Growth conditions for uniform asymptotic stability of damped oscillators

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Abstract

The present paper is devoted to an investigation on the uniform asymptotic stability for the linear differential equation with a damping term,

$$x'' + h(t)x' + \omega^2 x = 0$$

and its generalization

$$(\phi_p(x'))' + h(t)\phi_p(x') + \omega^p \phi_p(x) = 0,$$

where $\omega > 0$ and $\phi_p(z) = |z|^{p-2}z$ with $p > 1$. Sufficient conditions are obtained for the equilibrium $(x, x') = (0, 0)$ to be uniformly asymptotically stable under the assumption that the damping coefficient $h(t)$ is integrally positive. The obtained condition for the damped linear differential equation is given by the form of a certain uniform growth condition on $h(t)$. Another representation which is equivalent to this uniform growth condition is also given. Our results assert that the equilibrium can be uniformly asymptotically stable even if $h(t)$ is unbounded. An example is attached to show this fact. In addition, easy-to-use conditions are given to guarantee that the uniform growth condition is satisfied. Moreover, a sufficient condition expressed by an infinite series is presented. The relation between the representation of an infinite series and the uniform growth condition is also clarified. Finally, our results are extended to be able to apply to the above-mentioned nonlinear differential equation.

Key words: Growth condition; Uniform asymptotic stability; Damped linear oscillator, Damped half-linear oscillator, Integrally positive
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1. Introduction

The equations considered in this paper are

$$x'' + h(t)x' + \omega^2 x = 0, \tag{1.1}$$

and its generalization, where the prime denotes d/dt, the coefficient $h(t)$ is continuous and nonnegative for $t \geq 0$, and the number ω is positive. Equation (1.1) is called the *damped linear oscillator*. The only equilibrium of (1.1) is the origin $(x, x') = (0, 0)$. Our objective is to establish sufficient conditions on the damping coefficient $h(t)$ for the equilibrium to be uniformly asymptotically stable.

As is well known, the concept of uniform asymptotic stability is greatly different from the concept of (merely) asymptotic stability; that is,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$$

for every solution $x(t)$ of (1.1). To verify that the equilibrium is asymptotically stable, we have only to show that each solution of (1.1) and its derivative tend to zero as time t increases. It is not necessary to care about the asymptotic speed of each pair $(x(t), x'(t))$. On the other hand, we have to confirm that each pair $(x(t), x'(t))$ approaches the origin at the speed of the same level in order to prove that the equilibrium is uniformly asymptotically stable (see Section 2 about the strict definitions of asymptotic stability and uniform asymptotic stability). Here is the difficulty of the research of uniform asymptotic stability.

Uniform asymptotic stability concerning nonlinear differential equations has been investigated by many authors in relation to Lyapunov's direct method. Here, to explain an importance of the research of uniform asymptotic stability briefly, we consider the linear time-varying system given by

$$\mathbf{x}' = A(t)\mathbf{x} \tag{1.2}$$

with $A(t)$ being an $n \times n$ continuous matrix. System (1.2) has the zero solution, which is equivalent to the equilibrium of the corresponding n-order linear differential equation. Let $\|\mathbf{x}\|$ be the Euclidean norm of a vector $\mathbf{x}$. We denote the solution of (1.2) passing through a point $\mathbf{x}_0 \in \mathbb{R}^2$ at a time $t_0 \geq 0$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. It is well-known that the zero solution of (1.2) is uniformly asymptotically stable if and only if it is exponentially asymptotically stable (or exponentially stable); namely, there exists a $\kappa > 0$ and, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta(\varepsilon)$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon \exp(-\kappa(t - t_0))$ for all $t \geq t_0$. Thanks to this characteristic of solutions of (1.2), we can obtain converse theorems on uniform asymptotic stability that guarantee the existence of a good Lyapunov function. The good Lyapunov function $V(t, \mathbf{x})\colon [0, \infty) \times \mathbb{R}^n$ satisfies

(i) $a(\|\mathbf{x}\|) \leq V(t, \mathbf{x}) \leq b(\|\mathbf{x}\|)$,

(ii) $\dot{V}_{(1.2)}(t, \mathbf{x}) \leq -c(\|\mathbf{x}\|)$ or $\dot{V}_{(1.2)}(t, \mathbf{x}) \leq -d\, V(t, \mathbf{x})$,

(iii) $|V(t, \mathbf{x}_1) - V(t, \mathbf{x}_2)| \leq f(t)\|\mathbf{x}_1 - \mathbf{x}_2\|$, where $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are continuous increasing and positive definite functions, d is a positive constant and $f(t)$ is a positive suitable function. In general, however, (merely) asymptotic stability of the zero solution of a time-varying system does not ensure the existence of any good Lyapunov function (see [24, Example 2]). This point is a big difference with uniform asymptotic stability and asymptotic stability. A function satisfying the above properties (i) and (ii) is often called a *strict* Lyapunov function in control theory (for example, see [3, pp. 101–103]). We can solve perturbation problems by utilizing such a good Lyapunov function. For example, if the zero solution of (1.2) is uniformly asymptotically stable and if $\mathbf{g}(t, \mathbf{x})$ and $\lambda(t)$ satisfy that $\|\mathbf{g}(t, \mathbf{x})\| \leq \lambda(t)\|\mathbf{x}\|$ for $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^2$, where

$$\int_0^\infty \lambda(s) ds < \infty,$$

then the zero solution of the quasi-linear system

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{g}(t, \mathbf{x})$$

is also uniformly asymptotically stable. However, even if the zero solution of (1.2) is (merely) asymptotically stable, the zero solution of the quasi-linear system is not always asymptotically stable. Perron [28] has clarified this fact by considerably complicated analysis. For example, the reader is referred to the classical books [5, pp. 42–43], [6, pp. 169–170], [8, p. 71]. It is also known that the zero solution of (1.2) is uniformly asymptotically stable if and only if it is totally stable which is closely related to robustness. For the definition of total stability, see [3, pp. 45] and [35, pp. 118–119].

Let $X(t)$ be a fundamental matrix for a general n-dimensional linear system satisfying $X(0) = E$, the unit matrix E. We define the norm of $X(t)$ to be

$$\|X(t)\| = \sup_{\|\mathbf{x}\|=1} \|X(t)\mathbf{x}\|.$$

It is well-known that the zero solution of (1.2) is asymptotically stable if and only if

$$\|X(t)\| \to 0 \quad \text{as} \ \ t \to \infty$$

and that the zero solution of (1.2) is uniformly asymptotically stable if and only if there exist positive constants K and κ such that

$$\|X(t)X^{-1}(s)\| \leq K \exp(-\kappa(t - s)) \quad \text{for} \ \ 0 \leq s \leq t < \infty$$

(for the proof, see the books [8, p. 54] or [17, p. 84]). If we can get a concrete expression of a fundamental matrix, we may be able to judge whether the zero solution is uniformly asymptotically stable (or asymptotically stable) by using the above-mentioned criterion. Unfortunately, however, we are almost unable to find a fundamental matrix. Therefore, these criteria are not useful for practical use though they are sharp.

Before going into the main theme, let us look at the results concerning the asymptotic stability. Many papers have been written to find out sufficient conditions and necessary conditions for the zero solution (or the equilibrium) to be asymptotically stable without using the information on a fundamental matrix (for example, see [2, 4, 14, 19, 20, 21, 22, 23, 31]). Historical progress of this research is briefly summarized in Sugie [32, Section 1]. Here, we will describe some results of not having written to the summary.

For this purpose, we need to introduce two families of functions as follows. The damping coefficient $h(t)$ is said to be *integrally positive* if

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n + \lambda < \sigma_n \le \tau_{n+1}$ for some $\lambda > 0$. For example, the function $\sin^2 t$ is integrally positive. It is known that $h(t)$ is integrally positive if and only if

$$\liminf_{t\to\infty} \int_t^{t+d} h(s)ds > 0$$

for every $d > 0$. Let $\{I_n\}$ be a sequence of disjoint intervals and suppose the width of I_n is larger than a positive number for all $n \in \mathbb{N}$. As can be seen from the definition above, if $h(t)$ is integrally positive, then the sum from n equals 1 to ∞ of the integral of $h(t)$ on I_n diverges to infinity even if intervals I_n and I_{n+1} gradually part as n increases. Hence, the integral positivity is considerably strong restriction than

$$\lim_{t\to\infty} H(t) = \infty,$$

where

$$H(t) = \int_0^t h(s)ds.$$

Note that any function converging to zero is not integrally positive. Let us define a family of functions which is wider than the family of integrally positive functions. The damping coefficient $h(t)$ is said to be *weakly integrally positive* if

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n + \lambda < \sigma_n \le \tau_{n+1} \le \sigma_n + \Lambda$ for some $\lambda > 0$ and $\Lambda > 0$. Here, in order to loosen the restriction of integrally positive functions, we eliminate the case that intervals $[\tau_n, \sigma_n]$ and $[\tau_{n+1}, \sigma_{n+1}]$ gradually part as n increases. The typical example of the weakly integrally positive function is $1/(1+t)$ or $\sin^2 t/(1+t)$ (for the proof, see [33, Proposition 2.1]).

Hatvani [18] has considered the two-dimensional linear system with time-varying coefficients,

$$\mathbf{x}' = B(t)\mathbf{x},$$

where

$$B(t) = \begin{pmatrix} -r(t) & q(t) \\ -q(t) & -p(t) \end{pmatrix}$$

and presented some sufficient conditions for the zero solution to be asymptotically stable. If we apply his results to the damped linear oscillator (1.1), the following results are obtained.

Theorem A. *If* $h(t)$ *is integrally positive and*

$$\int_0^\infty \frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\,dt = \infty, \tag{1.3}$$

then the equilibrium of (1.1) *is asymptotically stable*.

Theorem B. *If* $h(t)$ *is weakly integrally positive and*

$$\lim_{t\to\infty}\int_\sigma^{t+\sigma} \frac{\int_\sigma^s e^{H(\tau)}d\tau}{e^{H(s)}}\,ds = \infty \quad \textit{uniformly with respect to } \; \sigma \geq 0, \tag{1.4}$$

then the equilibrium of (1.1) *is asymptotically stable*.

Needless to say, condition (1.4) is a restriction that is stronger than condition (1.3). On the other hand, if $h(t)$ is integrally positive, then it is weakly integrally positive. Thus, Theorems A and B have a good balance. Under the assumption that $h(t)$ is integrally positive, condition (1.3) is also necessary for the equilibrium of (1.1) to be asymptotically stable. To be precise, the following theorem holds (for the proof, see [32, Theorem 3.5]).

Theorem C. *Suppose that one of the following assumptions*

(i) $h(t)$ *is integrally positive,*

(ii) $h(t)$ *is uniformly continuous for* $t \geq 0$ *and weakly integrally positive*

holds. Then the equilibrium of (1.1) *is asymptotically stable if and only if condition* (1.3) *holds*.

It is known that the equilibrium of (1.1) does not become asymptotically stable when the damping coefficient $h(t)$ decreases rapidly or when it increases rapidly. Both the integral positivity and the weak integral positivity prohibit too fast decline of the damping coefficient $h(t)$. Conversely, conditions (1.3) and (1.4) prohibit too fast growth of the damping coefficient $h(t)$.

