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Global dynamics of Froude-type oscillators with superlinear damping terms

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Abstract This paper deals with the damped superlinear oscillator

$$x'' + a(t)\phi_p(x') + b(t)\phi_q(x') + \omega^2 x = 0,$$

where $a(t)$ and $b(t)$ are continuous and nonnegative for $t \geq 0$; p and q are real numbers greater than or equal to 2; $\phi_r(x') = |x'|^{r-2}x'$. This equation is a generalization of nonlinear ship rolling motion with Froude's expression, which is very familiar in marine engineering, ocean engineering and so on. Our concern is to establish a necessary and sufficient condition for the equilibrium to be globally asymptotically stable. In particular, the effect of the damping coefficients $a(t)$, $b(t)$ and the nonlinear functions $\phi_p(x')$, $\phi_q(x')$ on the global asymptotic stability is discussed. The obtained criterion is judged by whether the integral of a particular solution of the first-order nonlinear differential equation

$$u' + \omega^{p-2}a(t)\phi_p(u) + \omega^{q-2}b(t)\phi_q(u) + 1 = 0$$

is divergent or convergent. In addition, explicit sufficient conditions and explicit necessary conditions are given for the equilibrium of the damped superlinear oscillator to be globally attractive. Moreover, some examples are included to illustrate our results. Finally, our results are extended to be applied to a more complicated model.

Keywords Damped oscillator · Superlinear differential equations · Global asymptotic stability · Free rolling motion · Froude's expression

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1 Introduction

It is well-known that the resistance of free rolling motion of a small fishing vessel is mainly classified into three types. The first is frictional resistance produced when the surrounding water of a ship rubs the surface of a hull. The second is eddy-making resistance produced by vortices formed when the flow of water exfoliates from a hull. This is also called viscous pressure resistance. The third is wave-making resistance caused when a bow or a stern generates the propagating waves. In the latter half of the 19th century, William Froude who was an English engineer thought that frictional resistance and eddy-making resistance are proportional to the square of the angular velocity $\theta'(t)$, where $\theta(t)$ is the roll angle at $t \geq 0$ and the prime denotes d/dt, and that wave-making resistance is proportional to the angular velocity $\theta'(t)$. To verify his assertion, he repeated many experiments. Afterwards, by experiments, a lot of engineers examined causes that influence the extinction of free rolling motion (for example, see [4, 5, 9, 14, 18, 25, 31, 42]).

Let the initial time t_0 be zero and let us assume the inclination of a hull at the initial time to be θ_0; that is, $\theta(0) = \theta_0$ and $\theta'(0) = 0$. Denote by θ_n the absolute value of roll angle $\theta(t)$ at the time of the n-th extreme value. The value θ_n can be measured in an experiment of free rolling motion of a ship. Let $\Delta\theta$ and $\overline{\theta}_n$ be the difference and the average of θ_{n-1} and θ_n, respectively; namely,

$$\Delta\theta = \theta_{n-1} - \theta_n \quad \text{and} \quad \overline{\theta}_n = \frac{1}{2}(\theta_{n-1} + \theta_n)$$

with $n \in \mathbb{N}$. As a relation between $\Delta\theta$ and $\overline{\theta}_n$, Froude proposed

$$\Delta\theta = a\overline{\theta}_n + b\overline{\theta}_n^2,$$

where a and b are real positive numbers. This is Froude's expression which is famous in marine engineering. The numbers a and b are called extinction coefficients. The extinction coefficients a and b are presumed from experimental data by using the least squares method. Hence, we have to suppose the extinction coefficients to be positive fixed numbers.

According to the above-mentioned idea of Froude, the equation of rolling motion of a vessel on still water can be written as:

$$\theta'' + \alpha\theta' + \beta|\theta'|\theta' + \omega^2\theta = 0$$

with the initial condition $(\theta(0), \theta'(0)) = (\theta_0, 0)$, where α and β are the damping coefficients per unit of the virtual moment of inertia, and ω is the restoring coefficient per unit of the virtual moment of inertia. The relation between the damping coefficients α, β and the extinction coefficients a, b can be derived as follows: Let t_n be the time of the n-th extreme value of roll angle $\theta(t)$. The time lag of t_{n-1} and t_n is called the natural period of roll by a technical term. Researchers of ocean engineering assume that $\{t_n\}$ is an arithmetic sequence with common difference $2\pi/\omega$; namely, $t_{n+1} - t_{n-1} = 2\pi/\omega$ for arbitrary $n \in \mathbb{N}$. Such an approximation has validity when the damping coefficients α and β are relatively smaller than ω. Multiplying $\theta'(t)$ in the both sides of the above equation of motion and integrating from 0 to π/ω, we obtain

$$\int_0^{\pi/\omega} \theta''(t)\theta'(t)\,dt + \alpha\int_0^{\pi/\omega} (\theta'(t))^2\,dt + \beta\int_0^{\pi/\omega} |\theta'(t)|(\theta'(t))^2\,dt + \omega^2\int_0^{\pi/\omega} \theta(t)\theta'(t)\,dt = 0.$$

The first term of this integral equation means the kinetic energy of a hull. Since $\theta'(0) = 0$ and $\theta'(\pi/\omega) = \theta'(t_1) = 0$, we see that the kinetic energy is zero. The total of the second and third terms means an energy dissipated by the damping force. Using the approximation formula $\theta(t) \simeq \overline{\theta}_n \cos\omega t$, we get

$$\int_0^{\pi/\omega} (\theta'(t))^2\, dt \simeq \frac{\pi}{2}\omega\overline{\theta}_n^2 \quad \text{and} \quad \int_0^{\pi/\omega} |\theta'(t)|(\theta'(t))^2\, dt \simeq \frac{4}{3}\omega^2\overline{\theta}_n^3.$$

The last term means the potential energy of a hull. We can estimate that

$$\int_0^{\pi/\omega} \theta(t)\theta'(t)\, dt = \int_{\theta_{n-1}}^{\theta_n} \theta\, d\theta = \frac{1}{2}\left(\theta_n^2 - \theta_{n-1}^2\right) = -\overline{\theta}_n \Delta\theta.$$

Arranging these evaluations, we obtain

$$\Delta\theta = \frac{\pi}{2\omega}\alpha\overline{\theta}_n + \frac{4}{3}\beta\overline{\theta}_n^2.$$

Hence, the damping coefficients α and β are given from the extinction coefficients a and b by a simple expression of relations

$$\alpha = \frac{2\omega}{\pi}a \quad \text{and} \quad \beta = \frac{3}{4}b.$$

We can refer to Himeno [25] for the details of how to derive the damping coefficients α and β. In this paper, however, we are not concerned with finding the relationship between the damping coefficients and the extinction coefficients. From a mathematical viewpoint, we intend to consider merely Froude-type equations with damping coefficients.

Clearly, the above-mentioned equation proposed by Froude is a second-order nonlinear differential equation with two constants coefficients. Here, some simple doubts arise: Will the damping force be really proportional to the square of the angular velocity or the angular velocity? Can the damping force always be expressed by only the linear or quadratic form of the angular velocity? Actually, some models where the damping force is shown by the cubic form of the angular velocity have already been researched in free rolling motion (for example, see [4, 8, 9, 12, 14, 18, 35, 42]). Then, may we think that the damping force has the polynomial expression of the angular velocity? However, it was reported that the damping force of an air spring model is proportional to a velocity exponent of 1.7 or 1.8 though it was a different equation of motion (see [6, 36]). The damping coefficients α and β of the classic Froude-type equation are constants, but may we really assume that the damping coefficient is a constant? Do not the damping coefficients change with time under some kind of influence? We will be able to cite aging deterioration of a vessel as an example. If a ship has not gone into dock for a long time, then seaweed, shellfishes, and others will adhere, and if the surface of the hull loses smoothness, then frictional resistance becomes large rapidly. Let us consider a submarine instead of a vessel. Wave-making resistance acts when the submarine surfaces, but it is lost when the submarine dives. It is well-known that eddy-making resistance changes depending on the density of fluid and the form of the object. The density of fluid is influenced by temperature and atmospheric pressure, which change with time. From this point of view, it would be reasonable to deal with Froude-type equations with time-varying damping terms.

Hereafter, we consider the following second-order differential equation:

$$x'' + a(t)\phi_p(x') + b(t)\phi_q(x') + \omega^2 x = 0, \tag{1.1}$$

where the damping coefficients $a(t)$ and $b(t)$ are continuous and nonnegative for $t \geq 0$, the restoring coefficient ω is positive, and the functions $\phi_p(z)$ and $\phi_q(z)$ are defined by

$$\phi_p(z) = |z|^{p-2}z \quad \text{and} \quad \phi_q(z) = |z|^{q-2}z \quad \text{for } z \in \mathbb{R}$$

with $p \geq 2$ and $q \geq 2$, respectively. It is clear that the only equilibrium of (1.1) is the origin $(x, x') = (0, 0)$. In the special case in which $a(t) \equiv \alpha$, $b(t) \equiv \beta$, $p = 2$ and $q = 3$, equation (1.1) coincides with the Froude's expression above. Since $p \geq 2$ and $q \geq 2$, we call equation (1.1) a damped superlinear *Froude-type oscillator*. The global existence and uniqueness of solutions of (1.1) are guaranteed for the initial value problem because $a(t) \geq 0$ and $b(t) \geq 0$ for $t \geq 0$.

The purpose of this paper is to present a necessary and sufficient condition for the equilibrium of (1.1) to be globally asymptotically stable. It is well-known that the damping coefficients changes according to the form and the design of a ship; namely, they are affected by the shapes of hull and the design of the antiroll apparatus (bilge keels, fins, or antiroll tanks, and so on). The result that the stability of a ship was very sensitive to the change in the damping coefficients was often reported. For example, see [17]. Therefore, it is safe to say that research of the stability of a ship is extremely important in determining the form and the design of the ship.

To begin with, we give a set of definitions concerning stability. Let $\mathbf{x}(t) = (x(t), x'(t))$ and $\mathbf{x}_0 \in \mathbb{R}^2$, and let $\|\cdot\|$ be any suitable norm. We denote the solution of (1.1) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. The equilibrium is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The equilibrium is said to be *globally attractive* if, for any $t_0 \geq 0$, any $\eta > 0$, and any $\mathbf{x}_0 \in \mathbb{R}^2$, there is a $T(t_0, \eta, \mathbf{x}_0) > 0$ such that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T(t_0, \eta, \mathbf{x}_0)$. Roughly speaking, if $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \to 0$ as $t \to \infty$ for any $t_0 \geq 0$ and any $\mathbf{x}_0 \in \mathbb{R}^2$, then the equilibrium is globally attractive. The equilibrium is *globally asymptotically stable* if it is stable and globally attractive. With respect to the various definitions of stability, the reader may refer to the books [2, 7, 10, 13, 19, 20, 30, 34, 44] for example.

The study of global asymptotic stability is one of main themes in the qualitative theory of differential equations. Many efforts have been poured to find sufficient conditions and/or necessary conditions which guarantee that the equilibrium (or the zero solution) of nonlinear differential equations (or systems) is globally asymptotically stable (for example, see [1, 3, 15, 21–24, 27, 29, 32, 33, 37, 38, 40, 41]). The historical development of this research is concisely summarized in Sugie [38, Section 1].

Recently, the present authors [41] have discussed the stability problem for the damped superlinear oscillator

$$x'' + a(t)\phi_p(x') + \omega^2 x = 0, \tag{1.2}$$

and reported the following result.

Theorem A *Suppose that there exists a* γ_1 *with* $0 < \gamma_1 < \pi/\omega$ *such that*

$$\liminf_{t\to\infty} \int_t^{t+\gamma_1} a(s)ds > 0. \tag{1.3}$$

Then the equilibrium of (1.2) *is globally asymptotically stable if and only if*

$$\int_0^\infty u(t)dt = -\infty, \tag{1.4}$$

where $u(t)$ is the solution of

$$u' + \omega^{p-2}a(t)\phi_p(u) + 1 = 0$$

satisfying $u(0) = 0$.

