Optimality conditions for nonlinear and nonconvex programming problems

Yusuke Saeki

Interdisciplinary Graduate School of Science and Engineering
Shimane University

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Preface

In the study of mathematical programming, many mathematicians have considered optimality conditions for mathematical programming problems. In differentiable programming problems [2, 4, 6, 13, 16] which include linear programming problems and are nonlinear programming problems, the Karush-Kuhn-Tucker condition is the best-known necessary optimality condition under certain qualifications imposed on the constraints. Such constraint qualifications have been proposed, for example, linear independence constraint qualification, Cottle's constraint qualification, Abadie's constraint qualification, and Guignard's constraint qualification. In particular, Guignard's constraint qualification is necessary and sufficient for the Karush-Kuhn-Tucker necessary optimality conditions in differentiable programming problems, see [2].

In convex programming problems [12, 18, 19, 21, 25, 26, 27, 28], Slater condition, Farkas-Minkowski and the basic constraint qualification are known as constraint qualifications. Farkas-Minkowski is a necessary and sufficient constraint qualification for the Lagrangian strong duality in convex programming problems, see [26]. Also, the basic constraint qualification is a necessary and sufficient constraint qualification for global optimality conditions in convex programming problems, see [28]. Nonconvex programming problems involving convex programming problems have been extensively researched, for example, DC programming problems [8, 11, 14, 23, 29, 30, 31], fractional programming problems [1, 11, 22] and mathematical programming problems with reverse convex constraints [7, 10, 20].

In this thesis, we consider optimality conditions for nonlinear and nonconvex programming problems appeared in the following chapters and constraint qualifications for these conditions. The purpose of the thesis is to consider necessary and sufficient constraint qualifications for optimality conditions in these mathematical programming problems. The thesis is divided into five chapters and is organized as follows.

In Chapter 1, we introduce a real topological vector space, and definitions and notations in convex analysis. Also, we introduce results related to convex analysis. Moreover, we introduce some definitions and notations in the Euclidean space.

In Chapter 2, we show that the basic constraint qualification is a necessary and sufficient constraint qualification for local optimality conditions in DC programming problems with convex inequality constraints in a real locally convex Hausdorff topological vector space. Also, we show that the basic constraint qualification is a necessary and sufficient constraint qualification for local optimality conditions in fractional programming problems and weakly convex programming problems.

In Chapter 3, we give necessary local and sufficient global optimality conditions for DC programming problems with reverse convex constraints in the Euclidean space. Also, we show that a certain condition is necessary and sufficient for sufficient global optimality conditions in DC programming problems with reverse convex constraints. Moreover, we give optimality conditions for DC programming problems and fractional programming problems.

In Chapter 4, we show that Farkas-Minkowski is a necessary and sufficient constraint qualification for ε-optimality conditions in DC programming problems with convex inequality constraints in a real locally convex Hausdorff topological vector space. Also, we show that Farkas-Minkowski is a necessary and sufficient constraint qualification for ε-optimality conditions in fractional programming problems.

In Chapter 5, we investigate a necessary and sufficient constraint qualification for sufficient optimality conditions in differentiable programming, where the objective function is pseudoconvex at a point. Also, we observe necessary and sufficient constraint qualifications for sufficient conditions for Pareto optimality and weak Pareto optimality in differentiable multiobjective programming, where the components of the objective function or the linear combination of them is assumed some convexity condition.

Contents

Chapter 1

Preliminaries

In this chapter, we introduce a real topological vector space and notations in convex analysis. Also, we introduce results related to convex analysis. Moreover, we introduce some definitions and notations in the Euclidean space.

1.1 Topological vector space and notations

A real topological vector space is a vector space over $\mathbb{R}$ equipped with a topology such that the vector space operations of addition and scalar multiplication are continuous. Also, let X^* be the topological dual space of the space X endowed with the weak*-topology, and let $\langle x^*, x\rangle$ denote the value of a functional $x^* \in X^*$ at $x \in X$, that is, $\langle x^*, x\rangle = x^*(x)$. A real locally convex topological vector space is a real topological vector space which has a local basis at 0 consisting of convex sets. Let X be a real locally convex Hausdorff topological vector space.

Definition 1.1. Let A be a subset of X and let B be a nonempty subset of X.

(i) A is said to be convex if $(1-\alpha)x+\alpha y \in A$ whenever $x, y \in A$ and $\alpha \in (0, 1)$,

(ii) B is said to be a cone if $\lambda k \in B$ whenever $k \in B$ and $\lambda \geq 0$.

It is clear that the whole space and the empty set are convex. For a set $A \subset X$, we denote the closure, the interior, the convex hull and the conical hull of A, by $\operatorname{cl} A$, $\operatorname{int} A$, $\operatorname{co} A$ and $\operatorname{cone} A$, respectively.

The effective domain and epigraph of $f : X \to \mathbb{R} \cup \{+\infty\}$ are defined by

$$\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\},$$

and

$$\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\},$$

respectively.

Definition 1.2. Let f be a function from X to $\mathbb{R} \cup \{+\infty\}$.

(i) f is said to be convex if $\operatorname{epi} f$ is convex,

(ii) f is said to be proper if $\operatorname{dom} f$ is nonempty.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. The conjugate function of f, $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) \mid x \in X\}$$

for each $x^* \in X^*$. For any $\varepsilon \geq 0$, the ε-subdifferential of f at $x \in \operatorname{dom} f$, denoted by $\partial_\varepsilon f(x)$, is defined by

$$\partial_\varepsilon f(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon \text{ for each } y \in X\}.$$

When $\varepsilon = 0$, $\partial_0 f(x)$ is the subdifferential of f at x, and is often denoted by $\partial f(x)$.

Let A be a convex set in X. The indicator function δ_A is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

For any $\varepsilon \geq 0$, the ε-normal cone of A at $x \in A$, denoted by $N_\varepsilon(A, x)$, is defined by

$$N_\varepsilon(A, x) = \partial_\varepsilon \delta_A(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq \varepsilon \text{ for each } y \in A\}.$$

When $\varepsilon = 0$, $N_0(A, x)$ is the normal cone of A at x and is often denoted by $N_A(x)$.

1.2 Results in convex analysis

We introduce important results in convex analysis. First, the following theorem was proved in [12].

Theorem 1.1 ([12]). Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (lsc) proper convex function and $x \in \operatorname{dom} f$. Then

$$\operatorname{epi} f^* = \bigcup_{\varepsilon \geq 0} \{(v, \varepsilon - f(x) + \langle v, x\rangle) \mid v \in \partial_\varepsilon f(x)\}.$$

Next, the following theorem was shown in [24].

Theorem 1.2 ([24]). Let f and g be lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Then the following statements are equivalent:

(a) $\operatorname{epi} f^* + \operatorname{epi} g^*$ is weak*-closed.

(b) for any $\epsilon \geq 0$ and $x \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\partial_\epsilon (f+g)(x) = \bigcup_{\substack{\epsilon_1, \epsilon_2 \geq 0 \\ \epsilon_1 + \epsilon_2 = \epsilon}} (\partial_{\epsilon_1} f(x) + \partial_{\epsilon_2} g(x)).$$

In particular, if $\operatorname{epi} f^* + \operatorname{epi} g^*$ is weak*-closed, then for any $x \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\partial(f+g)(x) = \partial f(x) + \partial g(x).$$

Finally, we introduce the following optimality conditions for unconstrained DC (difference of convex functions) programming problems by Hiriart-Urruty [8]. For a function $f : X \to \mathbb{R} \cup \{+\infty\}$, a set $A \subset X$ and $\varepsilon \geq 0$, we say that a point $\bar{x} \in A$ is an ε-minimizer of f in A if f is finite at $\bar{x}$ and $f(x) \geq f(\bar{x}) - \varepsilon$ for each $x \in A$.

Theorem 1.3 ([8]). Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc proper convex function and $g : X \to \mathbb{R}$ be a lsc convex function. The following statements hold.

(i) If $\bar{x} \in X$ is a local minimizer of $f - g$ in X, then

$$\partial g(\bar{x}) \subset \partial f(\bar{x}).$$

(ii) For any $\varepsilon \geq 0$, $\bar{x} \in X$ is an ε-minimizer of $f - g$ in X if and only if for each $\alpha \geq 0$,

$$\partial_\alpha g(\bar{x}) \subset \partial_{\alpha+\varepsilon} f(\bar{x}).$$

In particular, $\bar{x} \in X$ is a minimizer of $f - g$ in X if and only if for each $\alpha \geq 0$,

$$\partial_\alpha g(\bar{x}) \subset \partial_\alpha f(\bar{x}).$$

1.3 Notations in the Euclidean space

The usual inner product of two vectors x and y in the n-dimensional Euclidean space $\mathbb{R}^n$ is denoted by $\langle x, y \rangle$, and the norm of a vector x in $\mathbb{R}^n$ is denoted by $\|x\|$.

Remark 1.1. Since $(\mathbb{R}^n)^*$ can be identified with $\mathbb{R}^n$, for convenience, this inner product is denoted by $\langle x^*, x \rangle$ which is the value of a functional $x^* \in (\mathbb{R}^n)^*$ at $x \in \mathbb{R}^n$ introduced in Section 1.1.

Let A be a nonempty subset of $\mathbb{R}^n$. The positive polar cone A^+ and the negative polar cone A^- are defined by

$$A^+ = \{x^* \in \mathbb{R}^n \mid \langle x^*, x \rangle \geq 0 \text{ for each } x \in A\},$$

and

$$A^- = \{x^* \in \mathbb{R}^n \mid \langle x^*, x\rangle \leq 0 \text{ for each } x \in A\},$$

respectively. We note that A^+ and A^- are closed convex and

$$(A^+)^+ = (A^-)^- = \mathrm{cl}\ \mathrm{cone}\ \mathrm{co}\ A.$$

The tangent cone to A at $x \in A$, denoted by $T_A(x)$, is defined by

$$T_A(x) = \{d \in \mathbb{R}^n \mid \text{there exist } t_k \downarrow 0,\ d_k \to d \text{ such that } x + t_k d_k \in A\}.$$

The tangent cone $T_A(x)$ is always closed but not necessarily convex. If B is a subset of $\mathbb{R}^n$ and $x \in A \cap B$, then

$$A \subset B \Rightarrow T_A(x) \subset T_B(x).$$

The normal cone to A at $x \in A$, denoted by $N_A(x)$, is defined by

$$N_A(x) = (T_A(x))^-.$$

Remark 1.2. Since this normal cone becomes the normal cone introduced in Section 1.1 whenever A is convex, this normal cone is denoted by $N_A(x)$ for convenience.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Define level sets of f with respect to a binary relation $\diamond$ on $\mathbb{R} \cup \{+\infty\}$ as

$$L(f, \diamond, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \diamond \alpha\},$$

for each $\alpha \in \mathbb{R}$. The directional derivative of f at $x \in \mathrm{dom}\, f$ in direction of $d \in \mathbb{R}^n$ is defined by

$$f'(x; d) = \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t},$$

where $\mathrm{dom}\, f$ is the effective domain of f.

We introduce the concepts of convexity, quasiconvexity and pseudoconvexity at a point, see [5].

Definition 1.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. The function f is said to be

(i) convex at $\bar{x}$, if for each $\lambda \in (0, 1)$ and $x \in \mathbb{R}^n$,

$$f((1 - \lambda)\bar{x} + \lambda x) \leq (1 - \lambda) f(\bar{x}) + \lambda f(x);$$

(ii) strictly convex at $\bar{x}$, if for each $\lambda \in (0, 1)$ and $x \in \mathbb{R}^n \setminus \{\bar{x}\}$,

$$f((1 - \lambda)\bar{x} + \lambda x) < (1 - \lambda) f(\bar{x}) + \lambda f(x);$$

(iii) quasiconvex at $\bar{x}$, if for each $\lambda \in (0,1)$ and $x \in \mathbb{R}^n$,

$$f((1-\lambda)\bar{x} + \lambda x) \leq \max\{f(\bar{x}), f(x)\};$$

(iv) pseudoconvex at $\bar{x}$, if f is differentiable at $\bar{x}$, and for each $x \in \mathbb{R}^n$,

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0 \text{ implies that } f(x) \geq f(\bar{x});$$

(v) strictly pseudoconvex at $\bar{x}$, if f is differentiable at $\bar{x}$, and for each $x \in \mathbb{R}^n \setminus \{\bar{x}\}$,

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0 \text{ implies that } f(x) > f(\bar{x}).$$

Proposition 1.1. The following statements hold.

