sqrt{\nu v_0}/\omega$ and $|y(t_n)| = \sqrt{2\nu v_0}/\omega$, we see that

$$\begin{aligned}\nu v_0 = y^2(t_n) - y^2(\tau_n) &= \int_{\tau_n}^{t_n} \big(y^2(t)\big)' dt = 2\int_{\tau_n}^{t_n} y'(t)y(t)dt \\ &= 2\int_{\tau_n}^{t_n} (-\omega \sin x(t) - \omega h(t)y(t)|y(t)|)y(t)dt \le 2\omega \int_{\tau_n}^{t_n} |y(t)|dt.\end{aligned}$$

It follows from (4.5) that

$$\lambda \stackrel{\text{def}}{=} \frac{\sqrt{\nu v_0}}{2\sqrt{2}\omega^2} < t_n - \tau_n < \sigma_n - \tau_n \tag{4.7}$$

for each $n \in \mathbb{N}$.

Let us pay attention to the value of $h(t)$ at $t = \sigma_n$ for each $n \in \mathbb{N}$. Define

$$S = \{n \in \mathbb{N}\colon h(\sigma_n) \ge 2\}.$$

Suppose that the number of elements in S is infinite. Let $d = \min\{\kappa, \lambda\}$. Then, it follows from (4.2) that $n \in S$ implies that

$$h(t) > 1 \quad \text{for } \ \sigma_n - d < t < \sigma_n.$$

Also, it turns out from (4.4) and (4.7) that

$$|y(t)| > \sqrt{\nu v_0}/\omega \quad \text{for } \ \sigma_n - d < t < \sigma_n.$$

Hence, we obtain

$$\begin{aligned}\lim_{t\to\infty} v(t) - v(t_0) &= \int_{t_0}^{\infty} v'(t)dt = -\int_{t_0}^{\infty} \omega h(t)y^2(t)|y(t)|dt \\ &\le -\sum_{n\in S}\int_{\sigma_n - d}^{\sigma_n} \omega h(t)y^2(t)|y(t)|dt < -\sum_{n\in S}(\nu v_0)^{\frac{3}{2}} d/\omega^2 = -\infty.\end{aligned}$$

This contradicts the fact that $v(t) > v_0 > 0$ for $t \ge t_0$. Thus, the number of elements in the set S is finite, and therefore, there exists an $N \in \mathbb{N}$ such that

$$h(\sigma_n) < 2 \quad \text{for } \ n \ge N. \tag{4.8}$$

By (4.6), we have

$$1 - \cos x(t) = v(t) - \frac{1}{2}y^2(t) > (1 - \nu/\omega^2)v_0$$

for $\sigma_n \le t \le \tau_{n+1}$. Taking into account that $(1 - \nu/\omega^2)v_0 < 1$, we obtain

$$|x(t)| > \cos^{-1}\Big(1 - \Big(1 - \frac{\nu}{\omega^2}\Big)v_0\Big) \quad \text{for } \ \sigma_n \le t \le \tau_{n+1}. \tag{4.9}$$

Since the domain D is a positive invariant set of (2.2), we see that

$$|x(t)| < \frac{\pi}{2} \quad \text{for } \ t \ge t_0. \tag{4.10}$$

Let us estimate the distance between σ_n and τ_{n+1} for each $n \in \mathbb{N}$. Suppose that there exists an $n_0 \ge N$ such that $\tau_{n_0+1} - \sigma_{n_0} > \kappa$. Then, from (4.2) and (4.8) it follows that

$$h(t) < 1 + h(\sigma_{n_0}) < 3 \quad \text{for } \ \sigma_{n_0} \le t \le \sigma_{n_0} + \kappa.$$