Theorem A is a generalization of the results given to the damped linear oscillator

$$x'' + a(t)x' + \omega^2 x = 0.$$

by Smith [37, Theorems 1 and 2] and Hatvani and Totik [24, Theorem 3.1]. In the special case in which $p = 2$, condition (1.4) coincides with

$$\int_0^\infty \frac{\int_0^t e^{A(s)}ds}{e^{A(t)}}\,dt = \infty,$$

where

$$A(t) = \int_0^t a(s)ds.$$

Even if intervals where $a(t)$ is zero appear repeatedly many times, condition (1.3) may be satisfied if the lengths of intervals are less than π/ω. Hatvani and Totik [24, Example 3.2] pointed out that the requirement that $0 < \gamma_1 < \pi/\omega$ was best possible for the linear case ($p = 2$) in the meaning that it cannot be changed to $\gamma_1 = \pi/\omega$.

Unfortunately, however, Theorem A cannot be applied to the superlinear Froude-type oscillator (1.1), because it has two different kinds of damping terms. Of course, the method which was used to obtain Theorem A might give a resolution to the questions below, but a detailed analysis often will be required.

What conditions should the two coefficients satisfy in order to guarantee that all solutions of (1.1) converge to zero as time increases? From Theorem A, we see that all solutions of (1.1) with $b(t) \equiv 0$ approach the origin as t tends to ∞ if conditions (1.3) and (1.4) are satisfied. Then, do all solutions of (1.1) approach the origin as t tends to ∞ whenever $b(t) \geq 0$ for $t \geq 0$? The answer is no (for example, see Example 5.1). From Theorem A, we also see that under the assumption that

$$\liminf_{t\to\infty} \int_t^{t+\gamma_2} b(s)ds > 0 \tag{1.5}$$

for some γ_2 with $0 < \gamma_2 < \pi/\omega$, all solutions of (1.1) with $a(t) \equiv 0$ approach the origin as t tends to ∞ if

$$\int_0^\infty u(t)dt = -\infty,$$

where $u(t)$ is the solution of

$$u' + \omega^{q-2}b(t)\phi_q(u) + 1 = 0$$

satisfying $u(0) = 0$. Then, cannot we say that all solutions of (1.1) approach the origin as t tends to ∞ whenever neither condition (1.3) nor condition (1.5) is satisfied? The answer is also no. There are cases that all solutions of (1.1) approach the origin as t tends to ∞ even if both conditions are not necessarily satisfied (for example, see Example 4.2).

Our main theorem is as follows:

Theorem 1.1 *Suppose that there exists a γ_0 with $0 < \gamma_0 < \pi/\omega$ such that*

$$\liminf_{t\to\infty} \int_t^{t+\gamma_0} \big(a(s)+b(s)\big)ds > 0. \tag{1.6}$$

Then the equilibrium of (1.1) *is globally asymptotically stable if and only if*

$$\int_0^\infty u(t)dt = -\infty,$$

where $u(t)$ is the solution of

$$u' + \omega^{p-2}a(t)\phi_p(u) + \omega^{q-2}b(t)\phi_q(u) + 1 = 0 \tag{1.7}$$

satisfying $u(0) = 0$.

Note that condition (1.6) is weaker than conditions (1.3) and (1.5). In fact, let $a(t) = s(t)$ and $b(t) = s(t+\pi/\omega)$, where

$$s(t) = \begin{cases} \sin^2(\omega t) & \text{for } \dfrac{2(n-1)}{\omega}\pi \le t < \dfrac{2n-1}{\omega}\pi, \\ 0 & \text{for } \dfrac{2n-1}{\omega}\pi \le t < \dfrac{2n}{\omega}\pi \end{cases}$$

with $n \in \mathbb{N}$. Then, $a(t)+b(t) = \sin^2(\omega t)$ for $t \ge 0$ and

$$\int_t^{t+\gamma_0} \big(a(s)+b(s)\big)ds = \int_t^{t+\gamma_0} \sin^2(\omega s)ds \ge \frac{1}{2}\left(\gamma_0 - \frac{\sin(\omega\gamma_0)}{\omega}\right) > 0,$$

and therefore, condition (1.6) is satisfied. However, neither condition (1.3) nor condition (1.5) is satisfied (see Figure 1).

2 Equivalence relation

Let $u(t)$ be any solution of (1.7). Since $a(t) \ge 0$ and $b(t) \ge 0$ for $t \ge 0$, it is clear that $u'(t) \le -1$ as long as $u(t) \ge 0$. Hence, there exists a $T \ge 0$ such that $u(T) = 0$. Since $u'(T) = -1$, we see that $u(t) < 0$ in a right-hand neighborhood of T. Suppose that there exists a $t_1 > T$ such that $u(t_1) = 0$ and

$$u(t) < 0 \quad \text{for } T < t < t_1.$$

From $u(t_1) = 0$ it follows that $u'(t_1) = -1$. Since $u'(t)$ is continuous as long as it exists, there exists a small $\delta > 0$ such that $u'(t) < 0$ for $t \in [t_1-\delta, t_1]$. This means that $u(t_1-\delta) > u(t_1) = 0$, which contradicts the definition of t_1. Hence, $u(t)$ is negative for $t > T$ as long as it exists. It is also clear that $u'(t) \ge -1$ as long as $u(t) < 0$. We therefore conclude that $u(t)$ exists in the future and

$$u(t) < 0 \quad \text{for } t > T.$$

In addition, since $p \ge 2$ and $q \ge 2$, the uniqueness of solutions of (1.7) is guaranteed for the initial value problem.

Let T be a nonnegative number. We denote the solution $u(t)$ of (1.7) satisfying $u(T) = 0$ by $u(t;T)$. Then, we have the following equivalence relation, which plays an important role in this paper.

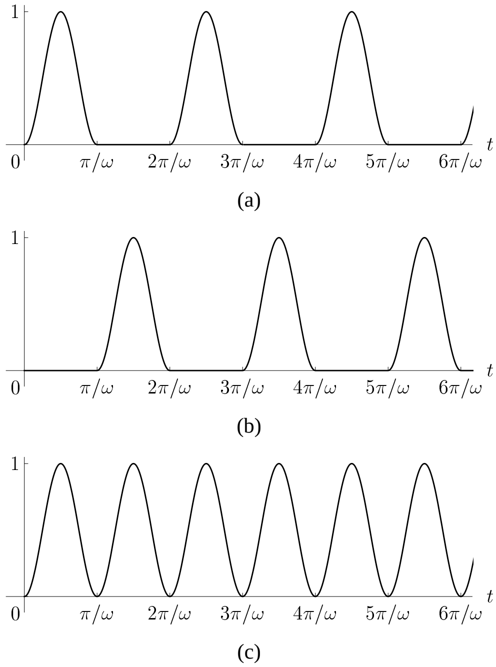

Fig. 1. (a) The graph of $a(t)$; (b) The graph of $b(t)$; (c) The graph of $a(t)+b(t)$.

Lemma 2.1 *For any* $T \geq 0$,

$$\int_T^\infty u(t;T)dt = -\infty$$

if and only if

$$\int_0^\infty u(t;0)dt = -\infty.$$

To prove Lemma 2.1, we need the following comparison results concerning the scalar differential equation

$$u' = f(t,u), \tag{2.1}$$

where $f(t,u)$ is continuous on $[0,\infty)\times\mathbb{R}$ and satisfies locally a Lipschitz condition with respect to u (for example, see [28, pp. 30–31] and [44, p. 5]).

Lemma 2.2 *Let* $u(t)$ *be a solution of* (2.1) *on an interval* $[a,b]$. *Suppose that* $\eta(t)$ *is continuous on* $[a,b]$ *and satisfies the inequality*

$$\eta'(t) \geq f(t,\eta(t)) \quad \text{for } a < t < b.$$

If $\eta(a) \geq u(a)$, *then* $\eta(t) \geq u(t)$ *for* $a \leq t \leq b$.

Lemma 2.3 *Let* $u(t)$ *be a solution of* (2.1) *on an interval* $[a,b]$. *Suppose that* $\eta(t)$ *is continuous on* $[a,b]$ *and satisfies the inequality*

$$\eta'(t) \leq f(t,\eta(t)) \quad \text{for } a < t < b.$$

If $\eta(a) \leq u(a)$, *then* $\eta(t) \leq u(t)$ *for* $a \leq t \leq b$.

3 Global asymptotic stability

Let $y = x'/\omega$ as a new variable. Then, Eq. (1.1) becomes the planar system

$$\begin{aligned} x' &= \omega y, \\ y' &= -\omega x - \omega^{p-2} a(t)\phi_p(y) - \omega^{q-2} b(t)\phi_q(y). \end{aligned} \tag{3.1}$$

The equilibrium of (1.1) corresponds to the zero solution of (3.1). Hence, in order to verify Theorem 1.1, we have only to discuss whether the zero solution of (3.1) is stable and globally attractive or not. For convenience's sake, we divide the whole x-y plane into four quadrants:

$$Q_1 = \{(x,y) : x \geq 0 \text{ and } y > 0\},$$

$$Q_2 = \{(x,y) : x < 0 \text{ and } y \geq 0\},$$

$$Q_3 = \{(x,y) : x \leq 0 \text{ and } y < 0\},$$

$$Q_4 = \{(x,y) : x > 0 \text{ and } y \leq 0\}.$$

We call the projection of a positive semitrajectory of (3.1) onto the x-y plane a *positive orbit* and we denote by $\Gamma^+(t_0, \mathbf{x}_0)$ the positive orbit of (3.1) starting from a point $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ at the initial time $t_0 \geq 0$.

As a suitable Lyapunov function for system (3.1), we choose the total energy

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right).$$

Then, we obtain

$$\dot{V}_{(3.1)}(t,x,y) = xx' + yy' = -\omega^{p-2} a(t)|y|^p - \omega^{q-2} b(t)|y|^q \leq 0$$

on $[0,\infty) \times \mathbb{R}^2$. This means the derivative of $V(x,y)$ along any solution of (3.1). Since $V(x,y)$ is positive definite and $\dot{V}_{(3.1)}(t,x,y)$ is nonpositive, we obtain the following result by means of a basic Lyapunov's direct method.

Proposition 3.1 *The zero solution of* (3.1) *is stable.*

Note that Proposition 3.1 can be led only under the assumption that $a(t) \geq 0$ and $b(t) \geq 0$ for $t \geq 0$. To be precise, the zero solution of (3.1) is uniformly stable.

Next, we discuss the global attractivity of the zero solution of (3.1). We first prove 'only if'-part of Theorem 1.1.

Theorem 3.2 *If the zero solution of* (3.1) *is attractive, then*

$$\int_0^\infty u(t)dt = -\infty, \tag{3.2}$$

where $u(t)$ *is the solution of* (1.7) *satisfying* $u(0) = 0$.

Proof. The proof is by contradiction. Suppose that (3.2) does not holds. Let $L = \max\{1, \omega\}$. Then, there exists a $T > 0$ such that

$$\int_T^\infty u(t)dt > -\frac{1}{2\omega L}.$$

Since

$$u(t) = u(t;0) \le u(t;T) < 0 \quad \text{for } t > T$$

as shown in the proof of Lemma 2.1, we see that

$$\int_T^\infty u(t;T)dt > -\frac{1}{2\omega L}. \tag{3.3}$$

Consider the positive orbit $\Gamma^+(t_0, \mathbf{x}_0)$, where $t_0 = T$ and $\mathbf{x}_0 = (1,0)$. Let $(x(t), y(t))$ be the solution of (3.1) corresponding to the positive orbit. Then, $x(T) = 1$ and $y(T) = 0$. Taking the vector field of (3.1) into account, we see that the positive orbit goes into Q_4 afterwards and it does not enter Q_1 passing through the positive x-axis. If

$$x(t) > \frac{1}{2} \quad \text{for } t \ge T, \tag{3.4}$$

then naturally the solution $(x(t), y(t))$ does not approach the origin; namely, the zero solution of (3.1) is not attractive. This completes the proof.