(i) If f is both convex and differentiable at $\bar{x}$, then for each $x \in \mathbb{R}^n$,

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq f(x) - f(\bar{x}).$$

(ii) If f is both strictly convex and differentiable at $\bar{x}$, then for each $x \in \mathbb{R}^n \setminus \{\bar{x}\}$,

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle < f(x) - f(\bar{x}).$$

(iii) If f is both quasiconvex and differentiable at $\bar{x}$, then for each $x \in \mathbb{R}^n$,

$$f(x) \leq f(\bar{x}) \text{ implies that } \langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq 0.$$

The above statements are analogous to well-known results in [5], and proofs are omitted. The following result is important to show theorems in Chapter 5; (iv) and (v) follow from Proposition 1.1 immediately.

Proposition 1.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. Then the following statements hold.

(i) If f is strictly convex at $\bar{x}$ then f is convex at $\bar{x}$.

(ii) If f is convex at $\bar{x}$ then f is quasiconvex at $\bar{x}$.

(iii) If f is strictly pseudoconvex at $\bar{x}$ then f is pseudoconvex at $\bar{x}$.

(iv) If f is convex and differentiable at $\bar{x}$ then f is pseudoconvex at $\bar{x}$.

(v) If f is strictly convex and differentiable at $\bar{x}$ then f is strictly pseudoconvex at $\bar{x}$.

Chapter 2

Local optimality for DC programming

In this chapter, we investigate a constraint qualification for local optimality conditions in DC programming problems with convex inequality constraints. Throughout this chapter, let X be a real locally convex Hausdorff topological vector space, and we consider mathematical programming problems under the following constraint set:

$$S = \{x \in X \mid h_i(x) \leq 0 \text{ for each } i \in I\},$$

where I is an arbitrary index set and $h_i : X \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, are lsc proper convex functions. This chapter is based on [32].

We introduce the basic constraint qualification (the BCQ) that is a necessary and sufficient constraint qualification for global optimality conditions in convex programming problems by Li, Ng and Pong [28].

Definition 2.1 ([28])**.** Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$. The family $\{h_i \mid i \in I\}$ is said to satisfy the BCQ at $\bar{x} \in S$ if

$$N_S(\bar{x}) = \operatorname{cone} \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial h_i(\bar{x}),$$

where $I(\bar{x}) = \{i \in I \mid h_i(\bar{x}) = 0\}$.

Theorem 2.1 ([28])**.** Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Then the following statements are equivalent:

(i) The family $\{h_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$.

(ii) For each lsc proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom} f \cap S \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, $\bar{x}$ is a minimizer of f in S if and only if there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial h_i(\bar{x}),$$

where $\mathbb{R}_+^{(I)}$ is the set of nonnegative real tuples $\lambda = (\lambda_i)_{i\in I}$ with only finitely many $\lambda_i \neq 0$.

This theorem shows that the BCQ is a necessary and sufficient constraint qualification for global optimality conditions in convex programming problems.

2.1 DC programming

In this section, we consider the following DC programming problem:

$$\begin{array}{ll} \text{minimize} & f(x) - g(x), \\ \text{subject to} & h_i(x) \leq 0, \ i \in I, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lsc proper convex function and $g : X \to \mathbb{R}$ is a lsc convex function.

Theorem 2.2. Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Then the following statements are equivalent:

(i) The family $\{h_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$.

(ii) For each lsc proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom} f \cap S \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, and lsc convex function $g : X \to \mathbb{R}$, if $\bar{x}$ is a local minimizer of $f - g$ in S, then for each $v \in \partial g(\bar{x})$, there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$v \in \partial f(\bar{x}) + \sum_{i\in I} \lambda_i \partial h_i(\bar{x}).$$

Proof. First, we prove (i) implies (ii). Assume that (i) holds. Let f be a lsc proper convex function from X to $\mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom} f \cap S \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, and g be a lsc convex function from X to $\mathbb{R}$. The point $\bar{x}$ is a local minimizer of $f - g$ in S if and only if $\bar{x}$ is a local minimizer of $(f + \delta_S) - g$ in X. We have from Theorem 1.3 that if $\bar{x}$ is a local minimizer of $(f + \delta_S) - g$ in X, then

$$\partial g(\bar{x}) \subset \partial (f + \delta_S)(\bar{x}).$$

By Theorem 1.2,

$$\partial (f + \delta_S)(\bar{x}) = \partial f(\bar{x}) + \partial \delta_S(\bar{x}).$$

Since $\partial \delta_S(\bar{x}) = N_S(\bar{x})$ and the assumption (i) holds,

$$\partial f(\bar{x}) + \partial \delta_S(\bar{x}) = \partial f(\bar{x}) + \operatorname{cone} \operatorname{co} \bigcup_{i\in I(\bar{x})} \partial h_i(\bar{x}).$$

Hence, if $\bar{x}$ is a local minimizer of $f-g$ in S, then

$$\partial g(\bar{x}) \subset \partial f(\bar{x}) + \operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial h_i(\bar{x}).$$

This implies that (ii) holds.

Next, we prove (ii) implies (i). Assume that (ii) holds and let $x^* \in N_S(\bar{x})$. Then $\bar{x}$ is a minimizer of $-x^*$ in S. By setting $f = -x^*$ and $g = 0$ in assumption (ii), there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$0 \in -x^* + \sum_{i \in I} \lambda_i \partial h_i(\bar{x}).$$

Therefore, we have

$$x^* \in \sum_{i \in I} \lambda_i \partial h_i(\bar{x}) = \sum_{i \in I(\bar{x})} \lambda_i \partial h_i(\bar{x}) \subset \operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial h_i(\bar{x}),$$

and hence $N_S(\bar{x}) \subset \operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial h_i(\bar{x})$ holds. Since the converse inclusion is always satisfied, (i) holds. This completes the proof. □

This theorem shows that the BCQ is a necessary and sufficient constraint qualification for local optimality conditions in DC programming problems with convex inequality constraints.

2.2 Applications

In this section, we apply the result of previous section to fractional programming problems and weakly convex programming problems. In particular, we consider weakly convex programming problems in a smooth real Banach space.

2.2.1 Fractional programming

We consider the following fractional programming problem:

$$\begin{array}{ll} \text{minimize} & f(x)/g(x), \\ \text{subject to} & h_i(x) \leq 0, \ i \in I, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lsc proper convex function and $g : X \to \mathbb{R}$ is a lsc convex function such that f is nonnegative and g is positive on S.

Theorem 2.3. Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Then the following statements are equivalent:

(i) The family $\{h_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$.

(ii) For each lsc proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom} f \cap S \neq \emptyset$, $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed and f is nonnegative on S, and lsc convex function $g : X \to \mathbb{R}$ such that g is positive on S, if $\bar{x}$ is a local minimizer of f/g in S, then there exists $\lambda_0 \geq 0$ such that for each $v \in \lambda_0 \partial g(\bar{x})$, there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$v \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial h_i(\bar{x}).$$

Proof. We first prove (i) implies (ii). Let f be a lsc proper convex function from X to $\mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom} f \cap S \neq \emptyset$, $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed and f is nonnegative on S, and g be a lsc convex function from X to $\mathbb{R}$ such that g is positive on S. In addition, let $\bar{x}$ be a local minimizer of f/g in S. By putting $\lambda_0 = f(\bar{x})/g(\bar{x})$, $\bar{x}$ is a local minimizer of $f - \lambda_0 g$ in S. Because $f - \lambda_0 g$ is a DC function, we can prove (i) implies (ii) by using Theorem 2.2. Also, it is clear that (ii) implies (i) by taking $f = -x^* + \langle x^*, \bar{x} \rangle$ and $g = 1$. □

This theorem shows that the BCQ is also a necessary and sufficient constraint qualification for the fractional programming problems.

2.2.2 Weakly convex programming

Let X be a real Banach space with norm $\|\cdot\|$. The norm of X^* is also denoted by $\|\cdot\|$ for convenience. The duality mapping of X, the multivalued operator $J : X \to X^*$, is defined by

$$J(x) = \{x^* \in X^* \mid \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in X$. Let $S(X)$ denote the unit sphere of X, that is, $S(X) = \{x \in X \mid \|x\| = 1\}$. Then X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(X)$. In this case, because the duality mapping J of X is single valued, $J(x)$ is identified with the element of $J(x)$ for each $x \in X$; see [15].

Recall that a function p is weakly convex if it can be written as $p = q - \frac{\rho}{2}\|\cdot\|^2$ for some convex function q and $\rho \geq 0$. We consider the following weakly convex programming problem:

$$\begin{array}{ll} \text{minimize} & f(x) - \frac{\rho}{2}\|x\|^2, \\ \text{subject to} & h_i(x) \leq 0, \ i \in I, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lsc proper convex function and $\rho \geq 0$.

We show the following theorem in a smooth real Banach space.

Theorem 2.4. Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Assume that X is smooth. Then the following statements are equivalent:

(i) The family $\{h_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$.

(ii) For each lsc proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom} f \cap S \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, and $\rho \geq 0$, if $\bar{x}$ is a local minimizer of $f - \frac{\rho}{2}\|\cdot\|^2$ in S, then there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and
$$\rho J(\bar{x}) \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial h_i(\bar{x}).$$

Proof. Since X is smooth, J is single valued. By taking g as $\frac{\rho}{2}\|\cdot\|^2$ in Theorem 2.2, we can prove (i) implies (ii) because $\partial g(\bar{x}) = \rho J(\bar{x})$. Also, it is clear that (ii) implies (i). $\square$

Example 2.1. Consider the problem:

$$\begin{array}{ll} \text{minimize} & \frac{1}{4}x^4 + |x| - x^2, \\ \text{subject to} & \max\{0, -x\} \leq 0. \end{array}$$

Let $X = \mathbb{R}$, $I = \{1\}$, $f(x) = \frac{1}{4}x^4 + |x|$, $\rho = 2$, $h_1(x) = \max\{0, -x\}$ and $S = [0, +\infty)$. Then f and h_1 are continuous convex functions and $\{h_i \mid i \in I\}$ satisfies the BCQ at each point of S. Let $\bar{x}$ be a local minimizer of $f(x) - \frac{\rho}{2}x^2$ in S. By Theorem 2.4, there exists $\lambda_1 \geq 0$ such that $\rho\bar{x} \in \partial f(\bar{x}) + \lambda_1 \partial h_1(\bar{x})$ and $\lambda_1 h_1(\bar{x}) = 0$, because J is an identity map for X. When $\bar{x} > 0$, since $\partial f(\bar{x}) = \bar{x}^3 + \{1\}$ and $\partial h_1(\bar{x}) = \{0\}$, $\bar{x}$ must be 1 or $\frac{-1+\sqrt{5}}{2}$. They also satisfy $\lambda_1 h_1(\bar{x}) = 0$. Otherwise, when $\bar{x} = 0$, since $\partial f(\bar{x}) = \bar{x}^3 + [-1, 1]$ and $\partial h_1(\bar{x}) = [-1, 0]$, $\bar{x} \in [-\lambda_1 - 1, 1]$ holds whenever $\lambda_1 \geq 0$. Also, $\bar{x} = 0$ satisfies $\lambda_1 h_1(\bar{x}) = 0$. Therefore 0, 1 and $\frac{-1+\sqrt{5}}{2}$ have possibilities for local minimizers, and actually, 0 is the global minimizer and 1 is a local minimizer. But $\frac{-1+\sqrt{5}}{2}$ is neither a minimizer nor a local minimizer.

Chapter 3

DC programming with reverse convex constraints

In this chapter, we consider optimality conditions for DC programming problems with reverse convex constraints in the Euclidean space $\mathbb{R}^n$. This chapter is based on [33].

3.1 The tangent cones to upper level sets

In this section, we consider the tangent cones to the upper level sets of pseudoconvex functions. Recall that a differentiable function h is pseudoconvex if $\langle \nabla h(x), y - x\rangle \geq 0$ implies $h(y) \geq h(x)$. We note that a differentiable convex function is pseudoconvex. We first introduce the following theorem established by Bazaraa, Goode and Nashed [3].