For example, consider the damped linear oscillators:

$$x'' + \frac{1}{1+t}x' + x = 0, \tag{1.5}$$

$$x'' + \frac{1}{(1+t)^2}x' + x = 0, \tag{1.6}$$

$$x'' + t\,x' + x = 0, \tag{1.7}$$

and

$$x'' + t^2 x' + x = 0. \tag{1.8}$$

Since the function $1/(1+t)$ is weakly integrally positive and condition (1.4) is satisfied with $h(t) = 1/(1+t)$, it follows from Theorem B that the equilibrium of (1.5) is asymptotically stable. It is easily check that $1/(1+t)^2$ is not weakly integrally positive. To tell the truth, the equilibrium of (1.6) is not asymptotically stable because $\lim_{t\to\infty} H(t) < \infty$. Since the function t is integrally positive and condition (1.3) is satisfied with $h(t) = t$, it follows from Theorem A that the equilibrium of (1.7) is asymptotically stable. Condition (1.3) is not satisfied when $h(t) = t^2$. Hence, from Theorem C we see that the equilibrium of (1.8) is not asymptotically stable (also refer to [20, Theorem 1.1]).

Restrictions on the damping coefficient $h(t)$ for the equilibrium of (1.1) to be uniformly asymptotically stable must be more stringent than restrictions for the equilibrium of (1.1) to be asymptotically stable. Then, are the equilibria of (1.5) and (1.7) uniformly asymptotically stable? The answer is no. Onitsuka [26] discussed the problem of non-uniform asymptotic stability for damped linear oscillators and showed that the equilibrium of the Bessel differential equation

$$x'' + \frac{1}{1+t}x' + \frac{(1+t)^2 - r^2}{1+t}x = 0, \qquad r \in \mathbb{R}$$

is asymptotically stable, but it is not uniformly asymptotically stable. Applying his result to equation (1.5), we can judge that the equilibrium is not uniformly asymptotically stable. On the other hand, fortunately, we can find a fundamental matrix for a system equivalent to equation (1.7). The fundamental matrix is given by

$$X(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix},$$

where

$$x_{11}(t) = e^{-\frac{t^2}{2}}, \qquad x_{12}(t) = e^{-\frac{t^2}{2}}\int_0^t e^{\frac{\tau^2}{2}}d\tau,$$
$$x_{21}(t) = -\,t\,e^{-\frac{t^2}{2}}, \qquad x_{22}(t) = 1 -\,t\,e^{-\frac{t^2}{2}}\int_0^t e^{\frac{\tau^2}{2}}d\tau.$$

Note that $X(0) = E$. Since

$$\lim_{t\to\infty} e^{-\frac{t^2}{2}}\int_0^t e^{\frac{\tau^2}{2}}d\tau = 0 \quad \text{and} \quad \lim_{t\to\infty} t\,e^{-\frac{t^2}{2}}\int_0^t e^{\frac{\tau^2}{2}}d\tau = 1,$$

it follows that $\|X(t)\| \to 0$ as $t \to \infty$. Hence, the equilibrium of (1.7) is asymptotically stable. However, it is not uniformly asymptotically stable. In fact, since $\det X(t) = e^{-\frac{t^2}{2}}$ and

$$X^{-1}(t) = e^{\frac{t^2}{2}}\begin{pmatrix} x_{22}(t) & -\,x_{12}(t) \\ -\,x_{21}(t) & x_{11}(t) \end{pmatrix},$$

we see that

$$X(t)X^{-1}(s) = \begin{pmatrix} y_{11}(t,s) & y_{12}(t,s) \\ y_{21}(t,s) & y_{22}(t,s) \end{pmatrix},$$

where

$$\begin{aligned}
y_{11}(t,s) &= e^{\frac{s^2}{2}}\big(x_{11}(t)x_{22}(s) - x_{12}(t)x_{21}(s)\big) = e^{\frac{s^2-t^2}{2}} + s\,e^{-\frac{t^2}{2}}\int_s^t e^{\frac{\tau^2}{2}}\,d\tau, \\
y_{12}(t,s) &= e^{\frac{s^2}{2}}\big(-x_{11}(t)x_{12}(s) + x_{12}(t)x_{11}(s)\big) = e^{-\frac{t^2}{2}}\int_s^t e^{\frac{\tau^2}{2}}\,d\tau, \\
y_{21}(t,s) &= e^{\frac{s^2}{2}}\big(x_{21}(t)x_{22}(s) - x_{22}(t)x_{21}(s)\big) = s - t\,e^{\frac{s^2-t^2}{2}} - ts\,e^{-\frac{t^2}{2}}\int_s^t e^{\frac{\tau^2}{2}}\,d\tau, \\
y_{22}(t,s) &= e^{\frac{s^2}{2}}\big(-x_{21}(t)x_{12}(s) + x_{22}(t)x_{11}(s)\big) = 1 - \,t\,e^{-\frac{t^2}{2}}\int_s^t e^{\frac{\tau^2}{2}}\,d\tau.
\end{aligned}$$

Let us pay attention to the $(1,2)$-element of $X(t)X^{-1}(s)$. For any $K > 0$ and $\kappa > 0$, there exists a $t^* = t^*(K,\kappa) > 0$ such that

$$e^{\kappa t}\int_0^t e^{\frac{\tau^2}{2}}\,d\tau > K\,e^{\frac{t^2}{2}} \quad \text{for } \ t \geq t^*.$$

Hence, we have

$$\|X(t)X^{-1}(0)\| \geq |y_{12}(t,0)| = e^{-\frac{t^2}{2}}\int_0^t e^{\frac{\tau^2}{2}}\,d\tau > K\,e^{-\kappa t}$$

for $t \geq t^*$. This means that the equilibrium of (1.7) is not uniformly asymptotically stable.

Of course, the equilibrium of (1.1) is uniformly asymptotically stable if $h(t)$ is a positive constant. As shown in the above-mentioned concrete examples, the equilibrium of (1.1) is not uniformly asymptotically stable no longer even when the damping coefficient $h(t)$ decays relatively slowly like $1/(1+t)$ or even when it grow relatively slowly like t. Here, simple questions arise. What kind of growth conditions on the damping coefficient $h(t)$ guarantee that the equilibrium of (1.1) is uniformly asymptotically stable? Does the equilibrium of (1.1) become uniformly asymptotically stable even if the damping coefficient $h(t)$ is unbounded? The following result is an answer to the first question above.

Theorem 1.1. *Suppose that $h(t)$ is integrally positive. If condition* (1.4) *is satisfied, then the equilibrium of* (1.1) *is uniformly asymptotically stable.*

Although there is an intimate relation between the statements of Theorems A, B and 1.1, the proof of Theorem 1.1 greatly differs from those of Theorems A and B, because the strictness is required more and more in order to demonstrate the uniform asymptotic stability.

The composition of this paper is as follows. In Section 2, we give the proof of Theorem 1.1. To this end, we consider the system which is equivalent to the damped linear oscillator (1.1) and analyze the asymptotic behavior of solutions of this linear system in details. The analytical procedure is divided into three parts. The last part is advanced in four steps. Section 3 is provided in order to reply to the second question mentioned above. To begin with, we introduce a characteristic equation for the damped linear oscillator (1.1) and give an equivalent condition to the uniform growth condition (1.4). Next, we present sufficient conditions for the equilibrium of (1.1) to be uniformly asymptotically stable, which is easy to check than condition (1.4). Finally, by using the presented result, we give an example that the equilibrium of (1.1) is uniformly asymptotically stable even if the damping coefficient $h(t)$ is unbounded. To facilitate an understanding of the example, we attach two graphs concerning $h(t)$ and a phase portrait of orbits. In Section 4, we give an infinite series representation which guarantees that the equilibrium of (1.1) is uniformly asymptotically stable. Also, we clarify the relation between the representation of an infinite series and sufficient conditions for uniform asymptotic stability given in Sections 2 and 3. In the final section, we extend Theorem 1.1 to be able to apply to a kind of non-linear equation called half-linear. As understood from the name, this equation is a natural generalization of the damped linear oscillator (1.1). Because the parameters are intertwined in a complex, the details of proof may be not easily imaginable though the proof policy is the same as that of Theorem 1.1. We give only a sketch of the proof.

2. Proof of Theorem 1.1

Let $y = x'/\omega$. Then, the damped linear oscillator (1.1) becomes the linear system

$$\begin{aligned} x' &= \omega y \\ y' &= -\omega x - h(t)y. \end{aligned} \tag{2.1}$$

Here, let us give some definitions about the zero solution of (2.1) which is equivalent to the equilibrium of (1.1). The zero solution of (2.1) is said to be *uniformly stable* if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta(\varepsilon)$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The zero solution is said to be *uniformly attractive* if there exists a $\delta_0 > 0$ and, for every $\eta > 0$, there exists a $T(\eta) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta_0$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T(\eta)$. The zero solution is *uniformly asymptotically stable* if it is uniformly stable and is uniformly attractive. For example, we can refer to the books [3, 7, 16, 17, 25, 29, 30, 35] for those definitions.

In the definition of uniform asymptotic stability, the numbers δ and T must not be dependent on t_0. Therefore, we have to find positive constants δ and T that are independent of t_0 in the proof of Theorem 1.1. This is an important point.

Before giving the full proof of Theorem 1.1, it is helpful to mention its broad outline. The proof is divided into three parts. First, we will show that

(a) the zero solution of (2.1) is uniformly stable.

To be precise, we verify that if $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta(\varepsilon) = \varepsilon$, then $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. This part is comparatively easy. We next show that the zero solution of (2.1) is uniformly attractive. For this purpose,

(b) we determine $T(\eta) > 0$ for an arbitrary $\eta > 0$,

and we prove that

(c) $\|\mathbf{x}(t^*; t_0, \mathbf{x}_0)\| < \delta(\eta)$ for some $t^* \in [t_0, t_0 + T]$.

Let $\mathbf{x}^* = \mathbf{x}(t^*; t_0, \mathbf{x}_0)$. Then, from the conclusion of parts (a) and (c), we see that

$$\|\mathbf{x}(t; t^*, \mathbf{x}^*)\| < \eta \quad \text{for } t \geq t^*,$$

where $\mathbf{x}(t; t^*, \mathbf{x}^*)$ is any solution of (2.1) passing through the point $\mathbf{x}^*$ at the time t^*. Part (c) is the core of the proof of Theorem 1.1. We prove part (c) by way of contradiction.

Proof of Theorem 1.1. Part (a): For any $\varepsilon > 0$ sufficiently small, we choose

$$\delta(\varepsilon) = \varepsilon.$$

Let $t_0 \geq 0$ and $\mathbf{x}_0 \in \mathbb{R}^2$ be given. We will show that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for $t \geq t_0$. For convenience, we write $(x(t), y(t)) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ and define

$$v(t) = \frac{x^2(t)}{2} + \frac{y^2(t)}{2} = \frac{1}{2}\|\mathbf{x}(t; t_0, \mathbf{x}_0)\|^2.$$

Then, $v'(t) = x(t)x'(t) + y(t)y'(t) = -\,h(t)y^2(t) \leq 0$ for $t \geq t_0$. Since $v(t)$ is decreasing for $t \geq t_0$, we see that

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| = \sqrt{2v(t)} \leq \sqrt{2v(t_0)} = \|\mathbf{x}_0\| < \delta = \varepsilon$$

for $t \geq t_0$; namely, the zero solution of (2.1) is uniformly stable. This completes the proof of part (a).