Hereafter, we will show that (3.4) holds. If (3.4) is not satisfied, we can find a $T_1 > T$ such that $x(T_1) = 1/2$ and $1/2 < x(t) \le 1$ for $T \le t < T_1$. Since the positive orbit does not enter Q_1 passing through the positive x-axis, we see that

$$y(t) < 0 \quad \text{for } T < t \le T_1.$$

Let $\eta(t) = y(t)/L$. We compare $\eta(t)$ with the solution $u(t;T)$ of (1.7) satisfying $u(T;T) = 0$. From the second equation of (3.1) it follows that

$$\begin{aligned}
\eta'(t) &= -\frac{\omega}{L}x(t) - \frac{\omega^{p-2}}{L}a(t)\phi_p(y(t)) - \frac{\omega^{q-2}}{L}b(t)\phi_q(y(t)) \\
&\ge -1 - \omega^{p-2}\frac{\phi_p(L)}{L}a(t)\phi_p(\eta(t)) - \omega^{q-2}\frac{\phi_q(L)}{L}b(t)\phi_q(\eta(t)) \\
&\ge -1 - \omega^{p-2}a(t)\phi_p(\eta(t)) - \omega^{q-2}b(t)\phi_q(\eta(t))
\end{aligned}$$

for $T \le t \le T_1$. Let $f(t,u) = -1 - \omega^{p-2}a(t)\phi_p(u) - \omega^{q-2}b(t)\phi_q(u)$. Then, we have

$$\eta'(t) \ge f(t, \eta(t)) \quad \text{for } T \le t \le T_1.$$

Since $\eta(T) = y(T)/L = 0$, it follows from Lemma 2.2 that

$$Lu(t;T) \le L\eta(t) = y(t) \le 0$$

for $T \le t \le T_1$. Hence, we have

$$x'(t) = \omega y(t) \ge \omega L u(t;T) \quad \text{for } T \le t \le T_1.$$

From this inequality and (3.3) it turns out that

$$x(T_1) \ge x(T) + \omega L\int_T^{T_1} u(t;T)dt > 1 + \omega L\int_T^\infty u(t;T)dt > \frac{1}{2} = x(T_1).$$

This is a contradiction.

We have thus proved the theorem. □

To examine the motion of positive orbits of (3.1), we transform system (3.1) into polar coordinates by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Then, we have

$$\begin{aligned} r' &= -\omega^{p-2}a(t)\phi_p(r)|\sin\theta|^p - \omega^{q-2}b(t)\phi_q(r)|\sin\theta|^q, \\ \theta' &= -\omega - \omega^{p-2}a(t)r^{p-2}\phi_p(\sin\theta)\cos\theta - \omega^{q-2}b(t)r^{q-2}\phi_q(\sin\theta)\cos\theta. \end{aligned} \tag{3.5}$$

Consider the positive orbit $\Gamma^+(t_0,\mathbf{x}_0)$ starting from a point $\mathbf{x}_0 \in Q_1 \cup Q_3$ at a time $t_0 \geq 0$. Let $(r(t),\theta(t))$ be the solution of (3.5) corresponding to $\Gamma^+(t_0,\mathbf{x}_0)$. The positive orbit $\Gamma^+(t_0,\mathbf{x}_0)$ moves clockwise around the origin as long as it is in $Q_1 \cup Q_3$. In fact,

$$r^2\theta' = -\omega\left(x^2+y^2\right) - \omega^{p-2}a(t)xy|y|^{p-2} - \omega^{q-2}b(t)xy|y|^{q-2} < 0$$

if $(x,y) \in Q_1 \cup Q_3$. Suppose that $\Gamma^+(t_0,\mathbf{x}_0)$ keeps staying in $Q_1 \cup Q_3$. Then,

$$\sin\theta(t)\cos\theta(t) \geq 0 \quad \text{for } t \geq t_0.$$

Hence, we obtain

$$\begin{aligned} \theta'(t) &= -\omega - \omega^{p-2}a(t)(r(t))^{p-2}\phi_p(\sin\theta(t))\cos\theta(t) \\ &\qquad - \omega^{q-2}b(t)(r(t))^{q-2}\phi_q(\sin\theta(t))\cos\theta(t) \\ &= -\omega - a(t)(\omega r(t)|\sin\theta(t)|)^{p-2}\sin\theta(t)\cos\theta(t) \\ &\qquad - b(t)(\omega r(t)|\sin\theta(t)|)^{q-2}\sin\theta(t)\cos\theta(t) \\ &\leq -\omega \end{aligned}$$

for $t \geq t_0$, and therefore,

$$\theta(t) \leq \theta(t_0) - \omega(t-t_0),$$

which tends to $-\infty$ as $t \to \infty$. This is a contradiction. Thus, we have the following result.

Lemma 3.3 *No positive orbit of* (3.1) *can continue staying in* $Q_1 \cup Q_3$ *ultimately*.

Judging from Lemma 3.3, system (3.1) has three types of positive orbits. Positive orbits of the first type keep rotating around the origin. Those of the second type remain in Q_4 (resp., Q_2) and approach the origin through Q_4 (resp., Q_2). Those of the third type stay in Q_4 (resp., Q_2) and tend to an interior point in Q_4 (resp., Q_2).

We are now ready to prove 'if'-part of Theorem 1.1.

Theorem 3.4 *Assume* (1.6) *and* (3.2). *Then the zero solution of* (3.1) *is globally attractive*.

Proof. Let $(x(t),y(t))$ be any solution of (3.1) with the initial time $t_0 \geq 0$. Define

$$v(t) = V(x(t),y(t)) \quad \text{for } t \geq t_0.$$

To prove the theorem, we have only to show that

$$v(t) \to 0 \quad \text{as } t \to \infty.$$

Since

$$v'(t) = \dot{V}_{(3.1)}(t,x(t),y(t)) = -\omega^{p-2}a(t)|y(t)|^p - \omega^{q-2}b(t)|y(t)|^q \leq 0$$

for $t \ge t_0$, the function $v(t)$ has the limit $v_0 \ge 0$. If $v_0 = 0$, then the proof is complete. We will show that the case of $v_0 > 0$ does not occur provided (1.6) and (3.2) hold.

Suppose that $v_0 > 0$. Consider the closed curve given by $V(x,y) = v_0$. This closed curve is the circumference of a circle whose center is at the origin and whose radius is $\sqrt{2v_0}$. Hence, this circle crosses with the x-axis only at two points $(\sqrt{2v_0},0)$ and $(-\sqrt{2v_0},0)$. Let $\mathbf{x}_0 = (x(t_0),y(t_0))$ and let $\Gamma^+(t_0,\mathbf{x}_0)$ be the positive orbit of (3.1), which corresponds to the solution $(x(t),y(t))$.

As already mentioned, all positive orbits of (3.1) are classified into three types. Hereafter, we will complete the proof in three steps as follows: (i) $\Gamma^+(t_0,\mathbf{x}_0)$ does not belong to the first type; (ii) $\Gamma^+(t_0,\mathbf{x}_0)$ does not belong to the second type; (iii) $\Gamma^+(t_0,\mathbf{x}_0)$ does not belong to the third type. This contradiction is caused from the assumption that $v_0 > 0$.

Step (i): Suppose that $\Gamma^+(t_0,\mathbf{x}_0)$ belongs to the first type; namely, it keeps rotating around the origin. Let ε be so small that

$$0 < \varepsilon < \frac{\pi - \omega\gamma_0}{2}, \tag{3.6}$$

where γ_0 is the number given in assumption (1.6). Consider the straight lines $y = (\tan\varepsilon)x$ and $y = (\tan(\pi-\varepsilon))x$. Naturally, $\Gamma^+(t_0,\mathbf{x}_0)$ crosses the two lines and the y-axis infinitely many times. Let $(r(t),\theta(t))$ be the solution of (3.5) corresponding to $\Gamma^+(t_0,\mathbf{x}_0)$. Then, we can find four divergent sequences $\{\tau_n\}$, $\{t_n\}$, $\{\sigma_n\}$ and $\{s_n\}$ with $t_0 \le \tau_n < t_n < \sigma_n < s_n$ such that $\theta(\tau_n) = 3\pi/2$, $\theta(t_n) = \pi - \varepsilon$, $\theta(\sigma_n) = \pi/2$ and $\theta(s_n) = \varepsilon$. The positive orbit $\Gamma^+(t_0,\mathbf{x}_0)$ moves clockwise around the origin when it passes through $(Q_1 \cup Q_3)$. However, $\Gamma^+(t_0,\mathbf{x}_0)$ does not always move clockwise when it is in $(Q_2 \cup Q_4)$, because $\theta'(t)$ may change the sign. Hence, $\Gamma^+(t_0,\mathbf{x}_0)$ might advance temporarily anti-clockwise. In such a case, we should select the supremum of all $t \in (\tau_n,\sigma_n)$ for which $\theta(t) \ge \pi-\varepsilon$ as the point t_n. Then, we have

$$\varepsilon < \theta(t) < \pi - \varepsilon \quad \text{for } t_n < t < s_n.$$

Since $v(t) \searrow v_0 > 0$ as $t \to \infty$, the positive orbit $\Gamma^+(t_0,\mathbf{x}_0)$ does not enter in the circle $\{(x,y)\colon x^2+y^2 \le 2v_0\}$. The circumference of the circle intersects with the half-line $\theta = \varepsilon$ at only one point. Let $h(\varepsilon)$ be the y-coordinate of the intersection. Then, it turns out that $y(t) = r(t)\sin\theta(t) > h > 0$ for $t_n \le t \le s_n$. Let $\mu = \min\{\omega^{p-2}h^p,\, \omega^{q-2}h^q\}$. Then, we obtain

$$\begin{aligned} v'(t) &= -\omega^{p-2}a(t)|y(t)|^p - \omega^{q-2}b(t)|y(t)|^q \\ &\le -\omega^{p-2}h^p a(t) - \omega^{q-2}h^q b(t) \le -\mu\big(a(t)+b(t)\big) \end{aligned} \tag{3.7}$$

for $t_n \le t \le s_n$. Needless to say, $v'(t)$ is nonpositive for $t \ge t_0$.

Suppose that there exists an $N \in \mathbb{N}$ such that $s_n - t_n \ge \gamma_0$ for $n \ge N$. Then, it follows from (3.7) that

$$v(s_n) - v(t_n) < -\mu\int_{t_n}^{s_n}\big(a(t)+b(t)\big)dt \le -\mu\int_{t_n}^{t_n+\gamma_0}\big(a(t)+b(t)\big)dt$$

for $n \ge N$. Since $v(t_{n+1}) - v(s_n) \le 0$ for $n \in \mathbb{N}$, we obtain

$$v(t_{n+1}) - v(t_n) < -\mu\int_{t_n}^{t_n+\gamma_0}\big(a(t)+b(t)\big)dt \quad \text{for } n \ge N,$$

and therefore,

$$v_0 - v(t_N) \le v(t_{n+1}) - v(t_N) < -\mu\sum_{i=N}^{n}\int_{t_i}^{t_i+\gamma_0}\big(a(t)+b(t)\big)dt.$$

However, from (1.6) it turns out that

$$\sum_{n=N}^{\infty}\int_{t_n}^{t_n+\gamma_0}\big(a(t)+b(t)\big)dt=\infty.$$

This is a contradiction. Thus, there exists a sequence $\{n_k\}$ with $n_k\in\mathbb{N}$ and $n_k\to\infty$ as $k\to\infty$ such that

$$s_{n_k}-t_{n_k}<\gamma_0. \tag{3.8}$$

Since $r'(t)=-\omega^{p-2}a(t)\phi_p(r(t))|\sin\theta(t)|^p-\omega^{q-2}b(t)\phi_q(r(t))|\sin\theta(t)|^q\le 0$ for $t\ge t_0$, we see that $r(t)\le r(t_0)$ for $t\ge t_0$. Hence,

$$\begin{aligned}\theta'(t)&\ge-\omega-a(t)(\omega r(t)|\sin\theta(t)|)^{p-2}|\sin\theta(t)||\cos\theta(t)|\\&\qquad-b(t)(\omega r(t)|\sin\theta(t)|)^{q-2}|\sin\theta(t)||\cos\theta(t)|\\&\ge-\omega-(\omega r(t_0))^{p-2}a(t)-(\omega r(t_0))^{q-2}b(t)\\&\ge-\omega-\nu\big(a(t)+b(t)\big)\end{aligned}$$

for $t\ge t_0$, where $\nu=\max\big\{(\omega r(t_0))^{p-2},\,(\omega r(t_0))^{q-2}\big\}$. From (3.8) it follows that

$$\begin{aligned}\varepsilon-(\pi-\varepsilon)=\theta(s_{n_k})-\theta(t_{n_k})&\\&\ge-\omega(s_{n_k}-t_{n_k})-\nu\int_{t_{n_k}}^{s_{n_k}}\big(a(t)+b(t)\big)dt\\&>-\omega\gamma_0-\nu\int_{t_{n_k}}^{s_{n_k}}\big(a(t)+b(t)\big)dt\end{aligned}$$

for each $k\in\mathbb{N}$; namely,

$$\nu\int_{t_{n_k}}^{s_{n_k}}\big(a(t)+b(t)\big)dt>\pi-\omega\gamma_0-2\varepsilon\quad\text{for }\ k\in\mathbb{N}.$$