Theorem 3.1 ([3])**.** Let $h : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function at $\bar{x} \in \mathbb{R}^n$. Assume that $\nabla h(\bar{x}) \neq 0$. Then

$$T_{L(h,\leq,h(\bar{x}))}(\bar{x}) = \{d \in \mathbb{R}^n \mid \langle \nabla h(\bar{x}), d\rangle \leq 0\},$$

$$T_{L(h,\geq,h(\bar{x}))}(\bar{x}) = \{d \in \mathbb{R}^n \mid \langle \nabla h(\bar{x}), d\rangle \geq 0\}.$$

In the following theorem, we prove a characterization of the tangent cone to the upper level set of a pseudoconvex function without $\nabla h(\bar{x}) \neq 0$.

Theorem 3.2. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a pseudoconvex function and $\bar{x} \in \mathbb{R}^n$. Then

$$T_{L(h,\geq,h(\bar{x}))}(\bar{x}) = \{d \in \mathbb{R}^n \mid \langle \nabla h(\bar{x}), d\rangle \geq 0\}.$$

Proof. When $\nabla h(\bar{x}) = 0$, by the pseudoconvexity of h, we have $L(h, \geq, h(\bar{x})) = \mathbb{R}^n$. Thus, $T_{L(h,\geq,h(\bar{x}))}(\bar{x}) = \mathbb{R}^n$ and $\{d \in \mathbb{R}^n \mid \langle \nabla h(\bar{x}), d\rangle \geq 0\} = \mathbb{R}^n$. When $\nabla h(\bar{x}) \neq 0$, the conclusion follows from Theorem 3.1. This completes the proof. $\square$

Theorem 3.3. Let $I = \{1, 2, \dots, m\}$, $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be pseudoconvex functions and $\bar{x} \in \mathbb{R}^n$. Then

$$T_{\cap_{i\in I} L(h_i,\geq,h_i(\bar{x}))}(\bar{x}) = \cap_{i\in I} T_{L(h_i,\geq,h_i(\bar{x}))}(\bar{x}).$$

Proof. For each $i \in I$, since $\cap_{j\in I} L(h_j, \geq, h_j(\bar{x})) \subset L(h_i, \geq, h_i(\bar{x}))$, we have

$$T_{\cap_{j\in I} L(h_j,\geq,h_j(\bar{x}))}(\bar{x}) \subset T_{L(h_i,\geq,h_i(\bar{x}))}(\bar{x}).$$

Therefore,

$$T_{\cap_{i\in I} L(h_i,\geq,h_i(\bar{x}))}(\bar{x}) \subset \cap_{i\in I} T_{L(h_i,\geq,h_i(\bar{x}))}(\bar{x}).$$

Conversely, let $d \in \cap_{i\in I} T_{L(h_i,\geq,h_i(\bar{x}))}(\bar{x})$. From Theorem 3.2, $\langle \nabla h_i(\bar{x}), d\rangle \geq 0$ for each $i \in I$. For each $k \in \mathbb{N}$ and $i \in I$, since $\langle \nabla h_i(\bar{x}), \bar{x} + (1/k)d - \bar{x}\rangle \geq 0$ and the pseudoconvexity of h_i, we have $h_i(\bar{x} + (1/k)d) \geq h_i(\bar{x})$, that is, $\bar{x} + (1/k)d \in L(h_i, \geq, h_i(\bar{x}))$. Thus $\bar{x} + (1/k)d \in \cap_{i\in I} L(h_i, \geq, h_i(\bar{x}))$ and $d \in T_{\cap_{i\in I} L(h_i,\geq,h_i(\bar{x}))}(\bar{x})$. Hence,

$$\cap_{i\in I} T_{L(h_i,\geq,h_i(\bar{x}))}(\bar{x}) \subset T_{\cap_{i\in I} L(h_i,\geq,h_i(\bar{x}))}(\bar{x}),$$

and this completes the proof. □

In the following example, we can see that Theorem 3.3 is satisfied.

Example 3.1. Let $n = 2$, $I = \{1, 2\}$, $h_1(x_1, x_2) = x_1^2 - x_2$, $h_2(x_1, x_2) = x_1^2 + x_2$ and $\bar{x} = (0, 0)$. Then h_1 and h_2 are differentiable convex functions, $L(h_1, \geq, h_1(\bar{x})) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 \geq x_2\}$ and $L(h_2, \geq, h_2(\bar{x})) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq -x_1^2\}$. Further, $T_{L(h_1,\geq,h_1(\bar{x}))}(\bar{x}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$ and $T_{L(h_2,\geq,h_2(\bar{x}))}(\bar{x}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$. Thus, $T_{\cap_{i=1}^2 L(h_i,\geq,h_i(\bar{x}))}(\bar{x}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0\} = \cap_{i=1}^2 T_{L(h_i,\geq,h_i(\bar{x}))}(\bar{x})$.

3.2 Local optimality conditions

In this section, we consider necessary local optimality conditions for DC programming problems with a set constraint and reverse convex constraints.

First, we consider the following DC programming problem with a set constraint:

$$\begin{array}{ll} \text{minimize} & f(x) - g(x), \\ \text{subject to} & x \in S, \end{array}$$

where f, $g : \mathbb{R}^n \to \mathbb{R}$ are convex functions and S is a subset of $\mathbb{R}^n$.

We show the following theorem by using some results related to convex analysis, for example, see [17].

Theorem 3.4. Let f, $g : \mathbb{R}^n \to \mathbb{R}$ be convex functions, S be a subset of $\mathbb{R}^n$ and $\bar{x} \in S$. Assume that $T_S(\bar{x})$ is a convex set. If $\bar{x}$ is a local minimizer of $f - g$ in S, then

$$\partial g(\bar{x}) \subset \partial f(\bar{x}) + N_S(\bar{x}).$$

Proof. Suppose that there exists $v_0 \in \partial g(\bar{x})$ such that $v_0 \notin \partial f(\bar{x}) + N_S(\bar{x})$. Since $\partial f(\bar{x}) + N_S(\bar{x})$ is a closed convex set, by the separation theorem, there exists $d_0 \in \mathbb{R}^n$ such that

$$\langle x^* + y^*, d_0 \rangle < \langle v_0, d_0 \rangle \,,$$

for each $x^* \in \partial f(\bar{x})$ and $y^* \in N_S(\bar{x})$. Therefore,

$$\langle x^* - v_0, d_0 \rangle < 0 \leq \langle -y^*, d_0 \rangle \,,$$

for each $x^* \in \partial f(\bar{x})$ and $y^* \in N_S(\bar{x})$. Because $\langle y^*, d_0 \rangle \leq 0$ for each $y^* \in N_S(\bar{x})$ and $T_S(\bar{x})$ is a convex set, $d_0 \in T_S(\bar{x})$. Then there exist $t_k \downarrow 0$ and $d_k \to d_0$ such that $\bar{x} + t_k d_k \in S$. Since $\bar{x}$ is a local minimizer of $f - g$ in S,

$$g(\bar{x} + t_k d_k) - g(\bar{x}) \leq f(\bar{x} + t_k d_k) - f(\bar{x}),$$

for large enough k.

Now, we show that $f'(\bar{x}; d_0) = \lim_{k\to\infty}(f(\bar{x}+t_k d_k) - f(\bar{x}))/t_k$. Since f is locally Lipschitz at $\bar{x}$, there exists $K > 0$ such that $|f(\bar{x}+t_k d_k) - f(\bar{x}+t_k d_0)| \leq K\|t_k(d_k - d_0)\|$ for large enough k. Therefore, $\lim_{k\to\infty}(f(\bar{x}+t_k d_k) - f(\bar{x}+t_k d_0))/t_k = 0$. Thus,

$$\begin{aligned}
\lim_{k\to\infty} \tfrac{f(\bar{x}+t_k d_k) - f(\bar{x})}{t_k} &= \lim_{k\to\infty} \tfrac{f(\bar{x}+t_k d_k) - f(\bar{x}+t_k d_0) + f(\bar{x}+t_k d_0) - f(\bar{x})}{t_k} \\
&= \lim_{k\to\infty} \tfrac{f(\bar{x}+t_k d_k) - f(\bar{x}+t_k d_0)}{t_k} + \lim_{k\to\infty} \tfrac{f(\bar{x}+t_k d_0) - f(\bar{x})}{t_k} \\
&= f'(\bar{x}; d_0).
\end{aligned}$$

Similarly, we can show that $g'(\bar{x}; d_0) = \lim_{k\to\infty}(g(\bar{x}+t_k d_k) - g(\bar{x}))/t_k$. Hence,

$$g'(\bar{x}; d_0) = \lim_{k\to\infty} \frac{g(\bar{x}+t_k d_k) - g(\bar{x})}{t_k} \leq \lim_{k\to\infty} \frac{f(\bar{x}+t_k d_k) - f(\bar{x})}{t_k} = f'(\bar{x}; d_0).$$

Since $f'(\bar{x}; d_0) = \sup_{x^* \in \partial f(\bar{x})} \langle x^*, d_0 \rangle$ and $\partial f(\bar{x})$ is compact, there exists $x_0^* \in \partial f(\bar{x})$ such that $f'(\bar{x}; d_0) = \langle x_0^*, d_0 \rangle$. Thus,

$$\langle v_0, d_0 \rangle \leq g'(\bar{x}; d_0) \leq f'(\bar{x}; d_0) = \langle x_0^*, d_0 \rangle \,,$$

because $v_0 \in \partial g(\bar{x})$. Then $\langle x_0^* - v_0, d_0 \rangle \geq 0$, and this is a contradiction. □

Next, we consider the following DC programming problem with reverse convex constraints:

$$\begin{array}{ll}
\text{minimize} & f(x) - g(x), \\
\text{subject to} & h_i(x) \geq 0, \ i \in I,
\end{array}$$

where $I = \{1, 2, \ldots, m\}$, f, $g : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are pseudoconvex functions.

Lemma 3.1. Let $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be pseudoconvex functions, $S = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0 \text{ for each } i \in I\}$ and $\bar{x} \in S$. Then

(i) $T_S(\bar{x}) = \cap_{i\in I(\bar{x})}\{d \in \mathbb{R}^n \mid \langle \nabla h_i(\bar{x}), d\rangle \geq 0\}$,

(ii) $N_S(\bar{x}) = \operatorname{cone}\operatorname{co} \cup_{i\in I(\bar{x})} \{-\nabla h_i(\bar{x})\}$,

where $I(\bar{x}) = \{i \in I \mid h_i(\bar{x}) = 0\}$.

Proof. (i) We first prove that

$$T_{\cap_{i\in I(\bar{x})}L(h_i,\geq,0)}(\bar{x}) \subset T_S(\bar{x}).$$

Let $d \in T_{\cap_{i\in I(\bar{x})}L(h_i,\geq,0)}(\bar{x})$, then there exist $t_k \downarrow 0$ and $d_k \to d$ such that $\bar{x} + t_kd_k \in \cap_{i\in I(\bar{x})}L(h_i,\geq,0)$. For each $i \notin I(\bar{x})$, since $h_i(\bar{x}) > 0$ and h_i is lsc, there exists $\delta_i > 0$ such that $h_i(x) \geq 0$ for each $x \in \mathbb{R}^n$ with $\|x - \bar{x}\| < \delta_i$. Put $\delta = \min_{i\notin I(\bar{x})} \delta_i > 0$, then $h_i(x) \geq 0$ for each $x \in \mathbb{R}^n$ with $\|x - \bar{x}\| < \delta$ and $i \notin I(\bar{x})$. Since $\bar{x} + t_kd_k \to \bar{x}$, for large enough $k \in \mathbb{N}$, $h_i(\bar{x} + t_kd_k) \geq 0$ for each $i \notin I(\bar{x})$, that is, $\bar{x} + t_kd_k \in \cap_{i\notin I(\bar{x})}L(h_i,\geq,0)$. Thus, $\bar{x} + t_kd_k \in S$ and $d \in T_S(\bar{x})$. Hence,

$$\begin{array}{rcl} T_S(\bar{x}) & = & T_{\cap_{i\in I(\bar{x})}L(h_i,\geq,0)}(\bar{x}) \\ & = & T_{\cap_{i\in I(\bar{x})}L(h_i,\geq,h_i(\bar{x}))}(\bar{x}) \\ & = & \cap_{i\in I(\bar{x})}T_{L(h_i,\geq,h_i(\bar{x}))}(\bar{x}) \\ & = & \cap_{i\in I(\bar{x})}\{d \in \mathbb{R}^n \mid \langle \nabla h_i(\bar{x}), d\rangle \geq 0\}, \end{array}$$

from Theorems 3.2 and 3.3.