Part (b): Let $\delta_0 = 1$. For every $\eta > 0$, we decide a number $T(\eta)$ as follows so that $\|\mathbf{x}_0\| < 1$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T$. As was mentioned in Section 1, since $h(t)$ is integrally positive, the inequality

$$\liminf_{t\to\infty} \int_t^{t+d} h(s)ds > 0$$

holds for every $d > 0$. Hence, we can find an $\ell > 0$ and a $\hat{t} > 0$ such that

$$\int_t^{t+1} h(s)ds \geq \ell \quad \text{for } t \geq \hat{t}.$$

We define

$$\mu = \min\left\{\frac{3\eta^2}{4}, \frac{\omega^2\eta^2}{16}\right\} \quad \text{and} \quad \tau_1 = \hat{t} + \left[\frac{1}{\ell\mu}\right] + 1,$$

where $[c]$ means the greatest integer that is less than or equal to a real number c. Since ω, ℓ and $\hat{t}$ are fixed positive constants, the numbers μ and τ_1 depend only on η. From condition (1.4) it turns out that there exists a τ_2 depending only on η such that

$$\int_\sigma^{t+\sigma} \frac{\int_\sigma^s e^{H(\tau)}d\tau}{e^{H(s)}} ds \geq \frac{4}{\omega^2\eta} \quad \text{for } t \geq \tau_2 - 1. \tag{2.2}$$

We may assume without loss of generality that $\tau_2 > 1$. Let

$$\nu = \liminf_{t\to\infty} \frac{\mu}{4}\int_t^{t+\mu/(4\omega)} h(s)ds.$$

Note that ν is a positive number and it also depends only on η. From the definition of ν, we can choose a positive number $\tau_3 > 0$ depending only on η such that

$$\int_t^{t+\mu/(4\omega)} h(s)ds \geq \frac{2\nu}{\mu} \quad \text{for} \;\; t \geq \tau_3. \tag{2.3}$$

Using numbers τ_1, τ_2, τ_3 and ν, we define

$$T = T(\eta) = \tau_3 + \left(\left[\frac{1}{\nu}\right] + 1\right)(\tau_1 + \tau_2).$$

Part (c): Consider a solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of (2.1) with $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta_0 = 1$. The purpose of part (c) is to prove that there exists a $t^* \in [t_0, t_0 + T]$ such that

$$\|\mathbf{x}(t^*; t_0, \mathbf{x}_0)\| < \eta \tag{2.4}$$

for every $\eta > 0$. By way of contradiction, we suppose that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \geq \eta$ for $t_0 \leq t \leq t_0 + T$. Then, we have

$$\frac{\eta^2}{2} \leq \frac{1}{2}\|\mathbf{x}(t; t_0, \mathbf{x}_0)\|^2 = v(t) \leq v(t_0) = \frac{1}{2}\|\mathbf{x}_0\|^2 < \frac{1}{2} \tag{2.5}$$

for $t_0 \leq t \leq t_0 + T$. Let us pay attention to the behavior of $y^2(t)$, which is the second component of $\mathbf{x}(t; t_0, \mathbf{x}_0)$.

Step 1: For any interval $[\alpha, \beta] \subset [t_0, t_0 + T]$, if $y^2(t) \geq \mu/2$ for $\alpha \leq t \leq \beta$, then the time width $\beta - \alpha$ is less than τ_1, where μ and τ_1 are numbers given in part (b). To show this, we suppose that there exists an interval $[\alpha_1, \beta_1] \subset [t_0, t_0 + T]$ with $\beta_1 - \alpha_1 \geq \tau_1$ such that $y^2(t) \geq \mu/2$ for $\alpha_1 \leq t \leq \beta_1$. Since $v'(t) = -\,h(t)y^2(t) \leq 0$ for $t \geq t_0$, by (2.5) we have

$$\frac{\mu}{2}\int_{\alpha_1}^{\beta_1} h(t)dt \leq \int_{\alpha_1}^{\beta_1} h(t)y^2(t)dt = -\int_{\alpha_1}^{\beta_1} v'(t)dt = v(\alpha_1) - v(\beta_1) < \frac{1}{2}. \tag{2.6}$$

On the other hand, since $\tau_1 = \hat{t} + [1/(\ell\mu)] + 1$, we see that

$$\begin{aligned}\int_{\alpha_1}^{\beta_1} h(t)dt &\geq \int_{\alpha_1}^{\alpha_1+\tau_1} h(t)dt = \int_{\alpha_1}^{\alpha_1+\hat{t}} h(t)dt + \int_{\alpha_1+\hat{t}}^{\alpha_1+\tau_1} h(t)dt \\ &\geq \int_{\alpha_1+\hat{t}}^{\alpha_1+\hat{t}+[1/(\ell\mu)]+1} h(t)dt = \sum_{i=0}^{[1/(\ell\mu)]} \int_{\alpha_1+\hat{t}+i}^{\alpha_1+\hat{t}+i+1} h(t)dt \\ &\geq \left(\left[\frac{1}{\ell\mu}\right] + 1\right)\ell \geq \frac{1}{\mu}.\end{aligned}$$

This contradicts (2.6). Thus, it turns out that the beginning sentence of this step is true.
Step 2: For any interval $[\alpha, \beta] \subset [t_0, t_0+T]$, if $y^2(t) \le \mu$ for $\alpha \le t \le \beta$, then the time width $\beta - \alpha$ is less than τ_2, where μ and τ_2 are numbers given in part (b). To show this, we suppose that there exists an interval $[\alpha_2, \beta_2] \subset [t_0, t_0+T]$ with $\beta_2 - \alpha_2 \ge \tau_2$ such that $y^2(t) \le \mu$ for $\alpha_2 \le t \le \beta_2$. Since $\mu \le 3\eta^2/4$, by (2.5) we have

$$|x(t)| = \sqrt{2v(t) - y^2(t)} \ge \sqrt{\eta^2 - \mu} \ge \frac{\eta}{2}$$

for $\alpha_2 \le t \le \beta_2$. Hence, there are two cases to consider: $x(t) \ge \eta/2$ for $\alpha_2 \le t \le \beta_2$; $x(t) \le -\eta/2$ for $\alpha_2 \le t \le \beta_2$. We consider only the former, because the latter is carried out in the same way. In the former, we have

$$\big(e^{H(t)}y(t)\big)' = \big(y'(t) + h(t)y(t)\big)e^{H(t)} = -\,\omega x(t)e^{H(t)} \le -\,\frac{\omega\eta}{2}e^{H(t)}$$

for $\alpha_2 \le t \le \beta_2$. Hence, we obtain

$$\begin{aligned} y(t) &\le y(\alpha_2)e^{H(\alpha_2)}e^{-H(t)} - \frac{\omega\eta}{2}e^{-H(t)}\int_{\alpha_2}^{t} e^{H(s)}ds \\ &\le \sqrt{\mu}\, e^{H(\alpha_2)}e^{-H(t)} - \frac{\omega\eta}{2}e^{-H(t)}\int_{\alpha_2}^{t} e^{H(s)}ds \end{aligned}$$

for $\alpha_2 \le t \le \beta_2$. Since $H(t)$ is increasing for $t \ge 0$, we see that

$$e^{H(\alpha_2)} \le e^{H(\alpha_2)}(t - \alpha_2) = e^{H(\alpha_2)}\int_{\alpha_2}^{t} ds \le \int_{\alpha_2}^{t} e^{H(s)}ds$$

for $t \ge \alpha_2 + 1$. Recall that $\tau_2 > 1$. Taking $\beta_2 \ge \alpha_2 + \tau_2 > \alpha_2 + 1$ and $\mu \le \omega^2\eta^2/16$ into account, we obtain

$$\begin{aligned} x'(t) = \omega y(t) &\le \omega\sqrt{\mu}\, e^{H(\alpha_2)}e^{-H(t)} - \frac{\omega^2\eta}{2}e^{-H(t)}\int_{\alpha_2}^{t} e^{H(s)}ds \\ &\le \omega\Big(\sqrt{\mu} - \frac{\omega\eta}{2}\Big)e^{-H(t)}\int_{\alpha_2}^{t} e^{H(s)}ds \le -\,\frac{\omega^2\eta}{4}e^{-H(t)}\int_{\alpha_2}^{t} e^{H(s)}ds \end{aligned}$$

for $\alpha_2 + 1 \le t \le \beta_2$. From (2.5) it follows that $x(\alpha_2 + 1) < 1$. In addition, $x(\alpha_2 + \tau_2) \ge \eta/2 > 0$. Hence, we have

$$-\,1 < -\,x(\alpha_2+1) < x(\alpha_2+\tau_2) - x(\alpha_2+1) = \int_{\alpha_2+1}^{\alpha_2+\tau_2} x'(t)dt,$$

and therefore,

$$\begin{aligned} -\,1 &< -\,\frac{\omega^2\eta}{4}\int_{\alpha_2+1}^{\alpha_2+\tau_2} e^{-H(t)}\int_{\alpha_2}^{t} e^{H(s)}ds\,dt \\ &= -\,\frac{\omega^2\eta}{4}\int_{\alpha_2+1}^{\alpha_2+\tau_2} e^{-H(t)}\bigg\{\int_{\alpha_2}^{\alpha_2+1} e^{H(s)}ds + \int_{\alpha_2+1}^{t} e^{H(s)}ds\bigg\}dt \\ &\le -\,\frac{\omega^2\eta}{4}\int_{\alpha_2+1}^{\alpha_2+\tau_2} e^{-H(t)}\int_{\alpha_2+1}^{t} e^{H(s)}ds\,dt. \end{aligned}$$

However, from (2.2) with $\sigma = \alpha_2 + 1$ and $t = \tau_2 - 1$, we see that

$$-1 < -\frac{\omega^2\eta}{4}\int_{\alpha_2+1}^{\alpha_2+\tau_2} e^{-H(t)}\int_{\alpha_2+1}^{t} e^{H(s)}ds\,dt \le -1.$$

This is a contradiction. Thus, it turns out that the beginning sentence of this step is true.

From the steps 1 and 2, we conclude that $y^2(t)$ cannot remain in the range from $\mu/2$ to μ and it passes through this range many times. Then, how much frequency does $y^2(t)$ go out of this range at?