Using this estimation and (3.7), we obtain

$$v(s_{n_k})-v(t_{n_k})<-\mu\int_{t_{n_k}}^{s_{n_k}}\big(a(t)+b(t)\big)dt<-\frac{\mu}{\nu}(\pi-\omega\gamma_0-2\varepsilon)$$

for $k\in\mathbb{N}$. Since $v(t_{n_{k+1}})-v(s_{n_k})\le 0$ for $k\in\mathbb{N}$, we see that

$$v(t_{n_{k+1}})-v(t_{n_k})<-\frac{\mu}{\nu}(\pi-\omega\gamma_0-2\varepsilon)\quad\text{for }\ k\in\mathbb{N}.$$

Taking (3.6) into consideration, we can conclude that

$$v_0-v(t_0)\le\sum_{k=1}^{\infty}\big(v(t_{n_{k+1}})-v(t_{n_k})\big)=-\infty,$$

which is a contradiction. Thus, $\Gamma^+(t_0,\mathbf{x}_0)$ does not belong to the first type.

Step (ii): Suppose that $\Gamma^+(t_0,\mathbf{x}_0)$ belongs to the second type; namely, it remains in Q_4 (resp., Q_2) and approaches the origin through Q_4 (resp., Q_2). Then, there exist a point $\mathbf{x}_1\in Q_4$ (resp., Q_2) and a time $T\ge t_0$ so that $\Gamma^+(t_0,\mathbf{x}_0)$ passes through $\mathbf{x}_1$ at T and remains in Q_4 (resp., Q_2) afterwards. We consider only the case in which $\Gamma^+(t_0,\mathbf{x}_0)$ remains in Q_4 ultimately, because the other case is carried out in the same way.

Since $(x(t),y(t)) \in Q_4$ for $t \ge T$, we see that $x'(t) = \omega y(t) < 0$ for $t \ge T$. Hence, there exists an $\alpha \ge 0$ such that $x(t) \to \alpha$ as $t \to \infty$, and therefore, it follows that

$$\frac{1}{2}y^2(t) \to v_0 - \frac{1}{2}\alpha^2 \ge 0 \quad \text{as } t \to \infty.$$

From the assumption of step (ii), the solution $(x(t),y(t))$ has to approach $(0,0)$ as $t \to \infty$. Hence, $\alpha = v_0 - \alpha^2/2 = 0$. This is impossible because $v_0 > 0$. Thus, $\Gamma^+(t_0,\mathbf{x}_0)$ does not belong to the second type.

If $v_0 > \alpha^2/2$, then we can choose a $T_1 \ge T$ so large that

$$y^2(t) > v_0 - \frac{1}{2}\alpha^2 > 0 \quad \text{for } t \ge T_1.$$

Hence, we have

$$\begin{aligned} v'(t) &= -\omega^{p-2}a(t)|y(t)|^p - \omega^{q-2}b(t)|y(t)|^q \\ &\le -\omega^{p-2}\big(v_0 - \alpha^2/2\big)^{p/2}a(t) - \omega^{q-2}\big(v_0 - \alpha^2/2\big)^{q/2}b(t) \\ &\le -\lambda\big(a(t)+b(t)\big) \end{aligned}$$

for $t \ge T_1$, where

$$\lambda = \min\left\{\omega^{p-2}\big(v_0-\alpha^2/2\big)^{p/2},\ \omega^{q-2}\big(v_0-\alpha^2/2\big)^{q/2}\right\}.$$

Integrating this inequality from T_1 to t, we obtain

$$v_0 - v(T_1) < v(t) - v(T_1) \le -\lambda\int_{T_1}^{t}\big(a(s)+b(s)\big)ds.$$

However, it follows from (1.6) that

$$\int_{T_1}^{\infty}\big(a(t)+b(t)\big)dt = \infty.$$

This is a contradiction. Thus, we see that $\alpha = \sqrt{2v_0}$. We therefore conclude that $\Gamma^+(t_0,\mathbf{x}_0)$ approaches the point $(\sqrt{2v_0},0)$ which is an intersection of the closed curve $V(x,y) = v_0$ and the x-axis.

Step (iii): Suppose that $\Gamma^+(t_0,\mathbf{x}_0)$ belongs to the third type; namely, it stays in Q_4 (resp., Q_2) and tends to an interior point in Q_4 (resp., Q_2). Then, as shown above, the interior point is $(\sqrt{2v_0},0)$ (resp., $(-\sqrt{2v_0},0)$). Let $\varepsilon_0 = \min\{1,\,\omega\sqrt{2v_0}\}$. Then, $\phi_p(\varepsilon_0) \le \varepsilon_0$ and $\phi_q(\varepsilon_0) \le \varepsilon_0$ because $p \ge 2$ and $q \ge 2$. Taking into account that

$$\sqrt{2v_0} < x(t) \le x(T) \quad \text{and} \quad y(t) < 0$$

for $t \ge T$, we can estimate that

$$\begin{aligned} \left(\frac{y(t)}{\varepsilon_0}\right)' &= -\frac{\omega x(t)}{\varepsilon_0} - \frac{\omega^{p-2}a(t)\phi_p(y(t))}{\varepsilon_0} - \frac{\omega^{q-2}b(t)\phi_q(y(t))}{\varepsilon_0} \\ &< -\frac{\omega\sqrt{2v_0}}{\varepsilon_0} - \frac{\omega^{p-2}a(t)\phi_p(y(t))}{\phi_p(\varepsilon_0)} - \frac{\omega^{q-2}b(t)\phi_q(y(t))}{\phi_q(\varepsilon_0)} \\ &\le -1 - \omega^{p-2}a(t)\phi_p\left(\frac{y(t)}{\varepsilon_0}\right) - \omega^{q-2}b(t)\phi_q\left(\frac{y(t)}{\varepsilon_0}\right) \end{aligned}$$

for $t \geq T$. Let $\eta(t) = y(t)/\varepsilon_0$ for $t \geq t_0$ and define

$$f(t,u) = -1 - \omega^{p-2}a(t)\phi_p(u) - \omega^{q-2}b(t)\phi_q(u).$$

Then, $\eta'(t) \leq f(t,\eta(t))$ for $t \geq T$. We compare $\eta(t)$ with the solution $u(t;T)$ of (1.7) satisfying $u(T;T) = 0$. Since $\eta(T) = y(T)/\varepsilon_0 < 0$, it follows from Lemma 2.3 that

$$\frac{y(t)}{\varepsilon_0} = \eta(t) \leq u(t;T) \leq 0$$

for $t \geq T$. Hence, we have

$$x'(t) \leq \omega\,\varepsilon_0\,u(t;T) \quad \text{for } t \geq T.$$

Integrate both sides of this inequality from T to t to obtain

$$-x(T) < x(t) - x(T) \leq \omega\,\varepsilon_0 \int_T^t u(s;T)ds.$$

By (3.2) and Lemma 2.1, however,

$$\int_T^t u(s;T)ds \to -\infty \quad \text{as } t \to \infty.$$

This is a contradiction. Thus, $\Gamma^+(t_0,\mathbf{x}_0)$ does not belong to the third type.

The proof of the theorem is now complete. □

We can obtain Theorem 1.1 by combining Theorems 3.2 and 3.4 with Proposition 3.1.

4 Sufficient conditions for global attractivity

In the special case in which $p = 2$ and $q = 2$, we can seek the solution $u(t)$ of (1.7) satisfying $u(0) = 0$ concretely. In fact,

$$u(t) = -\int_0^t \exp\left\{\int_t^s \big(a(u) + b(u)\big)du\right\} ds.$$

In general, however, it is difficult to confirm whether condition (3.2) is satisfied or not. For this reason, it is safe to say that Theorem 1.1 gives an implicit necessary and sufficient condition for global asymptotic stability. In this section, we give some explicit sufficient conditions for the equilibrium of (1.1) to be globally attractive.

Let p^* be the conjugate number of p; namely,

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

Since it is assumed throughout this paper that $p \geq 2$, the conjugate number p^* satisfies that $1 < p^* \leq 2$. Define

$$w = \phi_p(u) = \begin{cases} u^{p-1} & \text{if } u \geq 0, \\ -(-u)^{p-1} & \text{if } u < 0. \end{cases}$$

Then, w has the same sign as u and $u = \phi_{p^*}(w)$. In fact, since

$$u = \begin{cases} w^{1/(p-1)} & \text{if } w \geq 0, \\ -(-w)^{1/(p-1)} & \text{if } w < 0, \end{cases}$$

it follows from $(p-1)(p^*-1) = 1$ that $w^{1/(p-1)} = w^{p^*-1} = |w|^{p^*-2}w = \phi_{p^*}(w)$ if $w \geq 0$ and $-(-w)^{1/(p-1)} = -(-w)^{p^*-1} = (-w)^{p^*-2}w = |w|^{p^*-2}w = \phi_{p^*}(w)$ if $w < 0$. Hence, ϕ_{p^*} is the inverse function of ϕ_p. Similarly, ϕ_{q^*} is the inverse function of ϕ_q, where q^* is the number satisfying $1/q + 1/q^* = 1$.

Corollary 4.1 *Suppose that assumption* (1.6) *holds. Suppose also that there exist a $T > 0$ and differentiable functions $c(t)$ and $d(t)$ such that*

$$c(t) + d(t) > 0, \quad a(t) \leq c(t) \quad \textit{and} \quad b(t) \leq d(t) \tag{4.1}$$

for $t \geq T$. If, in addition, $c(t)$ and $d(t)$ are increasing for $t \geq T$ and

$$\int_T^\infty \frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\, dt = \infty, \tag{4.2}$$

then the equilibrium of (1.1) *is globally attractive.*

Proof. We divide the infinite interval $[0, \infty)$ as follows:

$$I = \{t \geq T : c(t) = 0\},$$

$$J = \{t \geq T : d(t) = 0\},$$

$$K = [T, \infty) \setminus (I \cup J).$$

From (4.1) it follows that the union of I and J is the empty set, and therefore,

$$c(t) = 0 \quad \text{and} \quad d(t) > 0 \quad \text{for } t \in I,$$

$$c(t) > 0 \quad \text{and} \quad d(t) = 0 \quad \text{for } t \in J,$$

$$c(t) > 0 \quad \text{and} \quad d(t) > 0 \quad \text{for } t \in K.$$

Define

$$g(t) = -\frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}$$

for $t \geq T$. Then, it is clear that $g(t) < 0$,

$$c(t)\phi_p(g(t)) \geq -1 \quad \text{and} \quad d(t)\phi_q(g(t)) \geq -1 \tag{4.3}$$

for $t \geq T$. We can rewrite $g(t)$ as

$$g(t) = \begin{cases} -\dfrac{1}{\phi_{q^*}(d(t))} & \text{if } t \in I, \\ -\dfrac{1}{\phi_{p^*}(c(t))} & \text{if } t \in J, \\ -\dfrac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))} & \text{if } t \in K. \end{cases}$$

Since $c(t)$ and $d(t)$ are differentiable and increasing for $t \ge T$, we see that

$$g'(t) = \frac{(q^*-1)d(t)^{q^*-2}d'(t)}{\phi_{q^*}(d(t))^2} \ge 0 \quad \text{for } t \in I,$$

$$g'(t) = \frac{(p^*-1)c(t)^{p^*-2}c'(t)}{\phi_{p^*}(c(t))^2} \ge 0 \quad \text{for } t \in J,$$

$$g'(t) = \frac{(p^*-1)c(t)^{p^*-2}c'(t) + (q^*-1)d(t)^{q^*-2}d'(t)}{\big(\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))\big)^2} \ge 0 \quad \text{for } t \in K.$$

In brief, $g(t)$ is negative, differentiable and increasing for $t \ge T$.