(ii) From (i),

$$\begin{array}{rcl} N_S(\bar{x}) & = & (T_S(\bar{x}))^- \\ & = & (\cap_{i\in I(\bar{x})}\{d \in \mathbb{R}^n \mid \langle \nabla h_i(\bar{x}), d\rangle \geq 0\})^- \\ & = & (\cap_{i\in I(\bar{x})}\{d \in \mathbb{R}^n \mid \langle -\nabla h_i(\bar{x}), d\rangle \leq 0\})^- \\ & = & (\cap_{i\in I(\bar{x})}\{-\nabla h_i(\bar{x})\}^-)^- \\ & = & ((\cup_{i\in I(\bar{x})}\{-\nabla h_i(\bar{x})\})^-)^- \\ & = & \operatorname{cl}\operatorname{cone}\operatorname{co} \cup_{i\in I(\bar{x})} \{-\nabla h_i(\bar{x})\} \\ & = & \operatorname{cone}\operatorname{co} \cup_{i\in I(\bar{x})} \{-\nabla h_i(\bar{x})\}, \end{array}$$

because $\operatorname{cone}\operatorname{co} \cup_{i\in I(\bar{x})} \{-\nabla h_i(\bar{x})\}$ is a finitely generated cone. □

Theorem 3.5. Let f, $g : \mathbb{R}^n \to \mathbb{R}$ be convex functions, $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be pseudoconvex functions, $S = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0 \text{ for each } i \in I\}$ and $\bar{x} \in S$. If $\bar{x}$ is a local minimizer of $f - g$ in S, then for each $v \in \partial g(\bar{x})$, there exists $\lambda \in \mathbb{R}^m_+$ such that $\lambda_ih_i(\bar{x}) = 0$ for each $i \in I$, and

$$v \in \partial f(\bar{x}) + \sum_{i\in I} \lambda_i(-\nabla h_i(\bar{x})),$$

where $\mathbb{R}^m_+ = \{\lambda \in \mathbb{R}^m \mid \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\}$.

Proof. From Lemma 3.1, $T_S(\bar{x})$ is a convex set. By Theorem 3.4, we have $\partial g(\bar{x}) \subset \partial f(\bar{x}) + N_S(\bar{x})$. Hence, from Lemma 3.1 again, we obtain $\partial g(\bar{x}) \subset \partial f(\bar{x}) + \operatorname{cone}\operatorname{co} \cup_{i\in I(\bar{x})} \{-\nabla h_i(\bar{x})\}$. This implies that the conclusion holds. □

Finally, we consider the following convex programming problem with reverse convex constraints:

$$\begin{array}{ll} \text{minimize} & f(x), \\ \text{subject to} & h_i(x) \geq 0, \ i \in I, \end{array}$$

where $I = \{1, 2, \ldots, m\}$, $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are pseudoconvex functions.

Theorem 3.6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be pseudoconvex functions, $S = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0 \text{ for each } i \in I\}$ and $\bar{x} \in S$. If $\bar{x}$ is a local minimizer of f in S, then there exists $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i(-\nabla h_i(\bar{x})).$$

Proof. The conclusion follows from Theorem 3.5 by taking $g = 0$. □

Example 3.2. Consider the problem:

$$\begin{array}{ll} \text{minimize} & \max\{x_1, x_2\}, \\ \text{subject to} & (x_1 + 1)^2 + x_2^2 - 1 \geq 0, \\ & x_1^2 + (x_2 + 1)^2 - 1 \geq 0. \end{array}$$

Let $n = 2$, $I = \{1, 2\}$, $f(x) = \max\{x_1, x_2\}$, $h_1(x) = (x_1 + 1)^2 + x_2^2 - 1$, $h_2(x) = x_1^2 + (x_2 + 1)^2 - 1$ and $\bar{x} \in S = \{x \in \mathbb{R}^2 \mid h_1(x) \geq 0, \ h_2(x) \geq 0\}$. Then f is a convex function and h_1, h_2 are differentiable convex functions. Let $\bar{x}$ be a local minimizer of f in S. By Theorem 3.6, there exist $\lambda_1, \ \lambda_2 \geq 0$ such that $0 \in \partial f(\bar{x}) + \lambda_1(-2(\bar{x}_1 + 1), -2\bar{x}_2) + \lambda_2(-2\bar{x}_1, -2(\bar{x}_2 + 1))$, $\lambda_1 h_1(\bar{x}) = 0$ and $\lambda_2 h_2(\bar{x}) = 0$. When $\lambda_1 = \lambda_2 = 0$, $0 \in \cup_{x \in S} \partial f(x) = \text{co}\{(1, 0), (0, 1)\}$. But this does not hold. When $\lambda_1 > 0$ and $\lambda_2 = 0$, $\bar{x}_1 \leq \bar{x}_2$ because $h_1(\bar{x}) = 0$ and $h_2(\bar{x}) \geq 0$. If $\bar{x}_1 = \bar{x}_2$, then $\bar{x}$ must be $(0, 0)$. If $\bar{x}_1 < \bar{x}_2$, then $\bar{x}$ must be $(-1, 1)$. Similarly, when $\lambda_1 = 0$ and $\lambda_2 > 0$, we have $\bar{x} = (0, 0)$, $(1, -1)$. When $\lambda_1, \ \lambda_2 > 0$, $\bar{x}$ must be $(0, 0)$. Therefore $(0, 0)$, $(1, -1)$ and $(-1, 1)$ have possibilities for local minimizers, and actually, $(0, 0)$ is a local minimizer and $(1, -1)$, $(-1, 1)$ are not local minimizers.

3.3 Global optimality conditions

In this section, we consider a necessary and sufficient global optimality condition for DC programming problems with reverse convex constraints.

Lemma 3.2. Let A be a closed convex cone of $\mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$ and $\epsilon \geq 0$. Then

$$N_\epsilon(\bar{x} + A, \bar{x}) = N_{(\bar{x}+A)}(\bar{x}).$$

Proof. It is clear that $N_{(\bar{x}+A)}(\bar{x}) = A^-$. Therefore it is sufficient to show that $N_\epsilon(\bar{x}+A, \bar{x}) = A^-$. Let $x^* \in N_\epsilon(\bar{x}+A, \bar{x})$ and $x \in A$. For each $\lambda > 0$, since $\bar{x} + \lambda x \in \bar{x} + A$,

$$\lambda \langle x^*, x \rangle = \langle x^*, \bar{x} + \lambda x - \bar{x} \rangle \leq \epsilon,$$

that is, $\langle x^*, x \rangle \leq \epsilon/\lambda$. Letting $\lambda \to \infty$, $\langle x^*, x \rangle \leq 0$ and thus $x^* \in A^-$. Hence we have $N_\epsilon(\bar{x}+A, \bar{x}) \subset A^-$. Also, the converse inclusion is satisfied since $N_{(\bar{x}+A)}(\bar{x}) \subset N_\epsilon(\bar{x}+A, \bar{x})$. This completes the proof. □

In the following theorem, we give a necessary global optimality condition for DC programming problems with reverse convex constraints.

Theorem 3.7. Let f, $g : \mathbb{R}^n \to \mathbb{R}$ be convex functions, $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be pseudoconvex functions, $S = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0 \text{ for each } i \in I\}$ and $\bar{x} \in S$. Assume that $\cup_{i \notin I(\bar{x})} \{\nabla h_i(\bar{x})\} \subset T_S(\bar{x})^+$. If $\bar{x}$ is a minimizer of $f - g$ in S, then for each $\epsilon \geq 0$ and $v \in \partial_\epsilon g(\bar{x})$, there exists $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$v \in \partial_\epsilon f(\bar{x}) + \sum_{i \in I} \lambda_i (-\nabla h_i(\bar{x})).$$

Proof. We first prove that $\bar{x} + T_S(\bar{x}) \subset S$. Let $x \in \bar{x} + T_S(\bar{x})$. Since Lemma 3.1 and $\cup_{i \notin I(\bar{x})} \{\nabla h_i(\bar{x})\} \subset T_S(\bar{x})^+$, $\langle \nabla h_i(\bar{x}), x - \bar{x} \rangle \geq 0$ for each $i \in I$. For each $i \in I$, by the pseudoconvexity of h_i, $h_i(x) \geq h_i(\bar{x}) \geq 0$, and thus $x \in S$. Assume that $\bar{x}$ is a minimizer of $f - g$ in S. Then $\bar{x}$ is a minimizer of $f - g$ in $\bar{x} + T_S(\bar{x})$, that is, $\bar{x}$ is a minimizer of $\left(f + \delta_{(\bar{x}+T_S(\bar{x}))}\right) - g$ in $\mathbb{R}^n$. From Theorem 1.3, for each $\epsilon \geq 0$,

$$\partial_\epsilon g(\bar{x}) \subset \partial_\epsilon \left(f + \delta_{(\bar{x}+T_S(\bar{x}))}\right)(\bar{x}).$$

It follows from [17, Theorem 2.8.7] that

$$\partial_\epsilon \left(f + \delta_{(\bar{x}+T_S(\bar{x}))}\right)(\bar{x}) = \cup_{\delta \in [0,\epsilon]} (\partial_\delta f(\bar{x}) + N_{(\epsilon-\delta)}(\bar{x} + T_S(\bar{x}), \bar{x})).$$

By Lemma 3.2,

$$\begin{aligned} \cup_{\delta \in [0,\epsilon]} (\partial_\delta f(\bar{x}) + N_{(\epsilon-\delta)}(\bar{x} + T_S(\bar{x}), \bar{x})) &= \cup_{\delta \in [0,\epsilon]} \partial_\delta f(\bar{x}) + N_{(\bar{x}+T_S(\bar{x}))}(\bar{x}) \\ &= \partial_\epsilon f(\bar{x}) + N_S(\bar{x}), \end{aligned}$$

and by Lemma 3.1,

$$\partial_\epsilon f(\bar{x}) + N_S(\bar{x}) = \partial_\epsilon f(\bar{x}) + \operatorname{cone} \operatorname{co} \cup_{i \in I(\bar{x})} \{-\nabla h_i(\bar{x})\}.$$

Hence, we have

$$\partial_\epsilon g(\bar{x}) \subset \partial_\epsilon f(\bar{x}) + \operatorname{cone} \operatorname{co} \cup_{i \in I(\bar{x})} \{-\nabla h_i(\bar{x})\}.$$

This implies that the conclusion holds. □

In the following theorem, we establish that the necessary global optimality condition of the above theorem is also sufficient.

Theorem 3.8. Let $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be pseudoconvex functions, $S = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0 \text{ for each } i \in I\}$ and $\bar{x} \in S$. Then the following statements are equivalent:

(i) $S \subset \bar{x} + T_S(\bar{x})$.

(ii) For each convex functions f, $g : \mathbb{R}^n \to \mathbb{R}$, assume that for each $\epsilon \geq 0$ and $v \in \partial_\epsilon g(\bar{x})$, there exists $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$v \in \partial_\epsilon f(\bar{x}) + \sum_{i \in I} \lambda_i(-\nabla h_i(\bar{x})).$$

Then $\bar{x}$ is a minimizer of $f - g$ in S.