Step 3: To divide the interval $[t_0 + \tau_3, t_0 + T]$ into some small intervals whose width is $\tau_1 + \tau_2$, we define

$$J_i = \bigl[t_0 + \tau_3 + (i-1)(\tau_1 + \tau_2),\, t_0 + \tau_3 + i\,(\tau_1 + \tau_2)\bigr]$$

for any $i \in \mathbb{N}$. Then, we can describe

$$[t_0 + \tau_3, t_0 + T] = J_1 \cup J_2 \cup \cdots \cup J_{[1/\nu]+1}.$$

Let us examine the behavior of $y^2(t)$ in the interval J_1 in detail. For this purpose, we subdivide J_1 into the intervals $[t_0 + \tau_3, t_0 + \tau_1 + \tau_3]$ and $[t_0 + \tau_1 + \tau_3, t_0 + \tau_1 + \tau_2 + \tau_3]$. Since the width of $[t_0 + \tau_3, t_0 + \tau_1 + \tau_3]$ is τ_1, it turns out from the the conclusion of step 1 that there exists a $\underline{t} \in [t_0 + \tau_3, t_0 + \tau_1 + \tau_3]$ such that $y^2\bigl(\underline{t}\bigr) < \mu/2$. Since the width of $[t_0 + \tau_1 + \tau_3, t_0 + \tau_1 + \tau_2 + \tau_3]$ is τ_2, it also turns out from the the conclusion of step 2 that there exists a $\overline{t} \in [t_0 + \tau_1 + \tau_3, t_0 + \tau_1 + \tau_2 + \tau_3]$ such that $y^2\bigl(\overline{t}\bigr) > \mu$. From the continuity of $y^2(t)$, we can find numbers t_1 and t_2 with $\underline{t} \le t_1 < t_2 \le \overline{t}$ such that $y^2(t_1) = \mu/2$, $y^2(t_2) = \mu$ and

$$\frac{\mu}{2} \le y^2(t) \le \mu \quad \text{for} \;\; t_1 \le t \le t_2. \tag{2.7}$$

Hence, we have

$$\begin{aligned}\frac{\mu}{2} &= y^2(t_2) - y^2(t_1) = \int_{t_1}^{t_2}\bigl(y^2(t)\bigr)'dt \\ &= -2\int_{t_1}^{t_2}\bigl(\omega x(t)y(t) + h(t)y^2(t)\bigr)dt \le 2\,\omega\int_{t_1}^{t_2}|x(t)y(t)|dt.\end{aligned}$$

It follows from (2.5) that

$$|x(t)| < 1 \quad \text{and} \quad |y(t)| < 1$$

for $t_0 \le t \le t_0 + T$. Consequently, we obtain

$$\frac{\mu}{4\omega} < t_2 - t_1. \tag{2.8}$$

Using the estimations given in the preceding step, we examine the amount of change of the total energy $v(t)$.

Step 4: From (2.7) and (2.8) it turns out that

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} v'(t)dt = -\int_{t_1}^{t_2} h(t)y^2(t)dt$$
$$\le -\frac{\mu}{2}\int_{t_1}^{t_2} h(t)dt \le -\frac{\mu}{2}\int_{t_1}^{t_1+\mu/(4\omega)} h(t)dt.$$

Hence, by (2.3) we have

$$v(t_2) - v(t_1) \le -\nu.$$

Since $v'(t) = -h(t)y^2(t) \le 0$ for $t \ge t_0$, it is clear that

$$v(t_1) - v(t_0 + \tau_3) \le 0 \quad \text{and} \quad v(t_0 + \tau_1 + \tau_2 + \tau_3) - v(t_2) \le 0.$$

We therefore conclude that

$$\int_{J_1} v'(t)dt = v(t_0 + \tau_1 + \tau_2 + \tau_3) - v(t_2) + v(t_2) - v(t_1) + v(t_1) - v(t_0 + \tau_3)$$
$$\le -\nu.$$

Repeating the same process as in the proof of step 3, we can estimate that

$$\int_{J_i} v'(t)dt \le -\nu$$

for $i = 2, 3, \ldots, [1/\nu]+1$. This means that the loss of the total energy $v(t)$ in each interval J_i is at least ν. Hence, we obtain

$$v(t_0 + T) - v(t_0 + \tau_3) = \sum_{i=1}^{[1/\nu]+1} \int_{J_i} v'(t)dt \le -\nu\left(\left[\frac{1}{\nu}\right] + 1\right) < -1,$$

and therefore, by (2.5) we have

$$v(t_0 + T) < v(t_0 + \tau_3) - 1 < 0.$$

This contradicts the fact that $v(t) \ge 0$ for $t \ge t_0$. Thus, inequality (2.4) was proved. The proof of Theorem 1.1 is now complete. □

3. Example with unbounded damping

In this section, we intend to give an affirmative answer to the question presented in Section 1; namely, we show that there is an example in which the equilibrium of (1.1) is uniformly asymptotically stable even if the damping coefficient $h(t)$ is not bounded.

To begin with, we define the function

$$u(t;\sigma) = -\frac{\int_\sigma^t e^{H(s)}ds}{e^{H(t)}}$$

for any $\sigma \geq 0$. Then, we see that $u(t;\sigma)$ is the particular solution of the scalar differential equation

$$u' + h(t)u + 1 = 0 \tag{3.1}$$

satisfying the initial condition $u(\sigma;\sigma) = 0$. It is clear that $u(t;\sigma)$ exists in the future. Using the particular solution $u(t;\sigma)$, we can replace condition (1.4) with

$$\lim_{t\to\infty}\int_{\sigma}^{t+\sigma} u(s;\sigma)ds = -\infty \quad \text{uniformly with respect to } \ \sigma \geq 0. \tag{3.2}$$

Since equation (3.1) bears a close relation with the damped linear oscillator (1.1), we call it a *characteristic equation*. Generally, it is difficult to confirm whether the integration

$$\int_{\sigma}^{t+\sigma} \frac{\int_{\sigma}^{s} e^{H(\tau)}d\tau}{e^{H(s)}}\, ds$$

is divergent or convergent even when we use a personal computer. On the other hand, we can examine whether the integration

$$\int_{\sigma}^{t+\sigma} u(s;\sigma)ds$$

diverges comparatively easily by numerical analysis, because much excellent software program are already developed for calculating the solutions of a differential equation such as (3.1). This is a strong point which expresses condition (3.2) by using the characteristic equation (3.1).

We give a result which is easier to check than Theorem 1.1.

Theorem 3.1. *Suppose that $h(t)$ is integrally positive and that*

$$0 \leq h(t) \leq k(t) \quad \textit{for} \ \ t \geq 0,$$

where $1/k(t)$ is bounded and $(1/k(t))'$ is bounded from above. If

$$\lim_{t\to\infty}\int_{\sigma}^{t+\sigma} \frac{1}{k(s)}\, ds = \infty \quad \textit{uniformly with respect to} \ \ \sigma \geq 0,$$

then the equilibrium of (1.1) *is uniformly asymptotically stable.*

Remark 3.1. Although the upper function $k(t)$ have to be differentiable, the damping coefficient $h(t)$ does not necessarily need to be differentiable.

Proof of Theorem 3.1. By assumption, there exist numbers $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{1}{k(t)} \leq c_1 \quad \text{and} \quad \left(\frac{1}{k(t)}\right)' \leq c_2 \tag{3.3}$$

for $t \geq 0$. Define

$$g(t) = -\,\frac{1}{k(t)} \quad \text{for} \;\; t \geq 0.$$

Then, it is clear that

$$-\,c_1 \leq g(t) < 0 \quad \text{and} \quad g'(t) \geq -\,c_2$$

for $t \geq 0$.

Consider the characteristic equation (3.1) and let $u(t;\sigma)$ be the solution of (3.1) satisfying the initial condition $u(\sigma;\sigma) = 0$. Then, we see that

$$u(t;\sigma) < 0 \quad \text{for} \;\; t > \sigma. \tag{3.4}$$

In fact, since $u(\sigma;\sigma) = 0$ and $u'(\sigma;\sigma) = -\,1$, we can find a $t_1 > \sigma$ such that $u(t;\sigma) < 0$ for $\sigma < t \leq t_1$. Suppose that there exists a $t_2 > t_1$ such that $u(t_2;\sigma) = 0$ and $u(t;\sigma) < 0$ for $\sigma < t < t_2$. Then, since $u'(t_2;\sigma) = -\,1$, it follows that $u(t;\sigma) > 0$ in a left-hand neighborhood of t_2. This contradicts the definition of t_2.

Let us compare $u(t;\sigma)$ with $g(t)$. Since $g(\sigma) < 0 = u(\sigma;\sigma)$, there are two cases to consider: (i) $g(t) < u(t;\sigma)$ for $t \geq \sigma$ and (ii) there exists a $t^* > \sigma$ such that $g(t^*) = u(t^*;\sigma)$ and $g(t) < u(t;\sigma)$ for $\sigma \leq t < t^*$; namely, the graph of $g(t)$ intersects the solution curve $u(t;\sigma)$ at $t = t^*$ for the first time. Hereafter, we will show that there exists a c_3 with $0 < c_3 < 1$ such that

$$c_3 g(t) \geq u(t;\sigma) \quad \text{for} \;\; t \geq \sigma + 1 \tag{3.5}$$

in both cases.

Case (i): Since $0 \leq h(t) \leq k(t)$ for $t \geq 0$, we see that

$$u(t;\sigma) > g(t) = -\,\frac{1}{k(t)} \geq -\,\frac{1}{h(t)}$$

for $t \geq \sigma$. Hence, we have

$$u'(t;\sigma) = -\,1 - h(t)u(t;\sigma) < 0;$$

that is, $u(t;\sigma)$ is strictly decreasing for $t \geq \sigma$. Let

$$c_3 = \min\left\{\frac{u(\sigma+1;\sigma)}{g(\sigma+1)},\; \frac{1}{1+c_2}\right\}.$$

Then, $0 < c_3 \leq 1/(1+c_2) < 1$. For simplicity, let $\zeta(t) = c_3 g(t)$. Then,

$$k(t)\zeta(t) = -\,c_3 > -\,1 \quad \text{for} \;\; t \geq 0.$$

Hence, it turns out from (3.3) that

$$\zeta'(t) = c_3 g'(t) \geq -\,c_2\, c_3 \geq -\,1 + c_3 = -\,1 - k(t)\zeta(t)$$

for $t \ge 0$. Let $f(t,u) = -1 - h(t)u$. Taking $\zeta(t) < 0$ for $t \ge 0$ into account, we obtain

$$\zeta'(t) \ge -1 - k(t)\zeta(t) \ge -1 - h(t)\zeta(t) = f(t,\zeta(t))$$

for $t \ge 0$. Since $c_3 \le u(\sigma+1;\sigma)/g(\sigma+1)$ and $g(\sigma+1) < 0$, we see that

$$\zeta(\sigma+1) = c_3 g(\sigma+1) \ge u(\sigma+1;\sigma).$$

Consequently, we can get (3.5) by virtue of a standard comparison theorem.
Case (ii): We subdivide this case as follows: (a) $t^* > \sigma+1$ and (b) $\sigma < t^* \le \sigma+1$. If $t^* > \sigma+1$, then $g(t) < u(t;\sigma)$ for $\sigma \le t \le \sigma+1$. Hence, by the same way as the case (i), we can get (3.5). If $\sigma < t^* \le \sigma+1$, then $g(t) < u(t;\sigma)$ for $\sigma \le t < t^*$, and therefore,

$$u(t;\sigma) \ge g(t) = -\frac{1}{k(t)} \ge -\frac{1}{h(t)}$$

for $\sigma \le t \le t^*$. Hence, we have

$$u'(t;\sigma) = -1 - h(t)u(t;\sigma) \le 0 \quad \text{for} \;\; \sigma \le t \le t^*.$$

Let $c_3 = 1/(1+c_2) < 1$ and $\zeta(t) = c_3 g(t)$. Then, by (3.3) we obtain

$$\begin{aligned}\zeta'(t) &= c_3 g'(t) \ge -c_2 c_3 = -1 + c_3 \\ &= -1 - k(t)\zeta(t) \ge -1 - h(t)\zeta(t) = f(t,\zeta(t))\end{aligned}$$

for $t \ge 0$, where $f(t,u)$ is the function given in the case (i). Since $0 < c_3 < 1$ and $u(t^*;\sigma) < 0$, we see that $\zeta(t^*) = c_3 u(t^*;\sigma) > u(t^*;\sigma)$. We therefore conclude that $\zeta(t) \ge u(t;\sigma)$ for $t \ge t^*$. Since $\sigma+1 \ge t^*$, we get (3.5).