Hence, using (4.3), (4.6), (4.9), (4.10) and the second equation of (2.2), we get

$$\begin{aligned}|y'(t)| &\ge \omega|\sin x(t)| - \omega h(t)y^2(t) \ge \frac{2\omega}{\pi}|x(t)| - \omega h(t)y^2(t) \\ &> \frac{2\omega}{\pi}\cos^{-1}\Big(1 - \Big(1 - \frac{\nu}{\omega^2}\Big)v_0\Big) - \frac{6\nu v_0}{\omega} \\ &> \frac{2\omega}{\pi}\frac{\pi}{2}\Big(1 - \frac{\nu}{\omega^2}\Big) - \frac{6\nu v_0}{\omega} \ge \frac{2\sqrt{2\nu v_0}}{\kappa\omega} > 0\end{aligned}$$

for $\sigma_{n_0} \le t \le \sigma_{n_0} + \kappa < \tau_{n_0+1}$. Integrate this inequality to obtain

$$\begin{aligned} |y(\sigma_{n_0} + \kappa)| + |y(\sigma_{n_0})| &\ge \left| \int_{\sigma_{n_0}}^{\sigma_{n_0}+\kappa} y'(t)dt \right| \\ &= \int_{\sigma_{n_0}}^{\sigma_{n_0}+\kappa} |y'(t)|dt > 2\sqrt{2\nu v_0}/\omega. \end{aligned}$$

This contradicts (4.6). We therefore conclude that

$$\Lambda \overset{\text{def}}{=} \kappa \ge \tau_{n+1} - \sigma_n \tag{4.11}$$

for each $n \in \mathbb{N}$.

From (4.7) and (4.11), we see that $\tau_n + \lambda < \sigma_n \le \tau_{n+1} \le \sigma_n + \Lambda$ for each $n \in \mathbb{N}$. Since $h(t)$ is weakly integrally positive, we obtain

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty. \tag{4.12}$$

However, it follows from (4.4) that

$$\begin{aligned} v(\sigma_n) - v(\tau_1) &= \int_{\tau_1}^{\sigma_n} v'(t)dt = -\int_{\tau_1}^{\sigma_n} \omega h(t)y^2(t)|y(t)|dt \\ &\le -\sum_{i=1}^{n} \int_{\tau_i}^{\sigma_i} h(t)y^2(t)|y(t)|dt \le -\frac{(\nu v_0)^{\frac{3}{2}}}{\omega^2} \sum_{i=1}^{n} \int_{\tau_i}^{\sigma_i} h(t)dt. \end{aligned}$$

This contradicts (4.12). We have thus proved Theorem 1.2. □

5. The case of bounded damping coefficient

We consider the case that $h(t)$ is bounded for $t \ge 0$, namely, there exists an $\overline{h} > 0$ such that $0 \le h(t) \le \overline{h}$ for $t \ge 0$. Then, we have the following lemma.

Lemma 5.1. *Let $u(t)$ be the solution of* (1.6) *satisfying $u(0) = 0$. If $h(t)$ is bounded for $t \ge 0$, then condition* (1.5) *holds.*

Proof. Since $u(0) = 0$ and $u'(0) = -1$, we see that $u(t) < 0$ in a right-hand neighborhood of $t = 0$. Since

$$u'(t) = -1 - h(t)u(t)|u(t)| = -1 + h(t)u^2(t) \le 0$$

as long as $-1/\sqrt{\overline{h}} \le u(t) \le 0$, there are two possibilities to consider: (i) $u(t) \searrow -\alpha$ as $t \to \infty$ for some positive α which is less than $1/\sqrt{\overline{h}}$; (ii) $u(t_1) = -1/\sqrt{\overline{h}}$ for some $t_1 > 0$. In the former, condition (1.5) is satisfied, because $u(t) < -\alpha/2$ for t sufficient large. In the latter, if $h(t) = \overline{h}$ for $t \ge t_1$, then $u(t) = -1/\sqrt{\overline{h}}$ for $t \ge t_1$. Hence, it is clear that condition (1.5) is satisfied. Otherwise, we can find a $t_2 > t_1$ such that $u(t_2) < -1/\sqrt{\overline{h}}$. Suppose that $u(t_3) > -1/\sqrt{\overline{h}}$ for some $t_3 > t_2$. Let