Proof. First, we prove that (i) implies (ii). Assume that (i) holds. In addition, let f and g be convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ and suppose that for each $\epsilon \geq 0$ and $v \in \partial_\epsilon g(\bar{x})$, there exists $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$v \in \partial_\epsilon f(\bar{x}) + \sum_{i \in I} \lambda_i(-\nabla h_i(\bar{x})).$$

Since

$$v + \sum_{i \in I(\bar{x})} \lambda_i \nabla h_i(\bar{x}) \in \partial_\epsilon f(\bar{x}),$$

for each $x \in \bar{x} + T_S(\bar{x})$,

$$\begin{array}{rcl} f(x) & \geq & f(\bar{x}) + \langle v, x - \bar{x} \rangle + \sum_{i \in I(\bar{x})} \lambda_i \langle \nabla h_i(\bar{x}), x - \bar{x} \rangle - \epsilon \\ & \geq & f(\bar{x}) + \langle v, x - \bar{x} \rangle - \epsilon, \end{array}$$

from Lemma 3.1. Therefore, for each $x \in \mathbb{R}^n$,

$$\left(f + \delta_{(\bar{x}+T_S(\bar{x}))}\right)(x) \geq \left(f + \delta_{(\bar{x}+T_S(\bar{x}))}\right)(\bar{x}) + \langle v, x - \bar{x} \rangle - \epsilon,$$

that is, $v \in \partial_\epsilon \left(f + \delta_{(\bar{x}+T_S(\bar{x}))}\right)(\bar{x})$. Thus, we have

$$\partial_\epsilon g(\bar{x}) \subset \partial_\epsilon \left(f + \delta_{(\bar{x}+T_S(\bar{x}))}\right)(\bar{x}).$$

From Theorem 1.3, $\bar{x}$ is a minimizer of $\left(f + \delta_{(\bar{x}+T_S(\bar{x}))}\right) - g$ in $\mathbb{R}^n$, that is, $\bar{x}$ is a minimizer of $f - g$ in $\bar{x} + T_S(\bar{x})$. Hence, the conclusion follows from (i).

Next, we prove that (ii) implies (i). Assume that (ii) holds and let $i \in I(\bar{x})$. Put $f = \langle \nabla h_i(\bar{x}), \cdot \rangle$ and $g = 0$, then f and g are convex functions. For each $\epsilon \geq 0$ and $v \in \partial_\epsilon g(\bar{x})$, put

$$\lambda_j = \begin{cases} 1 & j = i, \\ 0 & j \neq i, \end{cases}$$

then $\lambda \in \mathbb{R}^m_+$ and we have $\lambda_j h_j(\bar{x}) = 0$ for each $j \in I$. Also,

$$v + \sum_{j \in I} \lambda_j \nabla h_j(\bar{x}) = \nabla h_i(\bar{x}) \in \partial_\epsilon f(\bar{x}),$$

that is,

$$v \in \partial_\epsilon f(\bar{x}) + \sum_{j \in I} \lambda_j(-\nabla h_j(\bar{x})).$$

Therefore, $\bar{x}$ is a minimizer of $f - g$ in S from (ii). Since $\langle \nabla h_i(\bar{x}), x - \bar{x} \rangle \geq 0$ for each $x \in S$, $S \subset \bar{x} + \{d \in \mathbb{R}^n \mid \langle \nabla h_i(\bar{x}), d \rangle \geq 0\}$. Hence $S \subset \bar{x} + \cap_{i \in I(\bar{x})}\{d \in \mathbb{R}^n \mid \langle \nabla h_i(\bar{x}), d \rangle \geq 0\}$ and the conclusion follows from Lemma 3.1. This completes the proof. □

This theorem shows that the condition (i) of this theorem is necessary and sufficient for sufficient global optimality conditions in DC programming problems with reverse convex constraints, that is, the optimality condition of this theorem is derived for each DC objective function whenever the condition (i) holds, but the optimality condition is not derived for some DC objective function whenever the condition (i) does not hold. Therefore it is important whether the condition (i) holds or not. Let us see the following two examples.

Example 3.3. Let $h_i(x) = q_i(\langle a_i^*, x \rangle)$, $i \in I$, where $a_i^* \in \mathbb{R}^n$ and $q_i : \mathbb{R} \to \mathbb{R}$ is a nondecreasing differentiable function such that $\inf_{y \in \mathbb{R}} q_i(y) < q_i(x)$ implies $q_i'(x) > 0$. Also, let $S = \{x \in \mathbb{R}^n \mid q_i(\langle a_i^*, x \rangle) \geq 0$ for each $i \in I\}$ and $\bar{x} \in S$. Then h_i, $i \in I$, are pseudoconvex functions and $T_S(\bar{x}) = \cap_{i \in I(\bar{x})}\{d \in \mathbb{R}^n \mid q_i'(\langle a_i^*, \bar{x} \rangle) \langle a_i^*, d \rangle \geq 0\}$. Since

$$\begin{aligned} \bar{x} + T_S(\bar{x}) &= \cap_{i \in I(\bar{x})}\{x \in \mathbb{R}^n \mid q_i'(\langle a_i^*, \bar{x} \rangle) \langle a_i^*, x - \bar{x} \rangle \geq 0\} \\ &= \cap_{i \in I(\bar{x})}\{x \in \mathbb{R}^n \mid q_i(\langle a_i^*, x \rangle) \geq q_i(\langle a_i^*, \bar{x} \rangle)\} \\ &= \cap_{i \in I(\bar{x})}\{x \in \mathbb{R}^n \mid q_i(\langle a_i^*, x \rangle) \geq 0\}, \end{aligned}$$

the condition (i) of Theorem 3.8 holds. Thus for each convex functions f, $g : \mathbb{R}^n \to \mathbb{R}$, it is possible to characterize a sufficient condition for global optimality for the following problem by using Theorem 3.8,

$$\begin{array}{ll} \text{minimize} & f(x) - g(x), \\ \text{subject to} & h_i(x) \geq 0, \ i \in I. \end{array}$$

Example 3.4. Let $n = 2$, $I = \{1\}$, $h_1(x) = x_1 + x_2^2$, $S = \{x \in \mathbb{R}^2 \mid h_1(x) \geq 0\}$ and $\bar{x} = (0, 0)$. Then h_1 is a differentiable convex function. Since $T_S(\bar{x}) = \{x \in \mathbb{R}^2 \mid x_1 \geq 0\}$, the condition (i) of Theorem 3.8 does not hold. Thus for some convex functions f, $g : \mathbb{R}^2 \to \mathbb{R}$, it is impossible to characterize a sufficient condition for global optimality for the following problem by using Theorem 3.8,

$$\begin{array}{ll} \text{minimize} & f(x) - g(x), \\ \text{subject to} & h_1(x) \geq 0. \end{array}$$

Actually, let $f(x) = x_1$ and $g(x) = 0$. Let $\epsilon \geq 0$ and $v \in \partial_\epsilon g(\bar{x})$. Put $\lambda_1 = 1$, then we have that $\lambda_1 h_1(\bar{x}) = 0$ and $v = 0 = (1,0)+1(-(1,0)) \in \partial_\epsilon f(\bar{x})+\lambda_1(-\nabla h_1(\bar{x}))$. However, $\bar{x}$ is not a minimizer of $f - g$ in S.

In the following theorem, we obtain a sufficient global optimality condition for convex programming problems with reverse convex constraints.

Theorem 3.9. Let $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be pseudoconvex functions, $S = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0 \text{ for each } i \in I\}$ and $\bar{x} \in S$. Then the following statements are equivalent:

(i) $S \subset \bar{x} + T_S(\bar{x})$.

(ii) For each convex function $f : \mathbb{R}^n \to \mathbb{R}$, assume that there exists $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i(-\nabla h_i(\bar{x})).$$

Then $\bar{x}$ is a minimizer of f in S.

Proof. We first prove that (i) implies (ii). Assume that (i) holds. In addition, let f be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$ and suppose that there exists $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i(-\nabla h_i(\bar{x})).$$

Put $g = 0$, then g is a convex function. For each $\epsilon \geq 0$ and $v \in \partial_\epsilon g(\bar{x})$,

$$\begin{array}{rcll} v & = & 0 & \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i(-\nabla h_i(\bar{x})) \\ & & & \subset \partial_\epsilon f(\bar{x}) + \sum_{i \in I} \lambda_i(-\nabla h_i(\bar{x})). \end{array}$$

Thus the conclusion follows from (i) and Theorem 3.8. Also, it is clear that (ii) implies (i). This completes the proof. □

3.4 Applications

In this section, we apply the results of the previous sections to DC programming problems and fractional programming problems. Recall that a function h is polyhedral convex if it can be written as $h = \max_{j \in J}(\langle a_j^*, \cdot\rangle + b_j)$ for some finite set J, $a_j^* \in \mathbb{R}^n$ and $b_j \in \mathbb{R}$.

First, we consider the following DC programming problem:

$$\begin{array}{ll} \text{minimize} & f(x) - g(x), \\ \text{subject to} & f_i(x) - g_i(x) \leq 0, \ i \in I, \end{array}$$

where $I = \{1, 2, \ldots, m\}$, f, $g : \mathbb{R}^n \to \mathbb{R}$ are convex functions, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are polyhedral convex functions and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are differentiable convex functions.

Theorem 3.10. Let f, $g : \mathbb{R}^n \to \mathbb{R}$ be convex functions, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be polyhedral convex functions such that $f_i = \max_{j \in J_i}(\langle a^*_{(i,j)}, \cdot \rangle + b_{(i,j)})$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be differentiable convex functions, $S = \{x \in \mathbb{R}^n \mid f_i(x) - g_i(x) \leq 0 \text{ for each } i \in I\}$ and $\bar{x} \in S$. If $\bar{x}$ is a local minimizer of $f - g$ in S, then for each $v \in \partial g(\bar{x})$, there exists $\lambda \in \mathbb{R}^{|T|}_+$ such that $\lambda_t(\langle a^*_t, \bar{x} \rangle + b_t - g_i(\bar{x})) = 0$ for each $t \in T$, and

$$v \in \partial f(\bar{x}) + \sum_{t \in T} \lambda_t (a^*_t - \nabla g_i(\bar{x})).$$

where $T = \{t = (i,j) \mid i \in I,\ j \in J_i\}$, the cardinality of T is denoted by $|T|$ and $\mathbb{R}^{|T|}_+ = \{\lambda \in \mathbb{R}^{|T|} \mid \lambda_t \geq 0 \text{ for each } t \in T\}$.

Proof.

$$\begin{aligned} S &= \{x \in \mathbb{R}^n \mid \max_{j \in J_i}(\langle a^*_{(i,j)}, x \rangle + b_{(i,j)}) - g_i(x) \leq 0 \text{ for each } i \in I\} \\ &= \{x \in \mathbb{R}^n \mid \langle a^*_{(i,j)}, x \rangle + b_{(i,j)} - g_i(x) \leq 0 \text{ for each } i \in I,\ j \in J_i\} \\ &= \{x \in \mathbb{R}^n \mid g_i(x) - (\langle a^*_t, x \rangle + b_t) \geq 0 \text{ for each } t \in T\}. \end{aligned}$$

For each $t = (i,j) \in T$, put $h_t = g_i - (\langle a^*_t, \cdot \rangle + b_t)$. Then, h_t is a differentiable convex function and $\nabla h_t(\bar{x}) = \nabla g_i(\bar{x}) - a^*_t$. Thus the conclusion follows from Theorem 3.5. □

Theorem 3.11. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be polyhedral convex functions such that $f_i = \max_{j \in J_i}(\langle a^*_{(i,j)}, \cdot \rangle + b_{(i,j)})$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be differentiable convex functions, $S = \{x \in \mathbb{R}^n \mid f_i(x) - g_i(x) \leq 0 \text{ for each } i \in I\}$ and $\bar{x} \in S$. Then the following statements are equivalent:

(i) $S \subset \bar{x} + T_S(\bar{x})$.

(ii) For each convex functions f, $g : \mathbb{R}^n \to \mathbb{R}$, assume that for each $\epsilon \geq 0$ and $v \in \partial_\epsilon g(\bar{x})$, there exists $\lambda \in \mathbb{R}^{|T|}_+$ such that $\lambda_t\big(\langle a^*_t, \bar{x} \rangle + b_t - g_i(\bar{x})\big) = 0$ for each $t \in T$, and

$$v \in \partial_\epsilon f(\bar{x}) + \sum_{t \in T} \lambda_t (a^*_t - \nabla g_i(\bar{x})).$$

Then $\bar{x}$ is a minimizer of $f - g$ in S.

Proof. The proof follows from Theorem 3.8 and the arguments of Theorem 3.10. □

Next, we consider the following fractional programming problem:

$$\begin{array}{ll} \text{minimize} & f(x)/g(x), \\ \text{subject to} & f_i(x)/g_i(x) \leq c_i,\ i \in I, \end{array}$$

where $I = \{1, 2, \ldots, m\}$, f, $g : \mathbb{R}^n \to \mathbb{R}$ are convex functions, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are polyhedral convex functions, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are differentiable convex functions such that $g_i > 0$ and $c_i \geq 0$, $i \in I$. Also, g is positive on the constraint set. We investigate the above problem by using an approach due to Dinkelbach [1].