From (3.3)–(3.5) it turns out that for t sufficiently large,

$$\begin{aligned}\int_\sigma^{t+\sigma} u(s;\sigma)ds &= \int_\sigma^{\sigma+1} u(s;\sigma)ds + \int_{\sigma+1}^{t+\sigma} u(s;\sigma)ds \\ &\le \int_\sigma^{\sigma+1} u(s;\sigma)ds + \int_{\sigma+1}^{t+\sigma} \zeta(s)ds \\ &< \int_{\sigma+1}^{t+\sigma} \zeta(s)ds = \int_\sigma^{\sigma+1} \frac{c_3}{k(s)}\,ds - \int_\sigma^{t+\sigma} \frac{c_3}{k(s)}\,ds \\ &< c_1 c_3 - \int_\sigma^{t+\sigma} \frac{c_3}{k(s)}\,ds.\end{aligned}$$

Since

$$\lim_{t\to\infty} \int_\sigma^{t+\sigma} \frac{1}{k(s)}\,ds = \infty$$

uniformly with respect to $\sigma \ge 0$, condition (3.2) holds. Thus, by Theorem 1.1, the equilibrium of (1.1) is uniformly asymptotically stable □

We are now ready to present a desired example. For any $n \in \mathbb{N}$, let

$$I_n = \left[n - \frac{n+1}{2n},\, n \right]$$

and

$$\frac{1}{h(t)} = \begin{cases} 1 - \dfrac{n}{n+1} \sin^2\!\left\{ \dfrac{2n\pi}{n+1}\left(t - n + \dfrac{n+1}{2n} \right) \right\} & \text{if } \ t \in I_n, \\ \qquad\qquad 1 & \text{if } \ t \notin I_n. \end{cases} \tag{3.6}$$

Then, it is easily seen that the width of I_n becomes gradually narrow and approaches $1/2$ as n increases, and the damping coefficient $h(t)$ is greater than or equal to 1 and is continuously differentiable for $t \geq 0$. Since $h(t_n) = n + 1$, where

$$t_n = n - \frac{n+1}{4n},$$

we see that $h(t)$ is unbounded. We present the graphs of the function $1/h(t)$ and $h(t)$, respectively (see Figures 1 and 2).

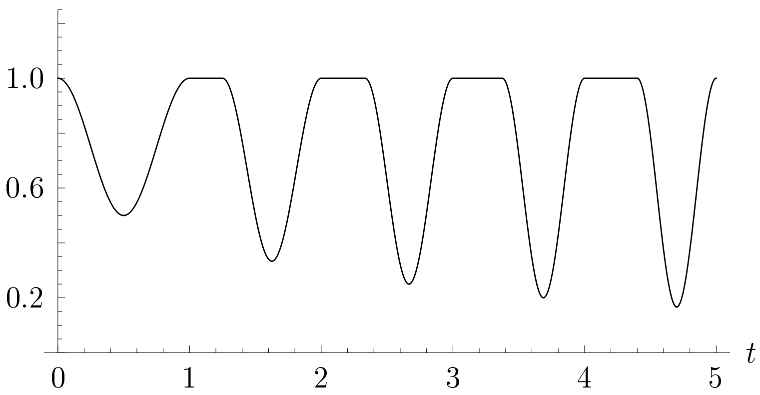

Figure 1: The value of $1/h(t_n)$ approaches zero as $n \to \infty$.

Since $h(t) \geq 1$ for $t \geq 0$, it follows that $h(t)$ is integrally positive. Let $k(t) = h(t)$ for $t \geq 0$. Then, $1/k(t) \leq 1$ for $t \geq 0$ and

$$\left(\frac{1}{k(t)} \right)' = \begin{cases} -\dfrac{2n^2\pi}{(n+1)^2} \sin\!\left\{ \dfrac{4n\pi}{n+1}\left(t - n + \dfrac{n+1}{2n} \right) \right\} & \text{if } \ t \in I_n, \\ \qquad\qquad 0 & \text{if } \ t \notin I_n. \end{cases}$$

Hence, $(1/k(t))' < 2\pi$ for $t \geq 0$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} \int_{n-1}^{n} \frac{1}{k(t)}\, dt &= 1 - \int_{I_n} \frac{n}{n+1} \sin^2\!\left\{ \frac{2n\pi}{n+1}\left(t - n + \frac{n+1}{2n} \right) \right\} dt \\ &= 1 - \frac{1}{2\pi} \int_0^{\pi} \sin^2 u\, du = \frac{3}{4}. \end{aligned}$$

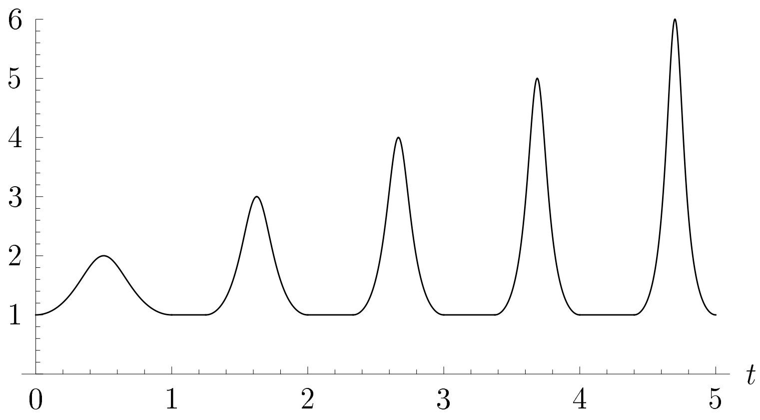

Figure 2: The value of $h(t_n)$ diverges to $+\infty$ as $n \to \infty$.

for $t > 0$ sufficiently large, there exists an n_1 such that $n_1 - 1 \leq t \leq n_1$. Of course, n_1 is a large integer. Similarly, for any $\sigma \geq 0$ there exists an $n_2 \in \mathbb{N}$ such that $n_2 - 1 \leq \sigma \leq n_2$. Hence, $n_2 < n_1 + n_2 - 2 \leq t + \sigma \leq n_1 + n_2$ and therefore,

$$\int_{\sigma}^{t+\sigma} \frac{1}{k(s)}\, ds \geq \int_{n_2}^{n_1+n_2-2} \frac{1}{k(s)}\, ds = \frac{3}{4}(n_1 - 2) > \frac{3}{4}(t-2).$$

This means that

$$\lim_{t\to\infty} \int_{\sigma}^{t+\sigma} \frac{1}{k(s)}\, ds = \infty$$

uniformly with respect to $\sigma \geq 0$. Thus, by means of Theorem 3.1, we obtain the following example with unbounded damping coefficient $h(t)$.

Example 3.2. Let $h(t)$ be the function defined by (3.6). Then the equilibrium of (1.1) is uniformly asymptotically stable.

We attach a phase portrait of positive orbits of Example 3.2 with $\omega = 1$ for a deeper understanding. In Figure 3, we draw four positive orbits starting at points $(0.3, 1.0)$, $(-1.0, 0.8)$, $(-0.3, -1.0)$ and $(1.0, -0.8)$, respectively. The four positive orbits have the same initial time $t_0 = 0$.

As was mentioned in Section 1, Hatvani [18] has first presented condition (1.4). To be precise, since he dealt with the general form of a two-dimensional linear system with time-varying coefficients, the obtained condition was a little more complicated than condition (1.4). Because it is hard to check condition (1.4) directly, he also gave easy-to-use conditions which guarantee that condition (1.4) holds. Using one of those conditions, we can lead the following result.

Theorem 3.2. *Suppose that $1/h(t)$ is bounded and $(1/h(t))'$ is bounded from below. If*

$$\lim_{t\to\infty} \int_{\sigma}^{t+\sigma} \frac{1}{h(s)}\, ds = \infty \quad \textit{uniformly with respect to} \;\; \sigma \geq 0, \tag{3.7}$$

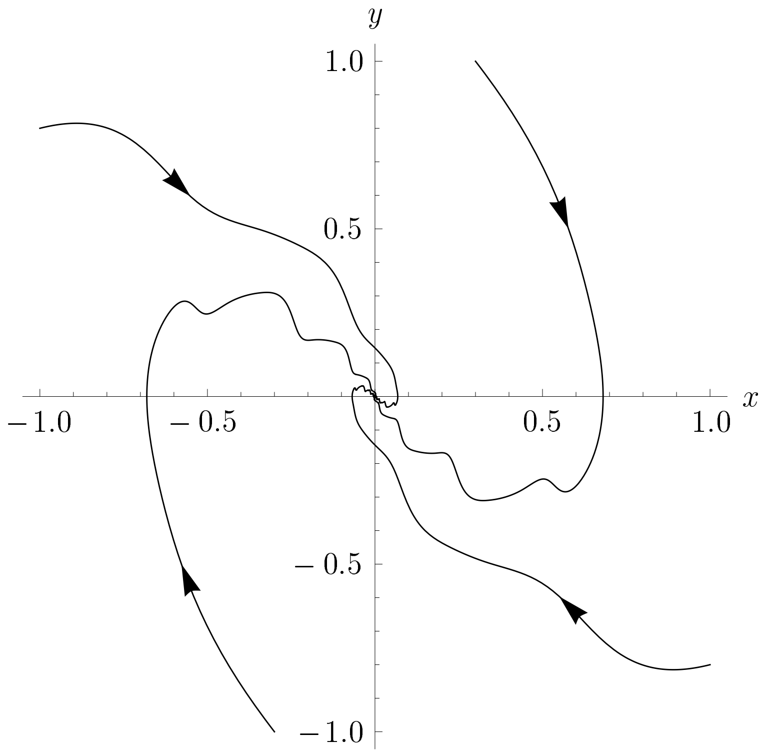

Figure 3: Every positive orbit moves round the origin in a clockwise and a counter-clockwise direction by turns and approach the origin windingly.

then the equilibrium of (1.1) *is uniformly asymptotically stable*.

Remark 3.3. If $1/h(t)$ is bounded, then $h(t)$ is integrally positive. Example 3.2 can be also confirmed by using Theorem 3.2.

4. Discrete condition for uniform asymptotic stability

Since $h(t) \geq 0$ for $t \geq 0$, the integral $H(t)$ is increasing for $t \geq 0$ (needless to say, $H(t)$ is not necessarily strictly increasing). Define

$$H^{-1}(r) = \min\{t \in \mathbb{R} \colon H(t) \geq r\}.$$

Then, the inverse function $H^{-1}(r)$ is also increasing for $r \geq 0$. Hatvani, Krisztin and Totik [20] have proved that condition (1.3) is equivalent to

$$\sum_{n=1}^{\infty} \left(H^{-1}(n) - H^{-1}(n-1)\right)^2 = \infty \tag{4.1}$$

under the assumption that $H(t)$ tends to ∞ as $t \to \infty$. If $h(t)$ is weakly integrally positive, then

$$\lim_{t\to\infty} H(t) = \infty.$$

Hence, combining their result and Theorem C, we obtain another criterion for the asymptotic stability as follows.