$$t_4 = \sup\left\{ t < t_3 : u(t) < -1/\sqrt{\overline{h}} \right\}.$$

Then, we see that $u(t_4) = -1/\sqrt{\overline{h}}$ and $u(t) > -1/\sqrt{\overline{h}}$ for $t_4 < t \le t_3$. By the mean value theorem, there exists a t_5 with $t_4 < t_5 < t_3$ such that $u'(t_5) > 0$. However, since $u(t_5) > -1/\sqrt{\overline{h}}$, it follows that

$$u'(t_5) = -1 + h(t_5)u^2(t_5) < -1 + \overline{h}\frac{1}{\overline{h}} = 0.$$

This is a contradiction. Thus, we conclude that $u(t) \le -1/\sqrt{\overline{h}}$ for $t \ge t_2$. It turns out from this inequality that condition (1.5) holds. □

In Theorem 1.2, we assumed that $h(t)$ is uniformly continuous for $t \geq 0$. Recall that the uniform continuity of $h(t)$ was used only to obtain estimation (4.8). This estimation is unnecessary when $h(t)$ is bounded for $t \geq 0$. Hence, by means of this fact and Lemma 5.1, we have the following consequence of Theorems 1.1 and 1.2.

Theorem 5.2. *Suppose that either $h(t)$ satisfies assumption (1.2) or that it is uniformly continuous for $t \geq 0$ and weakly integrally positive. If $h(t)$ is bounded for $t \geq 0$, then the origin of (P) is asymptotically stable.*

6. Discussion

As shown in the preceding section, condition (1.5) holds under the assumption that the damping coefficient $h(t)$ is bounded. The questions arise: Can condition (1.5) be satisfied for unbounded $h(t)$? Can condition (1.5) be not satisfied for unbounded $h(t)$? Theorems 6.1– 6.3 below answer this question. The proofs of Theorems 6.1–6.3 are carried out in the same manner as the proofs of Corollaries 4.1, 4.2 and 4.4 in [32], respectively. To save the space, we omit details.

Theorem 6.1. *Suppose that there exist a differentiable function $g(t)$ and a positive number T such that*

$$g(t) > 0 \quad \text{and} \quad h(t) \leq g(t)$$

for $t \geq T$. If $g'(t) \geq 0$ for $t \geq T$ and

$$\int_T^{\infty} \frac{1}{\sqrt{g(t)}}\, dt = \infty,$$

then condition (1.5) holds.

Theorem 6.2. *Suppose that there exist a differentiable function $g(t)$ and positive numbers $\underline{g}$ and T such that*

$$g(t) > \underline{g} \quad \text{and} \quad h(t) \leq g(t)$$

for $t \geq T$. If

$$\lim_{t\to\infty} \frac{g'(t)}{g(t)} = 0 \quad \text{and} \quad \int_T^{\infty} \frac{1}{\sqrt{g(t)}}\, dt = \infty,$$

then condition (1.5) holds.

Note that both Theorem 6.1 and Theorem 6.2 are generalization of Lemma 5.1.

Theorem 6.3. *Suppose that there exist a differentiable function $k(t)$ and positive numbers $\underline{k}$ and T such that*

$$\underline{k} \leq k(t) \leq h(t)$$

for $t \geq T$. If

$$\lim_{t\to\infty} \frac{k'(t)}{k(t)} = 0 \quad \text{and} \quad \int_T^{\infty} \frac{1}{\sqrt{k(t)}}\, dt < \infty,$$

then condition (1.5) fails to hold.

From Theorem 6.3, we see that condition (1.5) does not hold when the damping coefficient rapidly grows. Combining Theorems 6.1–6.3, we obtain the following simple necessary and sufficient condition for the origin of (P) to be asymptotically stable.

Corollary 6.4. *Suppose that there exist positive numbers γ and T such that*

$$h(t) = t^{\gamma} \quad \text{for } t \geq T.$$

Then the origin of (P) is asymptotically stable if and only if $\gamma \leq 2$.