Consider the solution $u(t;T)$ of (1.7) satisfying $u(T;T) = 0$. Since $u'(T;T) = -1$, there exists a $\delta > 0$ such that

$$u(t;T) < 0 \quad \text{for } T < t < T + \delta.$$

Taking into account that $g(T) < 0 = u(T;T)$, we can choose a $t^* \in (T, T+\delta)$ so that

$$g(t^*) \le u(t^*;T) < 0.$$

Let us compare $u(t;T)$ with $\eta(t) = \lambda\, g(t)$, where

$$\lambda = \min\left\{ \frac{u(t^*;T)}{g(t^*)},\ \phi_{p^*}\left(\frac{1}{2\omega^{p-2}}\right),\ \phi_{q^*}\left(\frac{1}{2\omega^{q-2}}\right) \right\}.$$

Note that $0 < \omega^{p-2}\phi_p(\lambda) \le 1/2$ and $0 < \omega^{q-2}\phi_q(\lambda) \le 1/2$. Using (4.3), we obtain

$$\omega^{p-2}c(t)\phi_p(\eta(t)) = \omega^{p-2}\phi_p(\lambda)c(t)\phi_p(g(t)) \ge -\omega^{p-2}\phi_p(\lambda) \ge -\frac{1}{2}$$

and

$$\omega^{q-2}d(t)\phi_q(\eta(t)) = \omega^{q-2}\phi_q(\lambda)d(t)\phi_q(g(t)) \ge -\omega^{q-2}\phi_q(\lambda) \ge -\frac{1}{2}$$

for $t \ge T$. From these estimations it follows that

$$\omega^{p-2}c(t)\phi_p(\eta(t)) + \omega^{q-2}d(t)\phi_q(\eta(t)) \ge -1 \quad \text{for } t \ge T.$$

Hence, by (4.1) and the fact that $\eta(t) = \lambda\, g(t) < 0$ for $t \ge T$, we have

$$\begin{aligned}\eta'(t) = \lambda\, g'(t) \ge 0 &\ge -1 - \omega^{p-2}c(t)\phi_p(\eta(t)) - \omega^{q-2}d(t)\phi_q(\eta(t)) \\ &\ge -1 - \omega^{p-2}a(t)\phi_p(\eta(t)) - \omega^{q-2}b(t)\phi_q(\eta(t))\end{aligned}$$

for $t \ge T$. Let $f(t,u) = -1 - \omega^{p-2}a(t)\phi_p(u) - \omega^{q-2}b(t)\phi_q(u)$. Then, it is continuous on $[0,\infty) \times \mathbb{R}$. Since $p \ge 2$ and $q \ge 2$, we see that $f(t,u)$ satisfies locally a Lipschitz condition with respect to u. In addition, we see that

$$\eta'(t) \ge f(t, \eta(t)) \quad \text{for } t \ge T.$$

Moreover, it follows from the definitions of $\eta(t)$ and λ that

$$\eta(t^*) = \lambda\, g(t^*) \ge u(t^*;T).$$

Hence, by means of Lemma 2.2, we have

$$\eta(t) \ge u(t;T) \quad \text{for } t \ge t^*,$$

and therefore,

$$\int_{t^*}^{t} \eta(s)ds \geq \int_{t^*}^{t} u(s;T)ds \quad \text{for } t \geq t^*.$$

It follows from this inequality and (4.2) that

$$\begin{aligned}
\int_T^\infty u(t;T)dt &= \int_T^{t^*} u(t;T)dt + \int_{t^*}^\infty u(t;T)dt \\
&\leq \int_T^{t^*} u(t;T)dt + \int_{t^*}^\infty \eta(t)dt \\
&= \int_T^{t^*} u(t;T)dt - \int_T^{t^*} \eta(t)dt + \int_T^\infty \eta(t)dt \\
&= \int_T^{t^*} \big(u(t;T) - \eta(t)\big)dt + \lambda \int_T^\infty g(t)dt \\
&= \int_T^{t^*} \big(u(t;T) - \eta(t)\big)dt - \lambda \int_T^\infty \frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\, dt = -\infty.
\end{aligned}$$

Hence, by Theorem 3.4 and Lemma 2.1, we conclude that the equilibrium of (1.1) is globally attractive. □

In Corollary 4.1, the functions $c(t)$ and $d(t)$ are assumed to be increasing. However, the increase properties of $c(t)$ and $d(t)$ are not always necessary for the equilibrium of (1.1) to be globally attractive. As shown by the following result, another condition on $c(t)$ and $d(t)$ can substitute for the increase properties.

Corollary 4.2 *Suppose that assumption* (1.6) *holds. Suppose also that there exist a* $T > 0$ *and differentiable functions* $c(t)$ *and* $d(t)$ *satisfying conditions* (4.1) *and* (4.2). *If, in addition,*

$$\lim_{t\to\infty} \left(\frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))} \right)' = 0, \tag{4.4}$$

then the equilibrium of (1.1) *is globally attractive.*

Proof. As in the proof of Corollary 4.1, we define

$$g(t) = -\frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))} < 0$$

for $t \geq T$. Then, it satisfies (4.3). From (4.4) it follows that

$$g'(t) \to 0 \quad \text{as } t \to \infty.$$

Hence, we can choose a $T_1 \geq T$ so that

$$g'(t) > -\frac{1}{2\mu} \quad \text{for } t \geq T_1, \tag{4.5}$$

where

$$\mu = \min\left\{ \phi_{p^*}\!\left(\frac{1}{4\omega^{p-2}}\right),\ \phi_{q^*}\!\left(\frac{1}{4\omega^{q-2}}\right) \right\}.$$

From the definition of μ it turns out that

$$0 < \omega^{p-2}\phi_p(\mu) \le \frac{1}{4} \quad \text{and} \quad 0 < \omega^{q-2}\phi_q(\mu) \le \frac{1}{4}. \tag{4.6}$$

Consider the solution $u(t;T_1)$ of (1.7) satisfying $u(T_1;T_1) = 0$. Since $u'(T_1;T_1) = -1$, we can find a $\delta > 0$ such that

$$u(t;T_1) < 0 \quad \text{for } T_1 < t < T_1 + \delta.$$

Taking into account that $g(T_1) < 0 = u(T_1;T_1)/\mu$, we can choose a $t^* \in (T_1, T_1 + \delta)$ so that

$$g(t^*) \le \frac{u(t^*;T)}{\mu} < 0.$$

Let

$$\nu = \frac{u(t^*;T_1)}{g(t^*)} \quad \text{and} \quad \eta(t) = \nu g(t).$$

Then, we see that $\nu \le \mu$ and $\eta(t) < 0$ for $t \ge T$. Hence, using (4.3) and (4.6), we obtain

$$\begin{aligned} \omega^{p-2}c(t)\phi_p(\eta(t)) &= \omega^{p-2}\phi_p(\nu)c(t)\phi_p(g(t)) \\ &\ge \omega^{p-2}\phi_p(\mu)c(t)\phi_p(g(t)) \ge -\omega^{p-2}\phi_p(\mu) \ge -\frac{1}{4} \end{aligned}$$

and

$$\begin{aligned} \omega^{q-2}d(t)\phi_q(\eta(t)) &= \omega^{q-2}\phi_q(\nu)d(t)\phi_q(g(t)) \\ &\ge \omega^{q-2}\phi_q(\mu)d(t)\phi_q(g(t)) \ge -\omega^{q-2}\phi_q(\mu) \ge -\frac{1}{4} \end{aligned}$$

for $t \ge T$. From these estimations it follows that

$$\omega^{p-2}c(t)\phi_p(\eta(t)) + \omega^{q-2}d(t)\phi_q(\eta(t)) \ge -\frac{1}{2} \quad \text{for } t \ge T.$$

Hence, by (4.5), we have

$$\begin{aligned} \eta'(t) = \nu g'(t) > -\frac{\nu}{2\mu} \ge -\frac{1}{2} &\ge -1 - \omega^{p-2}c(t)\phi_p(\eta(t)) - \omega^{q-2}d(t)\phi_q(\eta(t)) \\ &\ge -1 - \omega^{p-2}a(t)\phi_p(\eta(t)) - \omega^{q-2}b(t)\phi_q(\eta(t)) = f(t,\eta(t)) \end{aligned}$$

for $t \ge T_1$, where $f(t,u) = -1 - \omega^{p-2}a(t)\phi_p(u) - \omega^{q-2}b(t)\phi_q(u)$. Note that $f(t,u)$ is continuous on $[0,\infty) \times \mathbb{R}$ and satisfies locally a Lipschitz condition with respect to u. It also follows from the definitions of $\eta(t)$ and ν that

$$\eta(t^*) = \nu g(t^*) = u(t^*;T_1).$$

Hence, from Lemma 2.2 it turns out that

$$\eta(t) \ge u(t;T_1) \quad \text{for } t \ge t^*.$$

Using this inequality and following the same process as in the proof of Corollary 4.1, we can estimate that

$$\int_{T_1}^{\infty} u(t;T_1)dt = -\infty.$$

Hence, from Theorem 3.4 and Lemma 2.1, we see that the equilibrium of (1.1) is globally attractive. □

In Corollary 4.2, we assumed that there exist a $T > 0$ and differentiable functions $c(t)$ and $d(t)$ such that

$$c(t) \geq 0, \quad d(t) \geq 0 \quad \text{and} \quad c(t) + d(t) > 0$$

for $t \geq T$. When we strengthen this assumption somewhat, we can change condition (4.4) into a simple one.

Proposition 4.3 *Supose that there exist numbers $T > 0$ and $e > 0$ and differentiable functions $c(t)$ and $d(t)$ such that*

$$c(t) > 0, \quad d(t) > 0 \quad \textit{and} \quad c(t) + d(t) \geq e \tag{4.7}$$

for $t \geq T$. If

$$\lim_{t\to\infty} \frac{c'(t)}{c(t)} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{d'(t)}{d(t)} = 0, \tag{4.8}$$

then condition (4.4) *is satisfied*.

Proof. From (4.7) it turns out that there exists an $\widetilde{e} > 0$ satisfying

$$\phi_{p^*}(c(t)) + \phi_{q^*}(d(t)) \geq \widetilde{e} \quad \text{for } t \geq T_1. \tag{4.9}$$

Actually, otherwise we can choose the different sequence $\{t_n\}$ such that

$$\phi_{p^*}(c(t_n)) + \phi_{q^*}(d(t_n)) \to 0 \quad \text{as } n \to \infty.$$

Since $\phi_{p^*}(c(t_n)) > 0$ and $\phi_{q^*}(d(t_n)) > 0$ for $n \in \mathbb{N}$, we see that $\phi_{p^*}(c(t_n))$ and $\phi_{q^*}(d(t_n))$ tend to 0 as $t \to \infty$, and therefore,

$$c(t_n) + d(t_n) \to 0 \quad \text{as } n \to \infty.$$

This contradicts (4.7). From (4.8) it follows that for any $\varepsilon > 0$ there exists a $T_1(\varepsilon) > 0$ such that

$$\left|\frac{c'(t)}{c(t)}\right| < \frac{\widetilde{e}\varepsilon}{2(p^*-1)} \quad \text{and} \quad \left|\frac{d'(t)}{d(t)}\right| < \frac{\widetilde{e}\varepsilon}{2(q^*-1)} \tag{4.10}$$

for $t \geq T_1$. Let $T_2 = \max\{T, T_1\}$. Then, by (4.7), (4.9) and (4.10), we have

$$\begin{aligned}
\left|\left(\frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\right)'\right| &= \left|\frac{(p^*-1)c(t)^{p^*-2}c'(t) + (q^*-1)d(t)^{q^*-2}d'(t)}{\left(\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))\right)^2}\right| \\
&\leq \frac{(p^*-1)\phi_{p^*}(c(t))}{\left(\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))\right)^2}\left|\frac{c'(t)}{c(t)}\right| \\
&\quad + \frac{(p^*-1)\phi_{p^*}(d(t))}{\left(\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))\right)^2}\left|\frac{d'(t)}{d(t)}\right| \\
&\leq \frac{(p^*-1)\phi_{p^*}(c(t))}{\widetilde{e}\left(\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))\right)}\left|\frac{c'(t)}{c(t)}\right| \\
&\quad + \frac{(p^*-1)\phi_{p^*}(d(t))}{\widetilde{e}\left(\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))\right)}\left|\frac{d'(t)}{d(t)}\right| < \varepsilon
\end{aligned}$$

for $t \ge T_2$; namely, condition (4.4). $\square$

By Corollary 4.2 and Proposition 4.3, we obtain the following result.