Theorem D. *Suppose that one of the following assumptions*

(i) $h(t)$ *is integrally positive,*

(ii) $h(t)$ *is uniformly continuous for* $t \geq 0$ *and weakly integrally positive*

holds. Then the equilibrium of (1.1) *is asymptotically stable if and only if condition* (4.1) *holds.*

In this section, using their notation, we will give sufficient conditions for the equilibrium of (1.1) to be uniformly asymptotically stable. To this end, we prepare the following result.

Lemma 4.1. *Let* $a_n = H^{-1}(n) - H^{-1}(n-1)$ *for* $n \in \mathbb{N}$. *Suppose that* $h(t)$ *is integrally positive. Then the sequence* $\{a_n\}$ *is bounded.*

Proof. As was shown in the proof of Theorem 1.1, since $h(t)$ is integrally positive, we can find an $\ell > 0$ and a $\hat{t} > 0$ such that

$$H(t+1) - H(t) = \int_t^{t+1} h(s)ds \geq \ell \quad \text{for } \ t \geq \hat{t}. \tag{4.2}$$

By way of contradiction, we suppose that $\{a_n\}$ is unbounded. Then, we can choose a subsequence $\{a_{n_k}\} \subset \{a_n\}$ with $n_k \to \infty$ as $k \to \infty$ such that $\lim_{k\to\infty} a_{n_k} = \infty$; namely, for any $K > 0$, there exists an $N(K) \in \mathbb{N}$ such that $k \geq N$ implies $a_{n_k} > K$. In particular, let

$$K_* = \left[\frac{1}{\ell}\right] + 1 \in \mathbb{N},$$

where $[c]$ means the greatest integer that is less than or equal to a real number c. Then, there exists an $N_* = N(K_*) \in \mathbb{N}$ such that

$$H^{-1}(n_k) - H^{-1}(n_k - 1) = a_{n_k} > K_* \quad \text{for } \ k \geq N_*. \tag{4.3}$$

Since $H(t)$ tends to ∞ as $t \to \infty$, the inverse function $H^{-1}(r)$ also tends to ∞ as $r \to \infty$. Hence, $H^{-1}(n_k) \to \infty$ as $k \to \infty$. We therefore conclude that there exists an $N^* \geq N_*$ such that $k \geq N^*$ implies $H^{-1}(n_k - 1) > \hat{t}$. From this and (4.2), we see that

$$\begin{gathered}
H(H^{-1}(n_k-1)+1) - H(H^{-1}(n_k-1)) > \ell, \\
H(H^{-1}(n_k-1)+2) - H(H^{-1}(n_k-1)+1) > \ell, \\
H(H^{-1}(n_k-1)+3) - H(H^{-1}(n_k-1)+2) > \ell, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
H(H^{-1}(n_k-1)+K_*) - H(H^{-1}(n_k-1)+K_*-1) > \ell
\end{gathered}$$

for $k \geq N^*$. Adding these inequalities, we obtain

$$\begin{aligned} H(H^{-1}(n_k-1)+K_*) &> H(H^{-1}(n_k-1))+K_*\ell \\ &= n_k - 1 + K_*\ell = n_k - 1 + \left(\left[\frac{1}{\ell}\right]+1\right)\ell \\ &> n_k = H(H^{-1}(n_k)) \end{aligned}$$

for $k \geq N^*$. Since $H(t)$ is increasing for $t \geq 0$, it follows that $H^{-1}(n_k-1)+K_* > H^{-1}(n_k)$; namely,

$$a_{n_k} = H^{-1}(n_k) - H^{-1}(n_k-1) < K_* \quad \text{for} \;\; k \geq N^* \geq N_*.$$

This contradicts (4.3). □

By virtue of Lemma 4.1, it turns out that there exists an $\overline{a} > 0$ such that

$$0 < a_n \leq \overline{a} \quad \text{for} \;\; n \in \mathbb{N}.$$

We are now ready to state the following sufficient condition expressed by an infinite series, which guarantee uniform asymptotic stability for the damped linear oscillator (1.1) under the assumption that $h(t)$ is integrally positive.

Theorem 4.2. *Suppose that $h(t)$ is integrally positive. If*

$$\lim_{n\to\infty} \sum_{i=N}^{n+N} \left(H^{-1}(i) - H^{-1}(i-1)\right)^2 = \infty \quad \textit{uniformly with respect to} \;\; N \in \mathbb{N}, \tag{4.4}$$

then the equilibrium of (1.1) *is uniformly asymptotically stable.*

Proof. It follows from (4.4) that for any $L > 0$ there exists an $M(L) \in \mathbb{N}$ such that

$$\sum_{n=N}^{M+N} a_n^2 > L \quad \text{for} \;\; N \in \mathbb{N}. \tag{4.5}$$

For any $K > 0$, let $L = 2eK$. Then, there exists an $M_* = M(2eK) \in \mathbb{N}$ such that

$$\sum_{n=N}^{M_*+N} a_n^2 > 2eK \quad \text{for} \;\; N \in \mathbb{N}. \tag{4.6}$$

From the integral positivity of $h(t)$ and Lemma 4.1, we see that the the sequence $\{a_n\}$ has the upper bound $\overline{a} > 0$. Let $T = T(K) = \overline{a}(M_* + 2) = \overline{a}(M(2eK) + 2)$ and let $\sigma \geq 0$ be fixed arbitrarily. Define the domains

$$D = \left\{(s,\tau)\colon \sigma \leq \tau \leq s \leq \sigma + T\right\}$$

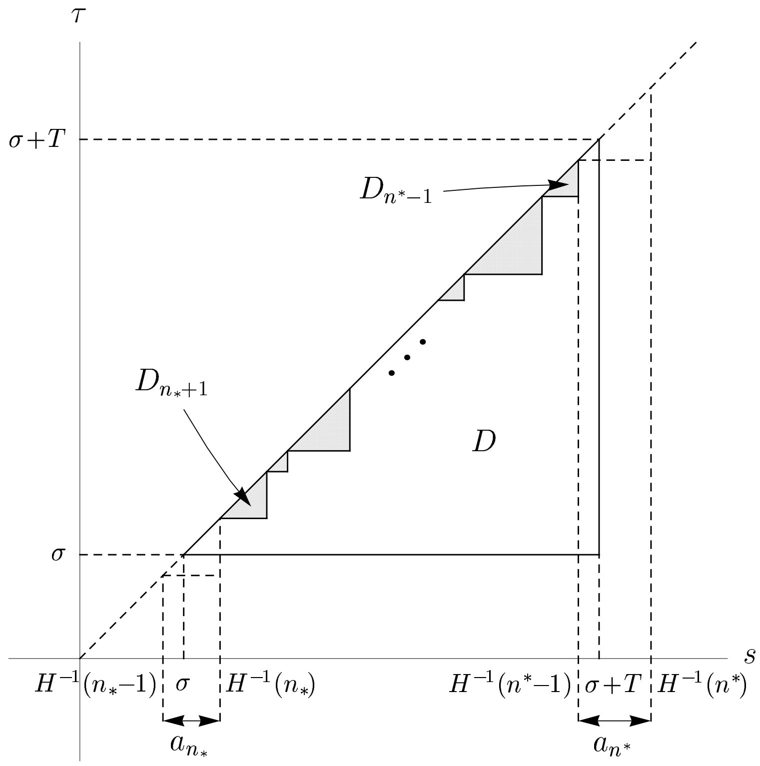

Figure 4: The triangle domain D contains small right isosceles triangles D_n for $n = n_* + 1, n_* + 2, \dots, n^* - 1$.

and

$$D_n = \big\{(s,\tau)\colon H^{-1}(n-1) \le \tau \le s \le H^{-1}(n)\big\} \quad \text{for } \ n \in \mathbb{N}$$

in the plane (see Figure 4). Note that D and D_n are right isosceles triangles whose legs are T and a_n, respectively. Since $T \ge 3\,\overline{a}$ and $H^{-1}(n)$ tends to ∞ as $n \to \infty$, we can choose two integers n_* and n^* so that

$$H^{-1}(n_* - 1) < \sigma \le H^{-1}(n_*) < H^{-1}(n^* - 1) < \sigma + T \le H^{-1}(n^*).$$

From these inequalities it follows that

$$H^{-1}(n_*) - \sigma < H^{-1}(n_*) - H^{-1}(n_* - 1) = a_{n_*} \le \overline{a}$$

and

$$\sigma + T - H^{-1}(n^* - 1) < H^{-1}(n^*) - H^{-1}(n^* - 1) = a_{n^*} \le \overline{a}.$$

Hence, we obtain

$$\begin{aligned}\sum_{n=n_*+1}^{n^*-1} a_n &= H^{-1}(n^* - 1) - H^{-1}(n_*) \\ &> \sigma + T - \overline{a} - (\sigma + \overline{a}) = \overline{a}(M_* + 2) - 2\,\overline{a} = \overline{a}M_*.\end{aligned}$$

We also obtain

$$\sum_{n=n_*+1}^{n^*-1} a_n \leq \overline{a}\big(n^* - 1 - (n_* + 1) + 1\big) = \overline{a}\,(n^* - n_* - 1).$$

We therefore conclude that $M_* < n^* - n_* - 1$. Since M_*, n^* and n_* are integers,

$$M_* \leq n^* - n_* - 2. \tag{4.7}$$

This means that the number of small triangles D_n which are included in the domain D is at least $M_* - 1$ pieces.

Let $U = D_{n_*+1} \cup D_{n_*+2} \cup \cdots \cup D_{n^*-1}$. Taking into account of

$$D_{n_*+1} \subset D, \quad D_{n_*+2} \subset D, \quad \cdots\cdots, \quad D_{n^*-1} \subset D,$$

we see that $U \subset D$. Hence, we have

$$\begin{aligned}\int_\sigma^{T+\sigma} \frac{\int_\sigma^s e^{H(\tau)} d\tau}{e^{H(s)}}\, ds &= \int_\sigma^{T+\sigma}\int_\sigma^s e^{-H(s)+H(\tau)}\, d\tau\, ds \\ &= \iint_D e^{-H(s)+H(\tau)}\, d\tau\, ds \geq \iint_U e^{-H(s)+H(\tau)}\, d\tau\, ds.\end{aligned}$$

For any $(s,\tau) \in U$, there exists an $n \in \mathbb{N}$ with $n_* + 1 \leq n \leq n^* - 1$ such that

$$n - 1 \leq H(\tau) \leq H(s) \leq n.$$

Hence, we see that

$$0 \geq -H(s) + H(\tau) \geq -n + n - 1 = -1,$$

and therefore,

$$\frac{1}{e} \leq e^{-H(s)+H(\tau)} \leq 1$$

for $(s,\tau) \in U$. Let $S(U)$ be the area of U, which is the union of the right isosceles triangles D_n $(n = n_* + 1, n_* + 2, \ldots, n^* - 1)$. Then, from (4.7) it turns out that

$$\iint_U e^{-H(s)+H(\tau)}\, d\tau\, ds \geq \frac{1}{e} S(U) = \frac{1}{2e} \sum_{n=n_*+1}^{n^*-1} a_n^2 \geq \frac{1}{2e} \sum_{n=n_*+1}^{M_*+n_*+1} a_n^2.$$

Consequently, using (4.6) with $N = n_* + 1$, we obtain

$$\int_\sigma^{T+\sigma} \frac{\int_\sigma^s e^{H(\tau)} d\tau}{e^{H(s)}}\, ds \geq \frac{1}{2e} \sum_{n=n_*+1}^{M_*+n_*+1} a_n^2 > \frac{1}{2e} 2eK = K.$$

We therefore obtain

$$\int_\sigma^{t+\sigma} \frac{\int_\sigma^s e^{H(\tau)} d\tau}{e^{H(s)}}\, ds \geq \int_\sigma^{T+\sigma} \frac{\int_\sigma^s e^{H(\tau)} d\tau}{e^{H(s)}}\, ds > K$$

for $t \geq T$. Since K does not depend on $\sigma \geq 0$, condition (1.4) holds. Thus, by means of Theorem 1.1, the equilibrium of (1.1) is uniformly asymptotically stable □

We consider the case that the function $1/h(t)$ is integrable on any bounded interval. Then, using the method of Hatvani, Krisztin and Totik [20, p. 215], we obtain the following relation.