Theorem 3.12. Let f, $g : \mathbb{R}^n \to \mathbb{R}$ be convex functions, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be polyhedral convex functions such that $f_i = \max_{j \in J_i}(\langle a^*_{(i,j)}, \cdot \rangle + b_{(i,j)})$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be differentiable convex functions such that $g_i > 0$, $c_i \geq 0$, $i \in I$, $S = \{x \in \mathbb{R}^n \mid f_i(x)/g_i(x) \leq c_i \text{ for each } i \in I\}$ and $\bar{x} \in S$. Assume that $f(\bar{x}) \geq 0$ and $g(x) > 0$ for each $x \in S$. If $\bar{x}$ is a local minimizer of f/g in S, then for each $v \in \lambda_0 \partial g(\bar{x})$, there exists $\lambda \in \mathbb{R}^{|T|}_+$ such that $\lambda_t \left(\langle a^*_t, \bar{x} \rangle + b_t - c_i g_i(\bar{x}) \right) = 0$ for each $t \in T$, and

$$v \in \partial f(\bar{x}) + \sum_{t \in T} \lambda_t (a^*_t - c_i \nabla g_i(\bar{x})),$$

where $\lambda_0 = f(\bar{x})/g(\bar{x})$.

Proof. We can verify that if $\bar{x}$ is a local minimizer of f/g in S, then $\bar{x}$ is a local minimizer of $f - \lambda_0 g$ in S. Further,

$$\begin{aligned} S &= \{x \in \mathbb{R}^n \mid f_i(x) \leq c_i g_i(x) \text{ for each } i \in I\} \\ &= \{x \in \mathbb{R}^n \mid f_i(x) - c_i g_i(x) \leq 0 \text{ for each } i \in I\}. \end{aligned}$$

Hence, the conclusion follows from Theorem 3.10 because $\lambda_0 g$ is convex and $c_i g_i$, $i \in I$, are differentiable convex. □

Theorem 3.13. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be polyhedral convex functions such that $f_i = \max_{j \in J_i}(\langle a^*_{(i,j)}, \cdot \rangle + b_{(i,j)})$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be differentiable convex functions such that $g_i > 0$, $c_i \geq 0$, $i \in I$, $S = \{x \in \mathbb{R}^n \mid f_i(x)/g_i(x) \leq c_i \text{ for each } i \in I\}$ and $\bar{x} \in S$. Then the following statements are equivalent:

(i) $S \subset \bar{x} + T_S(\bar{x})$.

(ii) For each convex functions f, $g : \mathbb{R}^n \to \mathbb{R}$ such that $f(\bar{x}) \geq 0$ and $g(x) > 0$ for each $x \in S$, assume that for each $\epsilon \geq 0$ and $v \in \partial_\epsilon(\lambda_0 g)(\bar{x})$, there exists $\lambda \in \mathbb{R}^{|T|}_+$ such that $\lambda_t \left(\langle a^*_t, \bar{x} \rangle + b_t - c_i g_i(\bar{x}) \right) = 0$ for each $t \in T$, and

$$v \in \partial_\epsilon f(\bar{x}) + \sum_{t \in T} \lambda_t (a^*_t - c_i \nabla g_i(\bar{x})).$$

Then $\bar{x}$ is a minimizer of f/g in S.

Proof. First, we prove that (i) implies (ii). Assume that (i) holds. In addition, let f and g be convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that $f(\bar{x}) \geq 0$ and $g(x) > 0$ for each $x \in S$, and suppose that for each $\epsilon \geq 0$ and $v \in \partial_\epsilon(\lambda_0 g)(\bar{x})$, there exists $\lambda \in \mathbb{R}^{|T|}_+$ such that $\lambda_t \left(\langle a^*_t, \bar{x} \rangle + b_t - c_i g_i(\bar{x}) \right) = 0$ for each $t \in T$, and

$$v \in \partial_\epsilon f(\bar{x}) + \sum_{t \in T} \lambda_t (a^*_t - c_i \nabla g_i(\bar{x})).$$

From Theorem 3.11 and (i), we have $\bar{x}$ is a minimizer of $f - \lambda_0 g$ in S. Hence the conclusion follows from the definition of λ_0.

Next, we prove that (ii) implies (i). Assume that (ii) holds. Put $T(\bar{x}) = \{t = (i,j) \in T \mid c_i g_i(\bar{x}) - (\langle a_t^*, \bar{x} \rangle + b_t) = 0\}$ and let $t = (i,j) \in T(\bar{x})$. Put $f = \langle c_i \nabla g_i(\bar{x}) - a_t^*, \cdot - \bar{x} \rangle$ and $g = 1$, then f and g are convex functions, $f(\bar{x}) \geq 0$ and $g(x) > 0$ for each $x \in S$. For each $\epsilon \geq 0$ and $v \in \partial_\epsilon g(\bar{x})$, put

$$\lambda_{(r,s)} = \begin{cases} 1 & (r,s) = t, \\ 0 & (r,s) \neq t, \end{cases}$$

then $\lambda \in \mathbb{R}_+^{|T|}$ and we have $\lambda_{(r,s)}(\langle a_{(r,s)}^*, \bar{x} \rangle + b_{(r,s)} - c_r g_r(\bar{x})) = 0$ for each $(r,s) \in T$. Also,

$$v + \sum_{(r,s) \in T} \lambda_{(r,s)}(c_r \nabla g_r(\bar{x}) - a_{(r,s)}^*) = c_i \nabla g_i(\bar{x}) - a_t^* \in \partial_\epsilon f(\bar{x}),$$

that is,

$$v \in \partial_\epsilon f(\bar{x}) + \sum_{(r,s) \in T} \lambda_{(r,s)}(a_{(r,s)}^* - c_r \nabla g_r(\bar{x})).$$

Therefore, $\bar{x}$ is a minimizer of f/g in S from (ii). Since $\left\langle c_i \nabla g_i(\bar{x}) - a_t^*, x - \bar{x} \right\rangle \geq 0$ for each $x \in S$, $S \subset \bar{x} + \{d \in \mathbb{R}^n \mid \langle c_i \nabla g_i(\bar{x}) - a_t^*, d \rangle \geq 0\}$. Thus $S \subset \bar{x} + \cap_{t \in T(\bar{x})} \{d \in \mathbb{R}^n \mid \langle c_i \nabla g_i(\bar{x}) - a_t^*, d \rangle \geq 0\}$. For each $t = (i,j) \in T$, put $h_t = c_i g_i - (\langle a_t^*, \cdot \rangle + b_t)$, then h_t is differentiable convex and $\nabla h_t(\bar{x}) = c_i \nabla g_i(\bar{x}) - a_t^*$. Since $S = \{x \in \mathbb{R}^n \mid h_t(x) \geq 0 \text{ for each } t \in T\}$ and $S \subset \bar{x} + \cap_{t \in T(\bar{x})} \{d \in \mathbb{R}^n \mid \langle \nabla h_t(\bar{x}), d \rangle \geq 0\}$, the conclusion follows from Lemma 3.1. This completes the proof. $\square$

Remark 3.1. When we consider fractional programming problems, we often add the assumption such as f is nonnegative and g is positive on the constraint set to a objective function f/g, for example, see [11, 20]. From Theorem 3.13, we notice that the assumption of f is strong to obtain these results and assumption $f(\bar{x}) \geq 0$ is enough.

Chapter 4

ε-Optimality for DC programming

In this chapter, we investigate a constraint qualification for ε-optimality conditions in DC programming problems with convex inequality constraints. Throughout this chapter, let X be a real locally convex Hausdorff topological vector space, and we consider mathematical programming problems under the following constraint set:

$$S = \{x \in X \mid h_i(x) \leq 0 \text{ for each } i \in I\},$$

where I is an arbitrary index set and $h_i : X \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, are lsc proper convex functions. This chapter is based on [34].

We introduce the notions of the conical epigraph hull property (conical EHP for short, see [28]) and Farkas-Minkowski (FM for short, see [26]) that are constraint qualifications for optimality conditions and duality in convex programming problems, respectively.

Definition 4.1 ([26, 28]). Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$. The family $\{h_i \mid i \in I\}$ is said to satisfy

(i) the conical EHP if

$$\operatorname{epi} \delta_S^* = \operatorname{cone} \operatorname{co} \bigcup_{i \in I} \operatorname{epi} h_i^*,$$

where $S \neq \emptyset$;

(ii) FM if

$$\operatorname{cone} \operatorname{co} \bigcup_{i \in I} \operatorname{epi} h_i^* + \{0\} \times [0, +\infty)$$

is weak*-closed.

The following characterization of the conical EHP was proved in [28].

Theorem 4.1 ([28])**.** Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $S \neq \emptyset$. Then the family $\{h_i \mid i \in I\}$ satisfies the conical EHP if and only if

$$\operatorname{cone}\operatorname{co}\bigcup_{i \in I} \operatorname{epi} h_i^*$$

is weak*-closed.

Remark 4.1. By Theorem 4.1, it can be verified that the family $\{h_i \mid i \in I\}$ satisfies FM if and only if the family $\{0, h_i \mid i \in I\}$ satisfies the conical EHP (see [28]).

4.1 Characterizations of the conical EHP and FM

In this section, we provide characterizations of the conical EHP and FM by using ε-subdifferentials and ε-normal cones.

Theorem 4.2. Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Then the following statements are equivalent:

(i) The family $\{h_i \mid i \in I\}$ satisfies the conical EHP.

(ii) For each $x \in S$ and $\varepsilon \geq 0$,

$$N_\varepsilon(S, x) = \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(I)},\ \mu \in \mathbb{R}_+^I \\ \sum_{i \in I} \lambda_i(\mu_i - h_i(x)) = \varepsilon}} \sum_{i \in I} \lambda_i \partial_{\mu_i} h_i(x).$$

(iii) For each $\varepsilon \geq 0$,

$$N_\varepsilon(S, \bar{x}) = \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(I)},\ \mu \in \mathbb{R}_+^I \\ \sum_{i \in I} \lambda_i(\mu_i - h_i(\bar{x})) = \varepsilon}} \sum_{i \in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}),$$

where $\mathbb{R}_+^I$ is the set of nonnegative real tuples $\lambda = (\lambda_i)_{i \in I}$, and $\mathbb{R}_+^{(I)}$ is the set of an element $\lambda = (\lambda_i)_{i \in I} \in \mathbb{R}_+^I$ with only finitely many $\lambda_i \neq 0$.