Corollary 4.4 *Suppose that assumptions* (1.6), (4.2) *and* (4.8) *hold. If there exist numbers* $T > 0$ *and* $e > 0$ *and differentiable functions* $c(t)$ *and* $d(t)$ *such that*

$$c(t) > 0, \quad d(t) > 0, \quad c(t) + d(t) \ge e, \quad a(t) \le c(t) \quad \textit{and} \quad b(t) \le d(t) \tag{4.11}$$

for $t \ge T$, *then the equilibrium of* (1.1) *is globally attractive.*

To compare Corollary 4.2 with Corollary 4.4, we give an example. Let

$$\begin{aligned} f(t) &= 1 + \frac{1}{4}\sin^3 t + \max\left\{0, \sqrt{t}\sin^3 t\right\} \\ &= \begin{cases} 1 + \dfrac{1}{4}\sin^3 t + \sqrt{t}\sin^3 t & \text{if } 2(n-1)\pi \le t < (2n-1)\pi, \\ 1 + \dfrac{1}{4}\sin^3 t & \text{if } (2n-1)\pi \le t < 2n\pi \end{cases} \end{aligned}$$

with $n \in \mathbb{N}$. Then, it is clear that $f(t) \ge 3/4$ for $t \ge 0$ and $f(t) \le 1$ for $t \in [(2n-1)\pi, 2n\pi]$. Since

$$f'(t) = \begin{cases} \dfrac{3}{4}\sin^2 t\cos t + \dfrac{\sin^3 t}{2\sqrt{t}} + 3\sqrt{t}\sin^2 t\cos t & \text{if } 2(n-1)\pi \le t < (2n-1)\pi, \\ \dfrac{3}{4}\sin^2 t\cos t & \text{if } (2n-1)\pi \le t < 2n\pi, \end{cases}$$

the function $f(t)$ is continuously differentiable for $t \ge 0$.

Example 4.1 Consider equation (1.1) with

$$a(t) = \phi_p(t^{f(t)}) \quad \text{and} \quad b(t) = \phi_q(t^{f(t)})$$

for any $p \ge 2$ and $q \ge 2$. Then the equilibrium is globally attractive.

It is obvious that assumption (1.6) is satisfied. Let $T = 1$. We define $c(t) = \phi_p(t^{f(t)})$ and $d(t) = \phi_q(t^{f(t)})$ for $t \ge T$. Then, condition (4.1) holds. We also see that

$$\begin{aligned} \int_T^\infty \frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\,dt &= \int_T^\pi \frac{1}{2\,t^{f(t)}}\,dt + \int_\pi^\infty \frac{1}{2\,t^{f(t)}}\,dt \\ &= \int_T^\pi \frac{1}{2\,t^{f(t)}}\,dt \\ &\quad + \frac{1}{2}\sum_{n=1}^\infty \left\{ \int_{(2n-1)\pi}^{2n\pi} \left(\frac{1}{t}\right)^{f(t)} dt + \int_{2n\pi}^{(2n+1)\pi} \left(\frac{1}{t}\right)^{f(t)} dt \right\} \\ &\ge \int_T^\pi \frac{1}{2\,t^{f(t)}}\,dt + \frac{1}{2}\sum_{n=1}^\infty \int_{(2n-1)\pi}^{2n\pi} \frac{1}{t}\,dt \\ &= \int_T^\pi \frac{1}{2\,t^{f(t)}}\,dt + \frac{1}{2}\sum_{n=1}^\infty \log\frac{2n}{2n-1} \\ &= \int_T^\pi \frac{1}{2\,t^{f(t)}}\,dt + \frac{1}{2}\lim_{n\to\infty} \log\frac{2}{1}\,\frac{4}{3}\,\frac{6}{5}\cdots\frac{2n}{2n-1}. \end{aligned}$$

For any $m \in \mathbb{N}$, let

$$I_m = \int_0^{\pi/2} \sin^m x \, dx.$$

Then, it is well-known that

$$I_{2n} = \frac{1}{2}\,\frac{4}{4}\,\frac{5}{6}\cdots\frac{2n-1}{2n}\,\frac{\pi}{2} \quad \text{and} \quad \lim_{n\to\infty}\sqrt{n}\, I_{2n} = \frac{\sqrt{\pi}}{2}.$$

Hence, I_{2n} tends 0 as $n \to \infty$, and therefore, we get

$$\int_T^\infty \frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\, dt = \infty;$$

namely condition (4.2). Moreover, we see that

$$\phi_{p^*}(c(t)) \ge t^{3/4} \quad \text{for } t \ge T. \tag{4.12}$$

Since $\log c(t) = (p-1) f(t) \log t$ for $t \ge T$, it turns out that

$$\begin{aligned}
\left|\frac{c'(t)}{c(t)}\right| = \left|(\log c(t))'\right| &= \left|(p-1) f'(t) \log t + (p-1)\frac{f(t)}{t}\right| \\
&\le (p-1)\left|f'(t)\right| |\log t| + (p-1)\frac{f(t)}{t} \\
&\le (p-1)\left(\frac{3}{4} + \frac{1}{2\sqrt{t}} + 3\sqrt{t}\right)\log t + (p-1)\left(\frac{5}{4} + \sqrt{t}\right)
\end{aligned}$$

for $t \ge T$. Using this estimation and (4.12), we obtain

$$\begin{aligned}
\left|\left(\frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\right)'\right| = \left|\left(\frac{1}{2\,\phi_{p^*}(c(t))}\right)'\right| &= \frac{p^*-1}{2\,\phi_{p^*}(c(t))}\left|\frac{c'(t)}{c(t)}\right| \\
&\le \frac{(p^*-1)(p-1)}{2}\,\frac{3/4 + 1/(2\sqrt{t}) + 3\sqrt{t}}{t^{3/4}}\log t \\
&\quad + \frac{(p^*-1)(p-1)}{2}\,\frac{5/4 + \sqrt{t}}{t^{3/4}} \\
&= \left(\frac{3}{8t^{3/4}} + \frac{1}{4t^{5/4}} + \frac{3}{2t^{1/4}}\right)\log t + \frac{5}{8t^{3/4}} + \frac{1}{2t^{1/4}},
\end{aligned}$$

which tends to 0 as $t \to \infty$. Thus, condition (4.4) is also satisfied. Hence, by Corollary 4.2, we conclude that the equilibrium is globally attractive (see Figure 2).

However, condition (4.8) is not satisfied when $c(t) = \phi_p(t^{f(t)})$ and $d(t) = \phi_q(t^{f(t)})$. In fact, let $t_n = (2n - 1/4)\pi$ for $n \in \mathbb{N}$. Then,

$$f'(t_n) = \frac{3\sqrt{2}}{16} \quad \text{and} \quad f(t_n) = 1 - \frac{\sqrt{2}}{16},$$

and therefore,

$$\frac{c'(t_n)}{c(t_n)} = (p-1)\frac{3\sqrt{2}}{16}\log\left((2n - \frac{1}{4})\pi\right) + (p-1)\frac{1-\sqrt{2}/16}{(2n-1/4)\pi},$$

which diverges to infinity as $n \to \infty$. It is difficult to find suitable upper functions $c(t)$ and $d(t)$ satisfying condition (4.8) in Example 4.1, because the damping coefficients $a(t)$ and $b(t)$ fluctuate intensely.

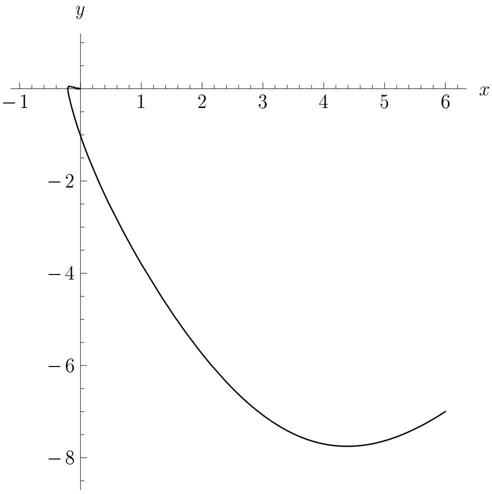

Fig. 2. The positive orbit of $x' = y,\ y' = -x - t^{f(t)}y - t^{2f(t)}|y|y$ starting from the point $(x_0, y_0) = (6, -7)$ at the initial time $t_0 = 0$.

Although condition (4.4) looks more complicated than condition (4.2), as shown in Example 4.1, it may be easy to check condition (4.4).

It is convenient to use the following result when the damping coefficients $a(t)$ and $b(t)$ are polynomial.

Corollary 4.5 *Suppose that assumption* (1.6) *holds. Suppose also that there exist numbers* γ, σ, ℓ_1, ℓ_2 *and* $T > 0$ *such that*

$$0 \le a(t) \le \ell_1 t^{\gamma} \quad \textit{and} \quad 0 \le b(t) \le \ell_2 t^{\sigma} \quad \textit{for } t \ge T. \tag{4.13}$$

If $0 \le \gamma \le p-1$ *and* $0 \le \sigma \le q-1$, *then the equilibrium of* (1.1) *is globally attractive.*

Proof. Let $c(t) = \ell_1 t^{\gamma}$ and $d(t) = \ell_2 t^{\sigma}$. Then, condition (4.1) is clearly satisfied. It is also clear that $c(t)$ and $d(t)$ are increasing for $t \ge T$ because $\gamma \ge 0$ and $\sigma \ge 0$. Let $T_1 = \max\{1, T\}$ and $\ell_3 = \max\{\ell_1^{p^*-1}, \ell_2^{q^*-1}\}$. Since $0 \le \gamma \le p-1$ and $0 \le \sigma \le q-1$, we see that

$$\frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))} = \frac{1}{c(t)^{p^*-1} + d(t)^{q^*-1}} = \frac{1}{\ell_1^{p^*-1} t^{\gamma/(p-1)} + \ell_2^{q^*-1} t^{\sigma/(q-1)}} \ge \frac{1}{2\,\ell_3 t}$$

for $t \ge T_1$. From this inequality, we can verify that

$$\int_T^{\infty} \frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\, dt \ge \int_T^{T_1} \frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\, dt + \frac{1}{2\,\ell_3} \int_{T_1}^{\infty} \frac{1}{t}\, dt = \infty;$$

namely, condition (4.2). Hence, by Corollary 4.1, we conclude that the equilibrium of (1.1) is globally attractive. □

In the proof of Corollary 4.5, we can also confirm condition (4.4). In fact, since

$$\left(\frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))} \right)' = -\frac{\frac{\gamma}{p-1}\ell_1^{p^*-1} t^{\gamma/(p-1)-1} + \frac{\sigma}{q-1}\ell_2^{q^*-1} t^{\sigma/(q-1)-1}}{\left(\ell_1^{p^*-1} t^{\gamma/(p-1)} + \ell_2^{q^*-1} t^{\sigma/(q-1)}\right)^2}$$

for $t \geq T_1$, it tends to 0 as $t \to \infty$ because $0 \leq \gamma \leq p-1$ and $0 \leq \sigma \leq q-1$. Hence, we can also lead Corollary 4.5 from Corollary 4.2.

Recall that we defined the function $s(t)$ in Section 1. Then, we can give the following example.