Theorem 4.3. *If* $h(t)$ *is integrally positive and* $1/h(t)$ *is integrable on any bounded interval, then condition* (4.4) *implies condition* (3.7).

Proof. From (4.4) it turns out that for any $L > 0$ there exists an $M(L) \in \mathbb{N}$ satisfying (4.5). Since $h(t)$ is integrally positive, Lemma 4.1 is available. Let $T = T(L) = \overline{a}(M(L)+2) \geq 3\,\overline{a}$, where $\overline{a}$ is the upper bound of the sequence $\{a_n\}$. For any $\sigma \geq 0$, we can choose two integers n_* and n^* so that

$$H^{-1}(n_* - 1) < \sigma \leq H^{-1}(n_*) < H^{-1}(n^* - 1) < \sigma + T \leq H^{-1}(n^*). \tag{4.8}$$

By the same manner as in the proof of Theorem 4.2, we obtain

$$M = M(L) \leq n^* - n_* - 2. \tag{4.9}$$

Since $1/h(t)$ is integrable on the interval $[H^{-1}(n-1), H^{-1}(n)]$ with $n \in \mathbb{N}$, it follows from the Cauchy-Bunyakovski-Schwarz inequality that

$$\begin{aligned} a_n^2 &= \big(H^{-1}(n) - H^{-1}(n-1)\big)^2 = \left(\int_{H^{-1}(n-1)}^{H^{-1}(n)} dt\right)^2 \\ &\leq \int_{H^{-1}(n-1)}^{H^{-1}(n)} h(t)dt \int_{H^{-1}(n-1)}^{H^{-1}(n)} \frac{1}{h(t)}\, dt \\ &= \big(H(H^{-1}(n)) - H(H^{-1}(n-1))\big) \int_{H^{-1}(n-1)}^{H^{-1}(n)} \frac{1}{h(t)}\, dt \\ &= \int_{H^{-1}(n-1)}^{H^{-1}(n)} \frac{1}{h(t)}\, dt \end{aligned}$$

for $n \in \mathbb{N}$. Consequently, using (4.5) with $N = n_* + 1$, we get

$$L < \sum_{n=n_*+1}^{M+n_*+1} a_n^2 \leq \sum_{n=n_*+1}^{M+n_*+1} \int_{H^{-1}(n-1)}^{H^{-1}(n)} \frac{1}{h(t)}\, dt = \int_{H^{-1}(n_*)}^{H^{-1}(M+n_*+1)} \frac{1}{h(t)}\, dt.$$

Since $H^{-1}(n)$ is increasing with respect to $n \in \mathbb{N}$, it turns out from (4.8) and (4.9) that

$$\int_{\sigma}^{T+\sigma} \frac{1}{h(t)}\, dt \geq \int_{\sigma}^{H^{-1}(n^*-1)} \frac{1}{h(t)}\, dt \geq \int_{H^{-1}(n_*)}^{H^{-1}(M+n_*+1)} \frac{1}{h(t)}\, dt > L.$$

Hence, we obtain

$$\int_{\sigma}^{t+\sigma} \frac{1}{h(t)}\, dt > L \quad \text{for } \ t \geq T.$$

Since T does not depend on $\sigma \geq 0$, we see that condition (3.7) holds. □

5. Extension to the damped half-linear oscillator

In this section, we consider the nonlinear second-order differential equation

$$(\phi_p(x'))' + h(t)\phi_p(x') + \omega^p\phi_p(x) = 0, \tag{5.1}$$

the function $\phi_p(z)$ is defined by

$$\phi_p(z) = |z|^{p-2}z, \qquad z \in \mathbb{R}$$

with $p > 1$. The only equilibrium of (5.1) is the origin $(x, x') = (0, 0)$. Equation (5.1) is often called *half-linear*, because it has only one of the two characteristics of the solution space of linear differential equations. To be precise, if $x(t)$ is a solution of (5.1), then the function $cx(t)$ is also a solution of (5.1) for an arbitrary constant $c \in \mathbb{R}$. However, the sum of two solutions is not a solution in general. Since equation (5.1) is a natural generalization of the damped linear oscillator (1.1), we will call it the *damped half-linear oscillator*.

We can rewrite (5.1) to the equation of self-adjoint type,

$$\left(e^{H(t)}\phi_p(x')\right)' + \omega^p e^{H(t)}\phi_p(x) = 0. \tag{5.2}$$

It is known that the global existence and uniqueness of solutions of (5.2) are guaranteed for the initial value problem. For details, see Došlý [9, p. 170] or Došlý and Řehák [13, pp. 8–10]. Many important studies on the half-linear differential equations have been performed over about forty years. Especially, there are good articles concerning oscillation theory. Especially, a lot of efforts were devoted to obtain good articles concerning oscillation theory. We can find those results in the books [1, 9, 13] and the references cited therein. Even after these books are published, the equation (5.2) is continuing being studied actively (for example, see [10, 11, 12, 15, 27, 34]). However, there is little research of stability theory of half-linear differential equations such as (5.1).

Let p^* be the conjugate number of p; namely,

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

then p^* is also greater than 1. Since the function $w = \phi_p(z)$ is strictly increasing, there exists the inverse function which is described by $z = \phi_{p^*}(w)$. As a new variable, let us introduce $y = \phi_p(x'/\omega)$. Then, equation (5.1) becomes the planar system

$$\begin{aligned} x' &= \omega\phi_{p^*}(y) \\ y' &= -\,\omega\phi_p(x) - h(t)y. \end{aligned} \tag{5.3}$$

System (5.3) has the zero solution which corresponds to the equilibrium of (5.1). We can define uniform asymptotic stability of the zero solution of (5.3) as well as (2.1). The following result is a generalization to Theorem 1.1.

Theorem 5.1. *Suppose that* $h(t)$ *is integrally positive. If*

$$\lim_{t\to\infty}\int_\sigma^{t+\sigma}\phi_{p^*}\left(\frac{\int_\sigma^s e^{H(\tau)}d\tau}{e^{H(s)}}\right)ds=\infty \quad \textit{uniformly with respect to} \;\; \sigma\ge 0, \tag{5.4}$$

is satisfied, then the equilibrium of (5.1) *is uniformly asymptotically stable.*

Since the proof is carried out in the same way as the proof of Theorem 1.1, we describe only an outline of the proof to focus on the differences.

Outline of the proof of Theorem 5.1. Part (a): Let $\overline{p}=\max\{p,p^*\}$ and $\underline{p}=\min\{p,p^*\}$. For any $\varepsilon\in(0,1)$, we choose

$$\delta(\varepsilon)=\left(\frac{\underline{p}\,\varepsilon^{\overline{p}}}{2^{1+\overline{p}/2}\,\overline{p}}\right)^{1/\underline{p}}.$$

Then, it follows that $\delta(\varepsilon)<\varepsilon<1$. Let $t_0\ge 0$ and $\mathbf{x}_0=(x_0,y_0)\in\mathbb{R}^2$ be given. We will show that $\|\mathbf{x}_0\|<\delta(\varepsilon)$ implies $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\|<\varepsilon$ for $t\ge t_0$. Define

$$v(t)=\frac{|x(t)|^p}{p}+\frac{|y(t)|^{p^*}}{p^*},$$

where $(x(t),y(t))=\mathbf{x}_0=(x_0,y_0)$. Then, $v'(t)=-\,h(t)|y(t)|^{p^*}\le 0$ for $t\ge t_0$. Hence, we obtain

$$\frac{1}{\overline{p}}\left(|x(t)|^p+|y(t)|^{p^*}\right)\le v(t)\le v(t_0)\le\frac{1}{\underline{p}}\left(|x_0|^p+|y_0|^{p^*}\right),$$

and therefore,

$$|x(t)|^p+|y(t)|^{p^*}\le\frac{\overline{p}}{\underline{p}}\left(|x_0|^p+|y_0|^{p^*}\right)<\frac{\overline{p}}{\underline{p}}\Big(\big(\delta(\varepsilon)\big)^p+\big(\delta(\varepsilon)\big)^{p^*}\Big)$$

for $t\ge t_0$. Since $\delta<1$, we see that

$$|x(t)|^p+|y(t)|^{p^*}<\frac{\overline{p}}{\underline{p}}\Big(\big(\delta(\varepsilon)\big)^{\underline{p}}+\big(\delta(\varepsilon)\big)^{\underline{p}}\Big)=\left(\frac{\varepsilon}{\sqrt{2}}\right)^{\overline{p}} \quad\text{for}\;\; t\ge t_0.$$

From this estimation it turns out that

$$|x(t)|<\left(\frac{\varepsilon}{\sqrt{2}}\right)^{\overline{p}/p}\le\frac{\varepsilon}{\sqrt{2}} \quad\text{and}\quad |y(t)|<\left(\frac{\varepsilon}{\sqrt{2}}\right)^{\overline{p}/p^*}\le\frac{\varepsilon}{\sqrt{2}}$$

for $t\ge t_0$. Hence, we obtain

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)\|=\sqrt{x^2(t)+y^2(t)}<\varepsilon \quad\text{for}\;\; t\ge t_0;$$

namely, the zero solution of (5.3) is uniformly stable. This completes the proof of part (a).