Needless to say, if the damping coefficient is a polynomial of power functions of t, then we merely have only to consider the largest exponent of the polynomial as γ in Corollary 6.4.

To illustrate our theorems, we give two examples in which $\liminf_{t\to\infty} h(t) = 0$ and $\limsup_{t\to\infty} h(t) = \infty$. In the first example, the set $\{t \ge 0\colon h(t) = 0\}$ is the union of infinitely many disjoint intervals whose length are $\pi/2$.

Example 6.1. Consider equation (P) with

$$h(t) = t\left(|\sin^3 2t| - \sin^3 2t\right).$$

Then the origin is asymptotically stable.

Let

$$I_n = [(n-1)\pi, (n-1/2)\pi] \quad \text{and} \quad J_n = [(n-1/2)\pi, n\pi]$$

for each $n \in \mathbb{N}$. Then

$$h(t) = \begin{cases} 0 & \text{if } t \in I_n, \\ -2t\sin^3 2t & \text{if } t \in J_n \end{cases}$$

with $n \in \mathbb{N}$. Assumption (1.2) is satisfied with $\gamma_0 = 3\pi/4$. In fact, since $h(t) \ge -2\sin^3 2t$ if $t \in J_n$ for each $n \in \mathbb{N}$, we see that

$$\liminf_{t\to\infty} \int_t^{t+3\pi/4} h(s)ds \ge \liminf_{t\to\infty} \int_t^{t+3\pi/4} \widetilde{h}(s)ds,$$

where

$$\widetilde{h}(t) = \begin{cases} 0 & \text{if } t \in I_n, \\ -2\sin^3 2t & \text{if } t \in J_n. \end{cases}$$

Note that $\widetilde{h}(t)$ is a periodic function with period π. Define

$$\varphi(t) = \int_t^{t+3\pi/4} \widetilde{h}(s)ds.$$

Then, by a straightforward calculation, we obtain

$$\varphi(t) = \begin{cases} \sin 2t - (\sin^3 2t)/3 + 2/3 & \text{for } 0 \le t < \pi/4, \\ 3/4 & \text{for } \pi/4 \le t < \pi/2, \\ -\cos 2t + (\cos^3 2t)/3 + 2/3 & \text{for } \pi/2 \le t < 3\pi/4, \\ \sin 2t - \cos 2t & \\ \quad -(\sin^3 2t)/3 + (\cos^3 2t)/3 + 4/3 & \text{for } 3\pi/4 \le t < \pi. \end{cases}$$

Hence, it turns out that $\varphi(t)$ is increasing for $0 \le t \le \pi/4$ and $7\pi/8 \le t \le \pi$, and decreasing for $\pi/2 \le t \le 7\pi/8$. In addition, $\varphi(0) = \varphi(3\pi/4) = \varphi(\pi) = 2/3$ and $\varphi(t) = 4/3$ for $\pi/4 \le t \le \pi/2$. Since $\varphi(t)$ is also a periodic function with period π, we see that

$$\liminf_{t\to\infty} \int_t^{t+3\pi/4} h(s)ds \ge \liminf_{t\to\infty} \varphi(t) \ge \varphi(7\pi/8) = -\frac{5}{6}\sqrt{2} + \frac{4}{3} > 0.$$

Let $g(t) = 2t$ and $T = 1$. Then, it is clear that

$$g(t) \ge 2, \quad h(t) \le g(t) \quad \text{and} \quad g'(t) = 2 > 0$$

for $t \ge T$, and

$$\int_T^\infty \frac{1}{\sqrt{g(t)}}dt = \int_1^\infty \frac{1}{\sqrt{2t}}dt = \infty.$$

Thus, from Theorem 6.1 it turns out that condition (1.5) holds. Hence, by means of Theorem 1.1, we conclude that the origin is asymptotically stable (see Figure 1).