Example 4.2 Consider equation (1.1) with

$$a(t) = t\,s(t) \quad \text{and} \quad b(t) = t\,s(t+\pi/\omega)$$

for any $p \geq 2$ and $q \geq 2$. Then the equilibrium is globally attractive.

Since $a(t)+b(t) = t\,\sin^2(\omega t)$ for $t \geq 0$, assumption (1.6) is satisfied. Condition (4.13) is also satisfied with $\gamma = \sigma = \ell_1 = \ell_2 = 1$. Since $p \geq 2$ and $q \geq 2$, it is clear that

$$0 \leq \gamma \leq p-1 \quad \text{and} \quad 0 \leq \sigma \leq q-1.$$

Hence, from Corollary 4.5, we see that the equilibrium is globally attractive (see Figure 3). Note that neither condition (1.3) nor condition (1.5) is satisfied.

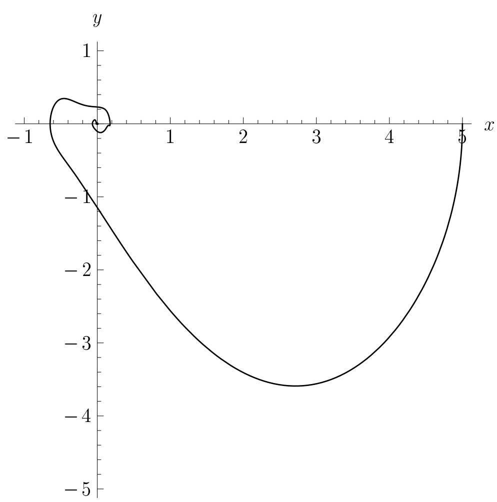

Fig. 3. The positive orbit of $x' = y,\ y' = -x - ts(t)y - ts(t+\pi)|y|y$ starting from the point $(x_0, y_0) = (5, 0)$ at the initial time $t_0 = 0$.

5 Necessary conditions for attractivity

In this section, we give some explicit necessary conditions for the equilibrium of (1.1) to be attractive. We judge that the equilibrium of (1.1) is not attractive by using lower functions instead of the damping coefficients $a(t)$ and $b(t)$.

Corollary 5.1 *Suppose that there exist a $T > 0$ and differentiable functions $c(t)$ and $d(t)$ satisfying condition* (4.4) *and*

$$c(t)+d(t) > 0, \quad 0 \leq c(t) \leq a(t) \quad \textit{and} \quad 0 \leq d(t) \leq b(t) \tag{5.1}$$

for $t \geq T$. If

$$\int_T^\infty \frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\,dt < \infty, \tag{5.2}$$

then the equilibrium of (1.1) *is not attractive*.

Proof. Let

$$g(t) = -\frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}$$

for $t \geq T$. Then, we can easily confirm that $g(t) < 0$ and

$$\phi_{p^*}(c(t)\phi_p(g(t))) + \phi_{q^*}(d(t)\phi_q(g(t))) = \phi_{p^*}(c(t))g(t) + \phi_{q^*}(d(t))g(t) = -1 \tag{5.3}$$

for $t \geq T$. From (5.3) it turns out that there exists a $\rho > 0$ such that

$$\omega^{p-2}c(t)\phi_p(g(t)) + \omega^{q-2}d(t)\phi_q(g(t)) \leq -\rho \quad \text{for } t \geq T. \tag{5.4}$$

Actually, otherwise we can find a divergent sequence $\{t_n\}$ such that

$$\omega^{p-2}c(t_n)\phi_p(g(t_n)) + \omega^{q-2}d(t_n)\phi_q(g(t_n)) \to 0 \quad \text{as } n \to \infty.$$

Since $c(t) \geq 0$, $d(t) \geq 0$ and $g(t) < 0$ for $t \geq T$, we see that

$$\omega^{p-2}c(t)\phi_p(g(t)) + \omega^{q-2}d(t)\phi_q(g(t)) \leq \omega^{p-2}c(t)\phi_p(g(t)) \leq 0$$

and

$$\omega^{p-2}c(t)\phi_p(g(t)) + \omega^{q-2}d(t)\phi_q(g(t)) \leq \omega^{q-2}d(t)\phi_q(g(t)) \leq 0.$$

Hence, both $\omega^{p-2}c(t_n)\phi_p(g(t_n))$ and $\omega^{q-2}d(t_n)\phi_q(g(t_n))$ tend to 0 as $n \to \infty$. This contradicts (5.3). Let $\chi = \max\{1, 2/\rho\}$. From (4.4) it follows that

$$g'(t) \to 0 \quad \text{as } t \to \infty.$$

Hence, there exists a $T_1 \geq T$ such that

$$g'(t) < \frac{1}{\chi} \quad \text{for } t \geq T_1. \tag{5.5}$$

Let $\eta(t) = \chi g(t)$. Since $p \geq 2$, $q \geq 2$ and $\chi \geq 1$, we see that $\chi \leq \phi_p(\chi)$ and $\chi \leq \phi_q(\chi)$. Hence, by (5.4), we have

$$\begin{aligned}
&\omega^{p-2}c(t)\phi_p(\eta(t)) + \omega^{q-2}d(t)\phi_q(\eta(t)) \\
&\qquad = \omega^{p-2}c(t)\phi_p(\chi)\phi_p(g(t)) + \omega^{q-2}d(t)\phi_q(\chi)\phi_q(g(t)) \\
&\qquad \leq \left(\omega^{p-2}c(t)\phi_p(g(t)) + \omega^{q-2}d(t)\phi_q(g(t))\right)\chi \leq -\rho\chi
\end{aligned}$$

for $t \geq T$. Since $\chi \geq 2/\rho$, it follows that

$$\omega^{p-2}c(t)\phi_p(\eta(t)) + \omega^{q-2}d(t)\phi_q(\eta(t)) \leq -2 \quad \text{for } t \geq T.$$

Using this inequality and (5.5), we obtain

$$\begin{aligned}
\eta'(t) = \chi g'(t) < 1 &\leq -1 - \omega^{p-2}c(t)\phi_p(\eta(t)) - \omega^{q-2}d(t)\phi_q(\eta(t)) \\
&\leq -1 - \omega^{p-2}a(t)\phi_p(\eta(t)) - \omega^{q-2}b(t)\phi_q(\eta(t))
\end{aligned}$$

for $t \geq T_1$. Let $f(t,u) = -1 - \omega^{p-2}a(t)\phi_p(u) - \omega^{q-2}b(t)\phi_q(u)$. Then, $\eta'(t) < f(t,\eta(t))$ for $t \geq T_1$. Consider the solution $u(t;T_1)$ of (1.7) satisfying $u(T_1;T_1) = 0$. Then, it is clear that

$$\eta(T_1) = \chi g(T_1) < 0 = u(T_1;T_1).$$

Hence, Lemma 2.3 shows that

$$\eta(t) \leq u(t;T_1) \quad \text{for } t \geq T_1.$$

Integrating both sides of this inequality from T_1 to t, we obtain

$$\int_{T_1}^{t} \eta(s)ds \leq \int_{T_1}^{t} u(s;T_1)ds \quad \text{for } t \geq T_1,$$

which yields that

$$\begin{aligned}
\int_{T_1}^{\infty} u(t;T_1)dt \geq \int_{T_1}^{\infty} \eta(t)dt = \chi \int_{T_1}^{\infty} g(t)dt \\
= -\chi \int_{T}^{T_1} g(t)dt + \chi \int_{T}^{\infty} g(t)dt \\
= -\chi \int_{T}^{T_1} g(t)dt - \chi \int_{T}^{\infty} \frac{1}{\phi_{p^*}(c(t)) + \phi_{q^*}(d(t))}\, dt.
\end{aligned}$$

From (5.2) it follows that

$$\int_{T_1}^{\infty} u(t;T_1)dt > -\infty.$$

Hence, by Theorem 3.2 and Lemma 2.1, we conclude that the equilibrium of (1.1) is not attractive. □

Let us give an example which is applicable to Corollary 5.1. For this purpose, we define

$$g(t) = 2\left(1 + \sqrt{2}\sin t\right) \quad \text{and} \quad h(t) = 2\left(1 - \sqrt{2}\sin t\right)$$

for $t \geq 0$. Note that

$$\begin{aligned}
&2(1-\sqrt{2}) \leq h(t) \leq 2 \leq g(t) \leq 2(1+\sqrt{2}) \quad \text{if } 2(n-1)\pi \leq t < (2n-1)\pi, \\
&2(1-\sqrt{2}) \leq g(t) \leq 2 \leq h(t) \leq 2(1+\sqrt{2}) \quad \text{if } (2n-1)\pi \leq t < 2n\pi
\end{aligned} \tag{5.6}$$

with $n \in \mathbb{N}$.

Example 5.1 Consider equation (1.1) with

$$a(t) = \phi_p(t^{g(t)}) \quad \text{and} \quad b(t) = \phi_q(t^{h(t)})$$

for any $p \geq 2$ and $q \geq 2$. Then the equilibrium is not attractive.

Let $T = \pi/4$. Define $c(t) = \phi_p(t^{g(t)})$ and $d(t) = \phi_q(t^{h(t)})$ for $t \geq T$. Then, $c(t)$ and $d(t)$ are differentiable for $t \geq T$. It is obvious that condition (5.1) is satisfied. Condition (4.4) is also satisfied. In fact, using the estimation (5.6), we obtain

$$\frac{\left(t^{g(t)} + t^{h(t)}\right)^2}{t^{2\max\{g(t),h(t)\}}} = \left(1 + t^{\min\{g(t),h(t)\} - \max\{g(t),h(t)\}}\right)^2 \to 1 \quad \text{as } t \to \infty$$

and

$$\frac{\left|\left(2\sqrt{2}\cos t\log t+2\left(1+\sqrt{2}\sin t\right)/t\right)t^{g(t)}+\left(-2\sqrt{2}\cos t\log t+2\left(1-\sqrt{2}\sin t\right)/t\right)t^{h(t)}\right|}{t^{2\max\{g(t),h(t)\}}}$$
$$\leq \frac{4\sqrt{2}\log t}{t^{\max\{g(t),h(t)\}}}+\frac{2\left(2+\sqrt{2}\right)}{t^{\max\{g(t),h(t)\}+1}}\to 0 \quad \text{as } t\to\infty.$$

Hence, we have

$$\left|\left(\frac{1}{\phi_{p^*}(c(t))+\phi_{q^*}(d(t))}\right)'\right| = \left|\left(\frac{1}{t^{g(t)}+t^{h(t)}}\right)'\right| = \frac{\left|(g(t)\log t)'t^{g(t)}+(h(t)\log t)'t^{h(t)}\right|}{\left(t^{g(t)}+t^{h(t)}\right)^2}$$
$$= \frac{\left|\left(2\sqrt{2}\cos t\log t+2\left(1+\sqrt{2}\sin t\right)/t\right)t^{g(t)}+\left(-2\sqrt{2}\cos t\log t+2\left(1-\sqrt{2}\sin t\right)/t\right)t^{h(t)}\right|}{\left(t^{g(t)}+t^{h(t)}\right)^2}$$

which tends to 0 as $t\to\infty$. Noticing that

$$\max\{g(t),h(t)\}\geq 2 \quad \text{for } t\geq 0,$$

we obtain

$$\int_T^\infty \frac{1}{\phi_{p^*}(c(t))+\phi_{q^*}(d(t))}\,dt = \int_T^\infty \frac{1}{t^{g(t)}+t^{h(t)}}\,dt$$
$$\leq \int_T^\infty \frac{1}{t^{\max\{g(t),h(t)\}}}\,dt \leq \int_T^\infty \frac{1}{t^2}\,dt < \infty;$$

namely, condition (5.2). Thus, from Corollary 5.1, we see that the equilibrium is not attractive (see Figure 4).