Part (b): Let $\delta_0 = 1/\overline{p}$ and

$$\gamma(\eta) = \frac{\big(\delta(\eta)\big)^{\overline{p}}}{2^{\overline{p}/2}\,\overline{p}}$$

for every $\eta \in (0,1)$, where $\delta(\cdot)$ is the number given in part (a). Note that $\overline{p}\,\gamma(\eta) < \big(\delta(\eta)\big)^{\overline{p}} < \eta^{\overline{p}} < 1$. Since $h(t)$ is integrally positive, there exist numbers $\ell > 0$ and $\hat{t} > 0$ such that

$$\int_t^{t+1} h(s)ds \geq \ell \quad \text{for } \; t \geq \hat{t}.$$

We define

$$\mu = \min\left\{\frac{p^*\gamma(\eta)}{2},\ \frac{p\,\omega^{p^*}\gamma(\eta)}{2^{1+p^*}}\right\} \quad \text{and} \quad \tau_1 = \hat{t} + \left[\frac{1}{\ell\mu}\right] + 1,$$

where $[c]$ means the greatest integer that is less than or equal to a real number c. From condition (5.4) it turns out that there exists a τ_2 such that

$$\int_\sigma^{t+\sigma} \phi_{p^*}\!\left(\frac{\int_\sigma^s e^{H(\tau)}d\tau}{e^{H(s)}}\right) ds \geq \frac{2^{p^*-1}}{\omega^{p^*}}\left(\frac{2}{p\gamma(\eta)}\right)^{1/p} \quad \text{for } \; t \geq \tau_2 - 1. \tag{5.5}$$

We may assume without loss of generality that $\tau_2 > 1$. Let

$$\nu = \liminf_{t\to\infty} \frac{\mu}{4}\int_t^{t+\mu/(2\,p^*\omega)} h(s)ds.$$

Then, we can choose a $\tau_3 > 0$ so that

$$\int_t^{t+\mu/(2\,p^*\omega)} h(s)ds \geq \frac{2\nu}{\mu} \quad \text{for } \; t \geq \tau_3. \tag{5.6}$$

From the definitions of μ, ν, τ_1, τ_2 and τ_3, we see that these numbers depends only on η. This is important in the proof. We define

$$T = \tau_3 + \left(\left[\frac{1}{\nu}\right] + 1\right)(\tau_1 + \tau_2).$$

Then, the number T also depends only on η.

Part (c): Consider a solution $\mathbf{x}(t;t_0,\mathbf{x}_0)$ of (5.3) with $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta_0 = 1/\overline{p}$. To complete the proof, we have only to show that there exists a $t^* \in [t_0, t_0+T]$ such that

$$|x(t^*)|^p + |y(t^*)|^{p^*} < \frac{\big(\delta(\eta)\big)^{\overline{p}}}{2^{\overline{p}/2}} = \overline{p}\gamma(\eta) \tag{5.7}$$

for every $\eta > 0$, because this inequality implies that

$$\|\mathbf{x}(t^*;t_0,\mathbf{x}_0)\| = \sqrt{x^2(t^*) + y^2(t^*)} < \eta.$$

By way of contradiction, we suppose that

$$|x(t)|^p + |y(t)|^{p^*} \geq \overline{p}\gamma(\eta) \quad \text{for } \; t_0 \leq t \leq t_0 + T.$$

Since $\sqrt{x_0^2 + y_0^2} = \|\mathbf{x}_0\| < \delta_0 = 1/\overline{p}$, we see that

$$\begin{aligned}\gamma(\eta) &\leq \frac{1}{\overline{p}}\big(|x(t)|^p + |y(t)|^{p^*}\big) \leq v(t) \\ &\leq v(t_0) = \frac{|x_0|^p}{p} + \frac{|y_0|^{p^*}}{p^*} < \frac{1}{\overline{p}}\left(\frac{1}{p} + \frac{1}{p^*}\right) = \frac{1}{\overline{p}} \end{aligned} \tag{5.8}$$

for $t_0 \leq t \leq t_0 + T$. Let us examine the behavior of $|y(t)|^{p^*}$ in four steps that were used to prove Theorem 1.1.

Step 1: For any interval $[\alpha, \beta] \subset [t_0, t_0 + T]$, if $|y(t)|^{p^*} \geq \mu/2$ for $\alpha \leq t \leq \beta$, then the time width $\beta - \alpha$ is less than τ_1. Using $|y(t)|^{p^*}$ instead of $y^2(t)$ and repeating the step 1 of Theorem 1.1, we can prove this to be correct.

Step 2: For any interval $[\alpha, \beta] \subset [t_0, t_0 + T]$, if $|y(t)|^{p^*} \leq \mu$ for $\alpha \leq t \leq \beta$, then the time width $\beta - \alpha$ is less than τ_2. To show this, we suppose that there exists an interval $[\alpha_2, \beta_2] \subset [t_0, t_0 + T]$ with $\beta_2 - \alpha_2 \geq \tau_2$ such that $|y(t)|^{p^*} \leq \mu$ for $\alpha_2 \leq t \leq \beta_2$. Since $\mu \leq p^*\gamma(\eta)/2$, by (5.8) we have

$$|x(t)|^p = p\left(v(t) - \frac{|y(t)|^{p^*}}{p^*}\right) \geq p\left(\gamma(\eta) - \frac{\mu}{p^*}\right) \geq \frac{p\gamma(\eta)}{2}$$

for $\alpha_2 \leq t \leq \beta_2$. Hence, there are two cases to consider: $x(t) \geq (p\,\gamma(\eta)/2)^{1/p}$ for $\alpha_2 \leq t \leq \beta_2$; $x(t) \leq -(p\gamma(\eta)/2)^{1/p}$ for $\alpha_2 \leq t \leq \beta_2$. We consider only the former, because the latter is carried out in the same way. In the former, we have

$$\begin{aligned}\big(e^{H(t)}y(t)\big)' &= \big(y'(t) + h(t)y(t)\big)e^{H(t)} \\ &= -\,\omega\phi_p(x(t))e^{H(t)} \leq -\,\omega\left(\frac{p\gamma(\eta)}{2}\right)^{1/p^*} e^{H(t)}\end{aligned}$$

for $\alpha_2 \leq t \leq \beta_2$. Integrating this inequality from α_2 to $t \leq \beta_2$, we obtain

$$\begin{aligned}y(t) &\leq y(\alpha_2)e^{H(\alpha_2)}e^{-H(t)} - \omega\left(\frac{p\gamma(\eta)}{2}\right)^{1/p^*} e^{-H(t)}\int_{\alpha_2}^t e^{H(s)}ds \\ &\leq \mu^{1/p^*}e^{H(\alpha_2)}e^{-H(t)} - \omega\left(\frac{p\gamma(\eta)}{2}\right)^{1/p^*} e^{-H(t)}\int_{\alpha_2}^t e^{H(s)}ds.\end{aligned}$$

Taking into account of

$$e^{H(\alpha_2)} \leq \int_{\alpha_2}^t e^{H(s)}ds \quad \text{for } \; t \geq \alpha_2 + 1,$$

$\beta_2 \geq \alpha_2 + \tau_2 > \alpha_2 + 1$ and $\mu \leq p\,\omega^{p^*}\gamma(\eta)/2^{1+p^*}$, we can estimate that

$$
\begin{aligned}
x'(t) = \omega\phi_{p^*}(y(t)) &\leq \omega\phi_{p^*}\!\left(\mu^{1/p^*}e^{H(\alpha_2)}e^{-H(t)} - \omega\left(\frac{p\gamma(\eta)}{2}\right)^{1/p^*}e^{-H(t)}\!\int_{\alpha_2}^{t}e^{H(s)}ds\right)\\
&\leq \omega\phi_{p^*}\!\left(\left(\mu^{1/p^*} - \omega\left(\frac{p\gamma(\eta)}{2}\right)^{1/p^*}\right)e^{-H(t)}\!\int_{\alpha_2}^{t}e^{H(s)}ds\right)\\
&\leq -\,\omega\phi_{p^*}\!\left(\frac{\omega}{2}\left(\frac{p\gamma(\eta)}{2}\right)^{1/p^*}\right)\phi_{p^*}\!\left(e^{-H(t)}\!\int_{\alpha_2}^{t}e^{H(s)}ds\right)\\
&= -\,\frac{\omega^{p^*}}{2^{p^*-1}}\left(\frac{p\gamma(\eta)}{2}\right)^{1/p}\phi_{p^*}\!\left(e^{-H(t)}\!\int_{\alpha_2}^{t}e^{H(s)}ds\right)
\end{aligned}
$$

for $\alpha_2 + 1 \leq t \leq \beta_2$. From (5.8) it follows that $x(\alpha_2+1) < 1$. Since $x(\alpha_2+\tau_2) \geq (p\gamma(\eta)/2)^{1/p} > 0$, we conclude that

$$
\begin{aligned}
-\,1 < -\,x(\alpha_2+1) < x(\alpha_2+\tau_2) - x(\alpha_2+1) &= \int_{\alpha_2+1}^{\alpha_2+\tau_2}x'(t)dt\\
&< -\,\frac{\omega^{p^*}}{2^{p^*-1}}\left(\frac{p\gamma(\eta)}{2}\right)^{1/p}\int_{\alpha_2+1}^{\alpha_2+\tau_2}\phi_{p^*}\!\left(e^{-H(t)}\!\int_{\alpha_2}^{t}e^{H(s)}ds\right)dt\\
&= -\,\frac{\omega^{p^*}}{2^{p^*-1}}\left(\frac{p\gamma(\eta)}{2}\right)^{1/p}\int_{\alpha_2+1}^{\alpha_2+\tau_2}\phi_{p^*}\!\left(e^{-H(t)}\left\{\int_{\alpha_2}^{\alpha_2+1}e^{H(s)}ds + \int_{\alpha_2+1}^{t}e^{H(s)}ds\right\}\right)dt\\
&\leq -\,\frac{\omega^{p^*}}{2^{p^*-1}}\left(\frac{p\gamma(\eta)}{2}\right)^{1/p}\int_{\alpha_2+1}^{\alpha_2+\tau_2}\phi_{p^*}\!\left(e^{-H(t)}\!\int_{\alpha_2+1}^{t}e^{H(s)}ds\right)dt.
\end{aligned}
$$

However, from (5.5) with $\sigma = \alpha_2 + 1$ and $t = \tau_2 - 1$, we see that

$$
-\,1 < -\,\frac{\omega^{p^*}}{2^{p^*-1}}\left(\frac{p\gamma(\eta)}{2}\right)^{1/p}\int_{\alpha_2+1}^{\alpha_2+\tau_2}\phi_{p^*}\!\left(e^{-H(t)}\!\int_{\alpha_2+1}^{t}e^{H(s)}ds\right)dt \leq -\,1.
$$

This is a contradiction. Thus, it turns out that the beginning sentence of this step is true.
Step 3: We can proceed the argument in the same way as the proof of Theorem 1.1 by using $|y(t)|^{p^*}$ instead of $y^2(t)$. As a result, we obtain

$$
\frac{\mu}{2\,p^*\omega} < t_2 - t_1, \tag{5.9}
$$

where t_1 and t_2 are numbers satisfying $|y(t_1)|^{p^*} = \mu/2$, $|y(t_2)|^{p^*} = \mu$ and

$$
\frac{\mu}{2} \leq |y(t_1)|^{p^*} \leq \mu \quad \text{for } \ t_1 \leq t \leq t_2. \tag{5.10}
$$

Step 4: Using (5.6), (5.9) and (5.10), we obtain

$$
\int_{J_i} v'(t)dt \leq -\,\nu,
$$

where

$$J_i = \big[t_0 + \tau_3 + (i-1)(\tau_1+\tau_2),\ t_0 + \tau_3 + i\,(\tau_1+\tau_2)\big]$$

for $i = 2, 3, \dots, [1/\nu] + 1$. We leave the detailed analysis to the reader.

This means that the loss of the total energy $v(t)$ in each interval J_i is at least ν. Hence, we obtain

$$\begin{aligned} -v(t_0+\tau_3) \le v(t_0+T) - v(t_0+\tau_3) &= \sum_{i=1}^{[1/\nu]+1} \int_{J_i} v'(t)dt \\ &\le -\nu\left(\left[\frac{1}{\nu}\right]+1\right) < -1, \end{aligned}$$

which contradicts (5.8). Thus, inequality (5.7) was proved. The proof of Theorem 5.1 is now complete. □

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