From Theorems 1.1 and 6.2, we can also confirm the asymptotic stability of the origin, because

$$\lim_{t\to\infty}\frac{g'(t)}{g(t)} = \lim_{t\to\infty}\frac{1}{t} = 0.$$

We will show that the major premises of Theorem 1.2 are not satisfied. The length of I_n is $\pi/2$ for each $n \in \mathbb{N}$. Hence, $h(t)$ is not weakly integrally positive. Since

$$h'(t) = \begin{cases} 0 & \text{if } t \in I_n \\ -2\sin^3 2t - 12\,t\sin^2 2t\cos 2t & \text{if } t \in J_n \end{cases}$$

with $n \in \mathbb{N}$, it is continuous for $t \geq 0$. However, $h'(t)$ is not bounded. Hence, $h(t)$ is not uniformly continuous. Thus, Theorem 1.2 cannot be applied to Example 6.1.

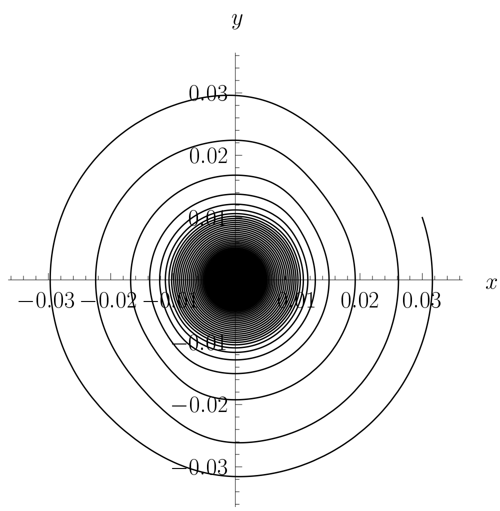

Fig. 1. The positive orbit of the system $x' = y$, $y' = -\omega^2\sin x - t\left(|\sin^3 2t| - \sin^3 2t\right) y\,|y|$ starting from the point $(x_0, y_0) = (0.03, 0.01)$ at the initial time $t_0 = 0$.

In contrast to Example 6.1, we next give an example that is applied to Theorem 1.2, but is not applied to Theorem 1.1.

Example 6.2. Consider equation (P) with

$$h(t) = \sqrt{t}\left(\left|\sin^3\sqrt{t}\right| - \sin^3\sqrt{t}\right) + \frac{\sin^2 t}{1+t}.$$

Then the origin is asymptotically stable.

Let

$$I_n = [4(n-1)^2\pi^2, (2n-1)^2\pi^2] \quad \text{and} \quad J_n = [(2n-1)^2\pi^2, 4n^2\pi^2]$$

for each $n \in \mathbb{N}$. Then

$$h(t) = \begin{cases} \sin^2 t/(1+t) & \text{if } t \in I_n, \\ -2\sqrt{t}\sin^3\sqrt{t} + \sin^2 t/(1+t) & \text{if } t \in J_n \end{cases}$$

with $n \in \mathbb{N}$. The function $h(t)$ is continuously differentiable for $t > 0$ and

$$h'(t) = \begin{cases} 2\sin t\cos t/(1+t) - \sin^2 t/(1+t)^2 & \text{if } t \in I_n, \\ -(\sin^3\sqrt{t})/\sqrt{t} - 3\sin^2\sqrt{t}\cos\sqrt{t} & \\ \quad + 2\sin t\cos t/(1+t) - \sin^2 t/(1+t)^2 & \text{if } t \in J_n \end{cases}$$

with $n \in \mathbb{N}$. Since

$$|h'(t)| \le \left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right| \left|\sin^2\sqrt{t}\right| + 3\left|\sin^2\sqrt{t}\right| \left|\cos\sqrt{t}\right| + \frac{|\sin 2t|}{1+t} + \frac{\sin^2 t}{(1+t)^2} \le 6$$

for $t \ge \pi^2$, we see that $h(t)$ is uniformly continuous for $t \ge 0$. As mentioned in Section 1, the function $\sin^2 t/(1+t)$ is weakly integrally positive (for the proof, see [31]). Taking the inequality