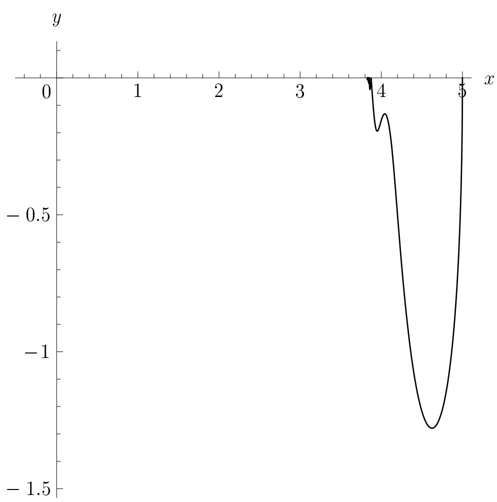

Fig. 4. The positive orbit of $x' = y$, $y' = -x - t^{g(t)}y - t^{2h(t)}|y|y$ starting from the point $(x_0, y_0) = (5, 0)$ at the initial time $t_0 = \pi/4$.

As shown in Example 5.1, the equilibrium of

$$x'' + \phi_p(t^{g(t)})\phi_p(x') + \phi_q(t^{h(t)})\phi_q(x') + \omega^2 x = 0 \tag{5.7}$$

is not attractive for any $\omega > 0$. To compare with (5.7), we consider the superlinear oscillators with a single damping term:

$$x'' + \phi_p(t^{g(t)})\phi_p(x') + \omega^2 x = 0, \tag{5.8}$$

$$x'' + \phi_q(t^{h(t)})\phi_q(x') + \omega^2 x = 0. \tag{5.9}$$

Since

$$g(t) = 2\Big(1+\sqrt{2}\sin t\Big) \le 0 \quad \text{if } \ (2n-3/4)\pi \le t \le (2n-1/4)\pi$$

with $n \in \mathbb{N}$, we obtain

$$\int_{\pi/4}^{\infty} \frac{1}{\phi_{p^*}(c(t))}\,dt = \int_{\pi/4}^{\infty} \frac{1}{t^{g(t)}}\,dt > \sum_{n=1}^{\infty} \int_{(2n-3/4)\pi}^{(2n-1/4)\pi} \frac{1}{t^{g(t)}}\,dt > \sum_{n=1}^{\infty} \frac{\pi}{2} = \infty.$$

Similarly, we can estimate that

$$\int_{\pi/4}^{\infty} \frac{1}{\phi_{q^*}(d(t))}\,dt = \int_{\pi/4}^{\infty} \frac{1}{t^{h(t)}}\,dt = \infty.$$

Hence, Corollary 5.1 is inapplicable to equations (5.8) and (5.9). To tell the truth, both equilibria of (5.8) and (5.9) with $\omega = 1$ are globally asymptotically stable (see Figures 5 and 6).

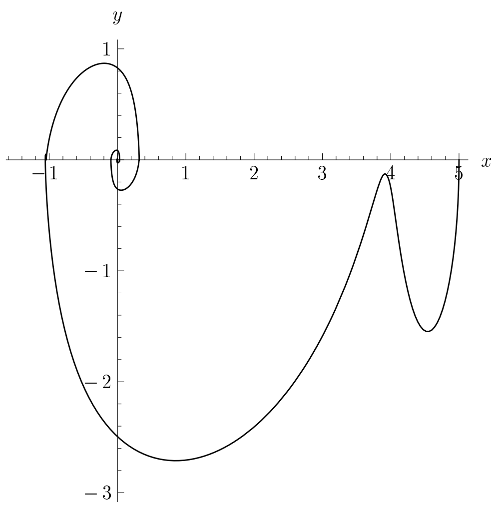

Fig. 5. The positive orbit of $x' = y,\ y' = -x - t^{g(t)}y$ starting from the point $(x_0, y_0) = (5, 0)$ at the initial time $t_0 = \pi/4$.

The following result is a direct consequence of Corollary 5.1.

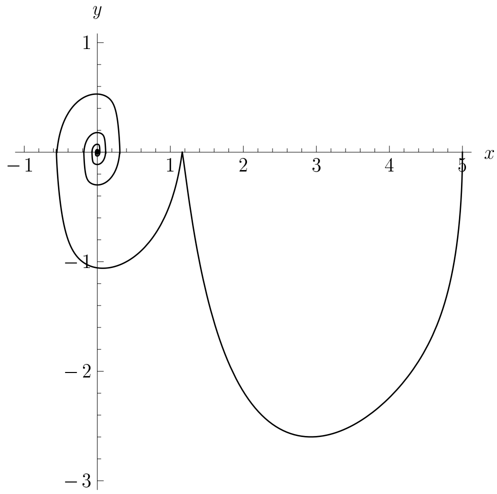

Fig. 6. The positive orbit of $x' = y$, $y' = -x - t^{2h(t)}|y|y$ starting from the point $(x_0, y_0) = (5, 0)$ at the initial time $t_0 = \pi/4$.

Corollary 5.2 *Suppose that there exist numbers γ, σ, ℓ_1, ℓ_2 and $T > 0$ such that*

$$\ell_1 t^\gamma \le a(t) \quad \textit{and} \quad \ell_2 t^\sigma \le b(t) \quad \textit{for } t \ge T.$$

If $p-1<\gamma$ or $q-1<\sigma$, then the equilibrium of (1.1) *is not attractive*.

Proof. We may assume without loss of generality that $T > 1$. Let $c(t) = \ell_1 t^\gamma$ and $d(t) = \ell_2 t^\sigma$. Then, it is clear that $c(t)+d(t) \ge \ell_1 T^\gamma + \ell_2 T^\sigma > 0$, $c(t) \le a(t)$ and $d(t) \le b(t)$ for $t \ge T$; that is, condition (5.1) is satisfied. Condition (4.4) is also satisfied. In fact,

$$\left|\left(\frac{1}{\phi_{p^*}(c(t))+\phi_{q^*}(d(t))}\right)'\right| = \left|\left(\frac{1}{\ell_1^{p^*-1}t^{\gamma(p^*-1)}+\ell_2^{q^*-1}t^{\sigma(q^*-1)}}\right)'\right|$$
$$= \frac{\gamma(p^*-1)\ell_1^{p^*-1}t^{\gamma(p^*-1)-1}+\sigma(q^*-1)\ell_2^{q^*-1}t^{\sigma(q^*-1)-1}}{\left(\ell_1^{p^*-1}t^{\gamma(p^*-1)}+\ell_2^{q^*-1}t^{\sigma(q^*-1)}\right)^2},$$

which tends to 0 as $t \to \infty$. If $p-1<\gamma$, then we can choose an $\varepsilon_1 > 0$ so that

$$1+\varepsilon_1 \le \frac{\gamma}{p-1} = \gamma(p^*-1).$$

Hence, we obtain

$$\int_T^\infty \frac{1}{\phi_{p^*}(c(t))+\phi_{q^*}(d(t))}\,dt = \int_T^\infty \frac{1}{\ell_1^{p^*-1}t^{\gamma(p^*-1)}+\ell_2^{q^*-1}t^{\sigma(q^*-1)}}\,dt$$
$$\le \frac{1}{\ell_1^{p^*-1}}\int_T^\infty \left(\frac{1}{t}\right)^{1+\varepsilon_1} dt < \infty.$$

Similarly, if $q-1<\sigma$, then we obtain

$$\int_T^\infty \frac{1}{\phi_{p^*}(c(t))+\phi_{q^*}(d(t))}\,dt \le \frac{1}{\ell_2^{q^*-1}}\int_T^\infty \left(\frac{1}{t}\right)^{1+\varepsilon_2} dt < \infty$$

for some $\varepsilon_2>0$. Thus, condition (5.2) holds. Consequently, by means of Corollary 5.1, we can conclude that the equilibrium of (1.1) is not attractive. □

6 Final comment

Combining Corollary 4.5 and Corollary 5.2, we obtain the following result.

Corollary 6.1 *Suppose that there exist numbers* γ, σ, ℓ_1, ℓ_2 *and* $T>0$ *such that*

$$a(t)=\ell_1 t^\gamma \quad \textit{and} \quad b(t)=\ell_2 t^\sigma \quad \textit{for}\ \ t\ge T.$$

Then, the equilibrium of (1.1) *is attractive if and only if*

$$0\le\gamma\le p-1 \quad \text{and} \quad 0\le\sigma\le q-1. \tag{6.1}$$

Proof. When $a(t)=\ell_1 t^\gamma$ and $b(t)=\ell_2 t^\sigma$ for $t\ge T$, assumption (1.6) is clearly satisfied for any $\gamma_0>0$. Hence, by virtue of Corollaries 4.5 and 5.2, we can conclude that (6.1) is a necessary and sufficient condition for the equilibrium of (1.1) to be globally asymptotically stable. □

Although William Froude paid his attention to two kinds of damping terms, three or more damping terms may act on a certain phenomenon. We can easily find models with the damping force has the cubic polynomial expression with respect to the angular velocity (for example. see [4, 8, 9, 12, 14, 18, 35, 42]). Himeno [25] proposed even the damping force with a power series expansion of the angular velocity (see also [16, 26, 43]). Such a model is described as follows:

$$x''+\sum_{i=1}^n a_i(t)\phi_{p_i}(x')+\omega^2 x=0, \tag{6.2}$$

where the damping coefficients $a_1(t)$, $a_2(t)$, $\dots$, $a_n(t)$ are continuous and nonnegative for $t\ge 0$, the restoring coefficient ω is positive, and the parameters $p_1\ge 2$, $p_2\ge 2$, $\dots$, $p_n\ge 2$. Our method in the present paper can be used even for this model. The following results are obtained (the proof is left to readers).

Theorem 6.2 *Suppose that there exists a* γ_0 *with* $0<\gamma_0<\pi/\omega$ *such that*

$$\liminf_{t\to\infty}\int_t^{t+\gamma_0}\sum_{i=1}^n a_i(s)\,ds>0. \tag{6.3}$$

Then the equilibrium of (6.2) *is globally asymptotically stable if and only if*

$$\int_0^\infty u(t)dt=-\infty,$$

where $u(t)$ *is the solution of*

$$u'+\sum_{i=1}^n \omega^{p_i-2}a_i(t)\phi_{p_i}(u)+1=0$$

satisfying $u(0) = 0$.

Let p_i^* be the conjugate number of p_i; namely,

$$\frac{1}{p_i} + \frac{1}{p_i^*} = 1,$$

where i is any integer satisfying $1 \le i \le n$. Then, we obtain explicit sufficient conditions and necessary conditions for the equilibrium of (6.2) to be globally attractive.

Corollary 6.3 *Suppose that assumption* (6.3) *holds. Suppose also that there exist a* $T > 0$ *and differentiable functions* $b_1(t)$, $b_2(t)$, $\ldots$, $b_n(t)$ *such that*

$$\sum_{i=1}^{n} b_i(t) > 0 \quad \text{and} \quad a_i(t) \le b_i(t) \quad (1 \le i \le n) \tag{6.4}$$

for $t \ge T$. *If, in addition,* $b_1(t)$, $b_2(t)$, $\ldots$, $b_n(t)$ *are increasing for* $t \ge T$ *and*

$$\int_T^\infty \frac{1}{\sum_{i=1}^{n} \phi_{p_i^*}(b_i(t))}\, dt = \infty, \tag{6.5}$$

then the equilibrium of (6.2) *is globally attractive.*

Corollary 6.4 *Suppose that assumption* (6.3) *holds. Suppose also that there exist a* $T > 0$ *and differentiable functions* $b_1(t)$, $b_2(t)$, $\ldots$, $b_n(t)$ *satisfying conditions* (6.4) *and* (6.5). *If, in addition,*

$$\lim_{t\to\infty} \left(\frac{1}{\sum_{i=1}^{n} \phi_{p_i^*}(b_i(t))} \right)' = 0, \tag{6.6}$$

then the equilibrium of (6.2) *is globally attractive.*

Corollary 6.5 *Suppose that there exist a* $T > 0$ *and differentiable functions* $b_1(t)$, $b_2(t)$, $\ldots$, $b_n(t)$ *satisfying condition* (6.6) *and*

$$\sum_{i=1}^{n} b_i(t) > 0 \quad \text{and} \quad 0 \le b_i(t) \le a_i(t) \quad (1 \le i \le n) \tag{6.7}$$

for $t \ge T$. *If*

$$\int_T^\infty \frac{1}{\sum_{i=1}^{n} \phi_{p_i^*}(b_i(t))}\, dt < \infty,$$

then the equilibrium of (6.2) *is not attractive.*

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