Proof. First, we prove (i) implies (ii). Assume that (i) holds. Let $x \in S$ and $\varepsilon \geq 0$. Take $x^* \in N_\varepsilon(S, x)$. Then we have $\delta_S^*(x^*) \leq \langle x^*, x \rangle + \varepsilon$, that is, $(x^*, \langle x^*, x \rangle + \varepsilon) \in \operatorname{epi} \delta_S^*$. Since the assumption (i) holds, we obtain

$$(x^*, \langle x^*, x \rangle + \varepsilon) \in \operatorname{cone}\operatorname{co}\bigcup_{i \in I} \operatorname{epi} h_i^*.$$

Thus, there exist $\bar\lambda \in \mathbb{R}^{(I)}_+$ and $(x_i^*, \alpha_i) \in \operatorname{epi} h_i^*$, $i \in I$, such that

$$(x^*, \langle x^*, x\rangle + \varepsilon) = \sum_{i\in I} \bar\lambda_i (x_i^*, \alpha_i).$$

By Theorem 1.1, there exist $\bar\mu_i \geq 0$, $i \in I$, such that for each $i \in I$, $x_i^* \in \partial_{\bar\mu_i} h_i(x)$ and $\alpha_i = \bar\mu_i - h_i(x) + \langle x_i^*, x\rangle$. Since

$$\sum_{i\in I} \bar\lambda_i(\bar\mu_i - h_i(x)) = \sum_{i\in I} \bar\lambda_i(\alpha_i - \langle x_i^*, x\rangle) = \langle x^*, x\rangle + \varepsilon - \langle x^*, x\rangle = \varepsilon,$$

we have

$$x^* = \sum_{i\in I} \bar\lambda_i x_i^* \in \sum_{i\in I} \bar\lambda_i \partial_{\bar\mu_i} h_i(x) \subset \bigcup_{\substack{\lambda\in\mathbb{R}^{(I)}_+,\ \mu\in\mathbb{R}^I_+ \\ \sum_{i\in I}\lambda_i(\mu_i - h_i(x))=\varepsilon}} \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(x),$$

and hence the following inclusion holds:

$$N_\varepsilon(S, x) \subset \bigcup_{\substack{\lambda\in\mathbb{R}^{(I)}_+,\ \mu\in\mathbb{R}^I_+ \\ \sum_{i\in I}\lambda_i(\mu_i - h_i(x))=\varepsilon}} \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(x).$$

Conversely, let $\lambda \in \mathbb{R}^{(I)}_+$ and $\mu \in \mathbb{R}^I_+$ such that $\sum_{i\in I} \lambda_i(\mu_i - h_i(x)) = \varepsilon$, and take $x^* \in \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(x)$. Then there exist $x_i^* \in \partial_{\mu_i} h_i(x)$, $i \in I$, such that $x^* = \sum_{i\in I} \lambda_i x_i^*$. Let $y \in S$. Since

$$\begin{aligned} \langle x^*, y - x\rangle &= \sum_{i\in I} \lambda_i \langle x_i^*, y - x\rangle \\ &\leq \sum_{i\in I} \lambda_i (h_i(y) - h_i(x) + \mu_i) \\ &\leq \sum_{i\in I} \lambda_i(\mu_i - h_i(x)) = \varepsilon, \end{aligned}$$

we obtain $x^* \in N_\varepsilon(S, x)$. Thus we have

$$\bigcup_{\substack{\lambda\in\mathbb{R}^{(I)}_+,\ \mu\in\mathbb{R}^I_+ \\ \sum_{i\in I}\lambda_i(\mu_i - h_i(x))=\varepsilon}} \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(x) \subset N_\varepsilon(S, x),$$

and hence (ii) holds.

Next, it is clear that (ii) implies (iii) by setting $x = \bar x$ in the assumption (ii).

Finally, we prove (iii) implies (i). Assume that (iii) holds and take $(x^*, \alpha) \in \operatorname{epi} \delta_S^*$. let $y \in S$. Since

$$\langle x^*, y - \bar x\rangle \leq \delta_S^*(x^*) - \langle x^*, \bar x\rangle \leq \alpha - \langle x^*, \bar x\rangle,$$

we obtain $x^* \in N_{\alpha - \langle x^*, \bar{x} \rangle}(S, \bar{x})$. By setting $\varepsilon = \alpha - \langle x^*, \bar{x} \rangle$ in the assumption (iii), there exist $\lambda \in \mathbb{R}^{(I)}_+$, $\mu \in \mathbb{R}^I_+$ and $x_i^* \in \partial_{\mu_i} h_i(\bar{x})$, $i \in I$, such that

$$x^* = \sum_{i \in I} \lambda_i x_i^* \text{ and } \sum_{i \in I} \lambda_i(\mu_i - h_i(\bar{x})) = \alpha - \langle x^*, \bar{x} \rangle .$$

Since

$$\alpha = \sum_{i \in I} \lambda_i(\mu_i - h_i(\bar{x})) + \langle x^*, \bar{x} \rangle = \sum_{i \in I} \lambda_i(\mu_i - h_i(\bar{x}) + \langle x_i^*, \bar{x} \rangle),$$

it follows from Theorem 1.1 that

$$(x^*, \alpha) = \sum_{i \in I} \lambda_i(x_i^*, \mu_i - h_i(\bar{x}) + \langle x_i^*, \bar{x} \rangle) \in \sum_{i \in I} \lambda_i \operatorname{epi} h_i^* \subset \operatorname{cone} \operatorname{co} \bigcup_{i \in I} \operatorname{epi} h_i^*.$$

Hence we have

$$\operatorname{epi} \delta_S^* \subset \operatorname{cone} \operatorname{co} \bigcup_{i \in I} \operatorname{epi} h_i^*.$$

Since the converse inclusion is always satisfied, (i) holds. This completes the proof. □

Corollary 4.1. Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Then the following statements are equivalent:

(i) The family $\{h_i \mid i \in I\}$ satisfies FM.

(ii) For each $x \in S$ and $\varepsilon \geq 0$,

$$N_\varepsilon(S, x) = \bigcup_{\substack{\lambda \in \mathbb{R}^{(I)}_+, \ \mu \in \mathbb{R}^I_+ \\ \sum_{i \in I} \lambda_i(\mu_i - h_i(x)) \leq \varepsilon}} \sum_{i \in I} \lambda_i \partial_{\mu_i} h_i(x).$$

(iii) For each $\varepsilon \geq 0$,

$$N_\varepsilon(S, \bar{x}) = \bigcup_{\substack{\lambda \in \mathbb{R}^{(I)}_+, \ \mu \in \mathbb{R}^I_+ \\ \sum_{i \in I} \lambda_i(\mu_i - h_i(\bar{x})) \leq \varepsilon}} \sum_{i \in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}).$$

Proof. Consider the family $\{0, \ h_i \mid i \in I\}$. Then

$$\{x \in X \mid 0(x) \leq 0, \ h_i(x) \leq 0 \text{ for each } i \in I\} = S.$$

By Theorem 4.2, the following statements are equivalent:

(a) The family $\{0, \ h_i \mid i \in I\}$ satisfies the conical EHP.

(b) For each $x \in S$ and $\varepsilon \geq 0$,

$$N_\varepsilon(S, x) = \bigcup_{\substack{\lambda', \mu' \geq 0,\ \lambda \in \mathbb{R}_+^{(I)},\ \mu \in \mathbb{R}_+^I \\ \lambda'(\mu' - 0(x)) + \sum_{i \in I} \lambda_i(\mu_i - h_i(x)) = \varepsilon}} (\lambda' \partial_{\mu'} 0(x) + \sum_{i \in I} \lambda_i \partial_{\mu_i} h_i(x)).$$

(c) For each $\varepsilon \geq 0$,

$$N_\varepsilon(S, \bar{x}) = \bigcup_{\substack{\lambda', \mu' \geq 0,\ \lambda \in \mathbb{R}_+^{(I)},\ \mu \in \mathbb{R}_+^I \\ \lambda'(\mu' - 0(\bar{x})) + \sum_{i \in I} \lambda_i(\mu_i - h_i(\bar{x})) = \varepsilon}} (\lambda' \partial_{\mu'} 0(\bar{x}) + \sum_{i \in I} \lambda_i \partial_{\mu_i} h_i(\bar{x})).$$

Hence, the conclusion follows from Remark 4.1. □

Remark 4.2. In [29], the equivalence between (i) and (ii) in Corollary 4.1 was shown, but the equivalence between (i) and (iii) was not. In the next section, the equivalence between (i) and (iii) in this corollary plays a key role.

4.2 Results of DC programming

In this section, we consider the following DC programming problem:

$$\begin{array}{ll} \text{minimize} & f(x) - g(x), \\ \text{subject to} & h_i(x) \leq 0,\ i \in I, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lsc proper convex function and $g : X \to \mathbb{R}$ is a lsc convex function.

Theorem 4.3. Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Then the following statements are equivalent:

(i) The family $\{h_i \mid i \in I\}$ satisfies FM.

(ii) For each lsc proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom} f \cap S \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, lsc convex function $g : X \to \mathbb{R}$ and $\varepsilon \geq 0$, $\bar{x}$ is an ε-minimizer of $f - g$ in S if and only if for each $\alpha \geq 0$ and $v \in \partial_\alpha g(\bar{x})$, there exist $\beta, \gamma \geq 0$, $\lambda \in \mathbb{R}_+^{(I)}$ and $\mu \in \mathbb{R}_+^I$ such that $\beta + \gamma = \alpha + \varepsilon$, $\sum_{i \in I} \lambda_i(\mu_i - h_i(\bar{x})) \leq \gamma$, and

$$v \in \partial_\beta f(\bar{x}) + \sum_{i \in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}).$$

Proof. First, we prove (i) implies (ii). Assume that (i) holds. Let f be a lsc proper convex function from X to $\mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom} f \cap S \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, g a lsc convex function from X to $\mathbb{R}$ and $\varepsilon \geq 0$. The point $\bar{x}$ is an ε-minimizer of $f - g$ in S if and only if $\bar{x}$ is an ε-minimizer of $(f + \delta_S) - g$ in X. From Theorem 1.3, $\bar{x}$ is an ε-minimizer of $(f + \delta_S) - g$ in X if and only if for each $\alpha \geq 0$,

$$\partial_\alpha g(\bar{x}) \subset \partial_{\alpha+\varepsilon}(f + \delta_S)(\bar{x}).$$

By Theorem 1.2, we have that for each $\alpha \geq 0$,

$$\partial_{\alpha+\varepsilon}(f + \delta_S)(\bar{x}) = \bigcup_{\substack{\beta,\gamma \geq 0 \\ \beta+\gamma=\alpha+\varepsilon}} (\partial_\beta f(\bar{x}) + \partial_\gamma \delta_S(\bar{x})).$$

Since the assumption (i) holds, it follows from Corollary 4.1 that for each $\gamma \geq 0$,

$$\partial_\gamma \delta_S(\bar{x}) = N_\gamma(S, \bar{x}) = \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(I)},\ \mu \in \mathbb{R}_+^I \\ \sum_{i\in I} \lambda_i(\mu_i - h_i(\bar{x})) \leq \gamma}} \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}).$$

Hence, $\bar{x}$ is an ε-minimizer of $f - g$ in S if and only if for each $\alpha \geq 0$,

$$\partial_\alpha g(\bar{x}) \subset \bigcup_{\substack{\beta,\gamma \geq 0 \\ \beta+\gamma=\alpha+\varepsilon}} (\partial_\beta f(\bar{x}) + \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(I)},\ \mu \in \mathbb{R}_+^I \\ \sum_{i\in I} \lambda_i(\mu_i - h_i(\bar{x})) \leq \gamma}} \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(\bar{x})).$$

This implies that (ii) holds.

Next, we prove (ii) implies (i). Assume that (ii) holds. By Corollary 4.1, it is sufficient to show that for each $\varepsilon \geq 0$,

$$N_\varepsilon(S, \bar{x}) = \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(I)},\ \mu \in \mathbb{R}_+^I \\ \sum_{i\in I} \lambda_i(\mu_i - h_i(\bar{x})) \leq \varepsilon}} \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}).$$

Let $\varepsilon \geq 0$. Take $x^* \in N_\varepsilon(S, \bar{x})$. Then $\bar{x}$ is an ε-minimizer of $-x^*$ in S. By setting $f = -x^*$, $g = 0$ and $\alpha = 0$ in the assumption (ii), there exist $\beta, \gamma \geq 0$, $\bar{\lambda} \in \mathbb{R}_+^{(I)}$ and $\bar{\mu} \in \mathbb{R}_+^I$ such that $\beta + \gamma = \varepsilon$, $\sum_{i\in I} \bar{\lambda}_i(\bar{\mu}_i - h_i(\bar{x})) \leq \gamma$, and

$$0 \in -x^* + \sum_{i\in I} \bar{\lambda}_i \partial_{\bar{\mu}_i} h_i(\bar{x}).$$

Since

$$\sum_{i\in I} \bar{\lambda}_i(\bar{\mu}_i - h_i(\bar{x})) \leq \gamma \leq \beta + \gamma = \varepsilon,$$

we have

$$x^* \in \sum_{i\in I} \bar{\lambda}_i \partial_{\bar{\mu}_i} h_i(\bar{x}) \subset \bigcup_{\substack{\lambda\in\mathbb{R}_+^{(I)},\ \mu\in\mathbb{R}_+^I \\ \sum_{i\in I}\lambda_i(\mu_i - h_i(\bar{x}))\le\varepsilon}} \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}),$$

and hence the following inclusion holds:

$$N_\varepsilon(S,\bar{x}) \subset \bigcup_{\substack{\lambda\in\mathbb{R}_+^{(I)},\ \mu\in\mathbb{R}_+^I \\ \sum_{i\in I}\lambda_i(\mu_i - h_i(\bar{x}))\le\varepsilon}} \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}).$$

Also, by the arguments of Theorem 4.2, it can be verified that the converse inclusion holds. This completes the proof. □

This theorem shows that FM is a necessary and sufficient constraint qualification for ε-optimality conditions in DC programming problems with convex inequality constraints.