$$h(t) \ge \frac{\sin^2 t}{1+t} \quad \text{for} \ \ t \ge 0,$$

we see that $h(t)$ is also a weakly integrally positive function. Thus, the major premises of Theorem 1.2 are satisfied. Let $g(t) = 3\sqrt{t}$ and $T = 1$. Then, it is clear that $g(t) \ge 3$, $g'(t) = 3/(2\sqrt{t}) > 0$ and

$$h(t) \le 2\sqrt{t} + \frac{1}{1+t} \le g(t)$$

for $t \ge T$. It is also cleat that

$$\int_T^\infty \frac{1}{\sqrt{g(t)}}\,dt = \int_1^\infty \frac{1}{\sqrt{3}\sqrt[4]{t}}\,dt = \infty.$$

Thus, from Theorem 6.1 it turns out that condition (1.5) holds. Hence, by means of Theorem 1.2, we conclude that the origin is asymptotically stable (see Figure 2).

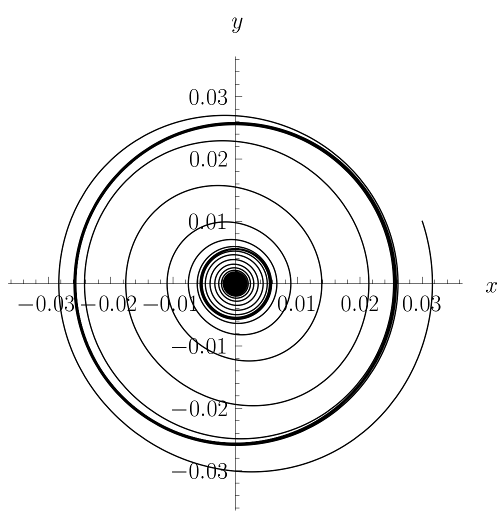

Fig. 2. The positive orbit of the system $x' = y$, $y' = -x - \left(\sqrt{t}\left(\left|\sin^3\sqrt{t}\right| - \sin^3\sqrt{t}\right) + \sin^2 t/(1+t)\right)|y|y$ starting from the point $(x_0, y_0) = (0.03, 0.01)$ at the initial time $t_0 = 0$.

From Theorems 1.2 and 6.2, we can also confirm the asymptotic stability of the origin, because

$$\lim_{t\to\infty} \frac{g'(t)}{g(t)} = \lim_{t\to\infty} \frac{1}{2t} = 0.$$

Assumption (1.2) is not satisfied. In fact, since the length of I_n is not less than π for each $n \in \mathbb{N}$, we see that

$$0 \le \liminf_{t\to\infty} \int_t^{t+\gamma_0} h(s)ds \le \lim_{n\to\infty} \int_{4(n-1)^2\pi^2}^{4(n-1)^2\pi^2+\pi} \frac{1}{1+t}dt$$
$$\le \lim_{n\to\infty} \frac{\pi}{1+4(n-1)^2\pi^2} = 0$$

for any γ_0 with $0 < \gamma_0 < \pi$. Thus, Theorem 1.1 cannot be applied to Example 6.2.

Finally, to apply to also a pendulum with a nonnegative real-power damping force, we extend the main results, namely, Theorems 1.1 and 1.2. Physical models whose damping force is neither linear nor quadratic have been reported in many papers (for example, see [4, 7, 20, 25]). For convenience, we define

$$\phi_q(y) = |y|^{q-2}y, \qquad y \in \mathbb{R}$$

with $q \ge 2$ (but may not be necessarily an integer) and consider the damped superlinear pendulum

$$x'' + h(t)\phi_q(x') + \omega^2 \sin x = 0. \tag{SP}$$

Then, combining mathematical ideas of the present paper and recent papers [32, 33], we can obtain the following results.

Theorem 6.5. *Suppose that either $h(t)$ satisfies assumption* (1.2) *or that it is uniformly continuous for $t \ge 0$ and weakly integrally positive. Let $u(t)$ is the solution of the characteristic equation*

$$u' + h(t)\phi_q(u) + 1 = 0$$

satisfying $u(0) = 0$. Then the origin of (SP) is asymptotically stable if and only if condition (1.5) *holds.*

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