Corollary 4.2. Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R}\cup\{+\infty\}$, $\bar{x}\in S$, $f : X \to \mathbb{R}\cup\{+\infty\}$ a lsc proper convex function such that $\operatorname{dom} f \cap S \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, and $g : X \to \mathbb{R}$ a lsc convex function. Assume that the family $\{h_i \mid i \in I\}$ satisfies FM. Then $\bar{x}$ is a minimizer of $f - g$ in S if and only if for each $\alpha \ge 0$ and $v \in \partial_\alpha g(\bar{x})$, there exist $\beta,\gamma \ge 0,\ \lambda \in \mathbb{R}_+^{(I)}$ and $\mu \in \mathbb{R}_+^I$ such that $\beta+\gamma=\alpha$, $\sum_{i\in I}\lambda_i(\mu_i - h_i(\bar{x})) \le \gamma$, and

$$v \in \partial_\beta f(\bar{x}) + \sum_{i\in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}).$$

Proof. The proof follows from Theorem 4.3 by setting $\varepsilon = 0$. □

Example 4.1. Consider the problem:

$$\begin{array}{ll} \text{minimize} & \frac{1}{4}x^4 + |x| - x^2, \\ \text{subject to} & \max\{0,-x\} \le 0. \end{array}$$

Let $X=\mathbb{R}$, $I=\{1\}$, $f(x)=\frac{1}{4}x^4+|x|$, $g(x)=x^2$, $h_1(x)=\max\{0,-x\}$ and $\bar{x}\in S=[0,+\infty)$. In Example 2.1, we have already seen that 0, 1 and $\frac{-1+\sqrt{5}}{2}$ have possibilities for local minimizers. We now find minimizers of $f-g$ in S. We can check that $\{h_i \mid i\in I\}$ satisfies FM. Let $\alpha\ge 0$ and $v\in\partial_\alpha g(0)=[-2\sqrt{\alpha},2\sqrt{\alpha}]$. Take $\beta\ge 0$ such that $\beta\le\alpha$ and $2\sqrt{\alpha}\le(\frac{4}{3}\beta)^{\frac{3}{4}}+1$, and put $\gamma=\alpha-\beta$ and $\lambda_1=\mu_1=0$. Then $\beta+\gamma=\alpha$ and $\lambda_1(\mu_1-h_1(0))\le\gamma$. Moreover, we have

$$v \in \big[-2\sqrt{\alpha},2\sqrt{\alpha}\big] \subset \big[-(\frac{4}{3}\beta)^{\frac{3}{4}}-1,(\frac{4}{3}\beta)^{\frac{3}{4}}+1\big]+0\big[-1,0\big] = \partial_\beta f(0)+\lambda_1\partial_{\mu_1}h_1(0).$$

By Corollary 4.2, 0 is a minimizer. However, since $f(0)-g(0)=0$, $f(1)-g(1)=\frac{1}{4}$ and $f(\frac{-1+\sqrt{5}}{2})-g(\frac{-1+\sqrt{5}}{2})=\frac{-9+5\sqrt{5}}{8}$, 1 and $\frac{-1+\sqrt{5}}{2}$ are not minimizers.

4.3 Results of fractional programming

In this section, we consider the following fractional programming problem:

$$\begin{array}{ll} \text{minimize} & f(x)/g(x), \\ \text{subject to} & h_i(x) \leq 0, \ i \in I, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lsc proper convex function and $g : X \to \mathbb{R}$ is a lsc convex function such that g is positive on S.

Theorem 4.4. Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Then the following statements are equivalent:

(i) The family $\{h_i \mid i \in I\}$ satisfies FM.

(ii) For each lsc proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$, lsc convex function $g : X \to \mathbb{R}$ and $\varepsilon \geq 0$ such that $\operatorname{dom} f \cap S \neq \emptyset$, $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, g is positive on S and $\nu_\varepsilon \geq 0$, $\bar{x}$ is an ε-minimizer of f/g in S if and only if for each $\alpha \geq 0$ and $v \in \partial_\alpha(\nu_\varepsilon g)(\bar{x})$, there exist $\beta, \gamma \geq 0$, $\lambda \in \mathbb{R}_+^{(I)}$ and $\mu \in \mathbb{R}_+^I$ such that $\beta + \gamma = \alpha + \varepsilon g(\bar{x})$, $\sum_{i \in I} \lambda_i(\mu_i - h_i(\bar{x})) \leq \gamma$, and

$$v \in \partial_\beta f(\bar{x}) + \sum_{i \in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}),$$

where $\nu_\varepsilon = f(\bar{x})/g(\bar{x}) - \varepsilon$.

Proof. We first prove (i) implies (ii). Assume that (i) holds. Let f be a lsc proper convex function from X to $\mathbb{R} \cup \{+\infty\}$, g a lsc convex function from X to $\mathbb{R}$ and $\varepsilon \geq 0$ such that $\operatorname{dom} f \cap S \neq \emptyset$, $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, g is positive on S and $\nu_\varepsilon \geq 0$. Then it can be verified that $\bar{x}$ is an ε-minimizer of f/g in S if and only if $\bar{x}$ is an $\varepsilon g(\bar{x})$-minimizer of $f - \nu_\varepsilon g$ in S. Thus, (ii) follows from Theorem 4.3. Also, it is clear that (ii) implies (i) by setting $f = -x^* + \langle x^*, \bar{x} \rangle + \varepsilon$, $g = 1$ and $\alpha = 0$ in the assumption (ii). This completes the proof. $\square$

This theorem shows that FM is also a necessary and sufficient constraint qualification for ε-optimality conditions in fractional programming problems with convex inequality constraints.

Corollary 4.3. Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, $\bar{x} \in S$, $f : X \to \mathbb{R} \cup \{+\infty\}$ a lsc proper convex function and $g : X \to \mathbb{R}$ a lsc convex function such that $\operatorname{dom} f \cap S \neq \emptyset$, $\operatorname{epi} f^* + \operatorname{epi} \delta_S^*$ is weak*-closed, g is positive on S and $\nu_0 \geq 0$. Assume that the family $\{h_i \mid i \in I\}$ satisfies FM. Then $\bar{x}$ is a minimizer of f/g in S if and only if for each $\alpha \geq 0$ and $v \in \partial_\alpha(\nu_0 g)(\bar{x})$, there exist $\beta, \gamma \geq 0$, $\lambda \in \mathbb{R}_+^{(I)}$ and $\mu \in \mathbb{R}_+^I$ such that $\beta + \gamma = \alpha$, $\sum_{i \in I} \lambda_i(\mu_i - h_i(\bar{x})) \leq \gamma$, and

$$v \in \partial_\beta f(\bar{x}) + \sum_{i \in I} \lambda_i \partial_{\mu_i} h_i(\bar{x}).$$

Proof. The proof follows from Theorem 4.4 by setting $\varepsilon = 0$. $\square$

Chapter 5

Optimality for differentiable programming

In this chapter, we consider a constraint qualification for sufficient optimality conditions in differentiable programming. Throughout this chapter, let g_i, $i \in I = \{1, 2, \ldots, m\}$, be functions from $\mathbb{R}^n$ to $\mathbb{R}$,

$$S = \{x \in \mathbb{R}^n : g_i(x) \leq 0 \text{ for each } i \in I\}$$

and

$$I(x) = \{i \in I : g_i(x) = 0\}$$

for each $x \in S$. We consider the following mathematical programming problem:

$$\begin{array}{ll} \text{minimize} & f(x), \\ \text{subject to} & g_i(x) \leq 0, \ i \in I, \end{array}$$

where f is a function from $\mathbb{R}^n$ to $\mathbb{R}$. This chapter is based on [35].

The following theorem shows that the assumption of quasiconvexity at a point of g_i is a constraint qualification for sufficient optimality conditions in a differentiable programming problem whose objective function f is pseudoconvex at a point, see [16].

Theorem 5.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be pseudoconvex at $\bar{x} \in S$, and g_i, $i \in I(\bar{x})$, differentiable at $\bar{x}$. Suppose that g_i, $i \in I(\bar{x})$, are quasiconvex at $\bar{x}$. In addition, assume that there exist $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that

$$\nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0. \tag{5.1}$$

Then $\bar{x}$ is a minimizer of f in S.

5.1 Results of differentiable programming

In this section, we consider constraint qualifications for sufficient optimality conditions in differentiable programming, where the objective function is pseudoconvex at a point.

In the whole of the chapter, suppose that $g_i,\ i \in I(\bar{x})$, are differentiable at $\bar{x} \in S$. From Theorem 5.1, the following assumption is a constraint qualification for sufficient optimality conditions:

$$\text{(CQ1)} \quad g_i,\ i \in I(\bar{x}), \text{ are quasiconvex at } \bar{x}.$$

The purpose of this chapter is to study the following assumption:

$$\text{(CQ2)} \quad S \subset \bar{x} + C_S(\bar{x}),$$

where

$$C_S(\bar{x}) = \{d \in \mathbb{R}^n : \langle \nabla g_i(\bar{x}), d \rangle \le 0 \text{ for each } i \in I(\bar{x})\}.$$

First we give the relation between (CQ1) and (CQ2).

Theorem 5.2. (CQ1) implies (CQ2).

Proof. Suppose that (CQ1) holds. Then, for each $i \in I(\bar{x})$,

$$g_i(x) \le g_i(\bar{x}) \text{ implies that } \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \le 0,$$

that is

$$g_i(x) \le 0 \text{ implies that } \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \le 0.$$

Therefore we have

$$\begin{aligned} S &\subset \{x \in \mathbb{R}^n : g_i(x) \le 0 \text{ for each } i \in I(\bar{x})\} \\ &\subset \{x \in \mathbb{R}^n : \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \le 0 \text{ for each } i \in I(\bar{x})\} \\ &= \bar{x} + C_S(\bar{x}), \end{aligned}$$

and then (CQ2) holds. □

The inverse of Theorem 5.2 is not true in general, see the next example:

Example 5.1. Let $n = 2$, $I = \{1, 2\}$, $g_1(x_1, x_2) = x_1^3 - x_2$, $g_2(x_1, x_2) = -x_1$ and $\bar{x} = (0, 0)$. Then g_1 and g_2 are differentiable at $\bar{x}$, $\nabla g_1(x_1, x_2) = (3x_1^2, -1)$, $\nabla g_2(x_1, x_2) = (-1, 0)$ and $I(\bar{x}) = \{1, 2\}$. Since $S = \{(x_1, x_2) : x_1 \ge 0,\ x_2 \ge x_1^3\}$ and $C_S(\bar{x}) = \{(x_1, x_2) : x_1 \ge 0,\ x_2 \ge 0\}$, (CQ2) holds. However (CQ1) does not hold because g_1 is not quasiconvex at $\bar{x}$.

The following theorem shows that (CQ2) is a constraint qualification for sufficient optimality conditions.

Theorem 5.3. The following statements are equivalent:

(i) (CQ2) is fulfilled.

(ii) For each $f : \mathbb{R}^n \to \mathbb{R}$ such that f is linear, assume that there exist $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that (5.1) is fulfilled. Then $\bar{x}$ is a minimizer of f in S.

(iii) For each $f : \mathbb{R}^n \to \mathbb{R}$ such that f is both strictly convex and differentiable at $\bar{x}$, assume that there exist $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that (5.1) is fulfilled. Then $\bar{x}$ is a minimizer of f in S.

(iv) For each $f : \mathbb{R}^n \to \mathbb{R}$ such that f is both convex and differentiable at $\bar{x}$, assume that there exist $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that (5.1) is fulfilled. Then $\bar{x}$ is a minimizer of f in S.

(v) For each $f : \mathbb{R}^n \to \mathbb{R}$ such that f is strictly pseudoconvex at $\bar{x}$, assume that there exist $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that (5.1) is fulfilled. Then $\bar{x}$ is a minimizer of f in S.

(vi) For each $f : \mathbb{R}^n \to \mathbb{R}$ such that f is pseudoconvex at $\bar{x}$, assume that there exist $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that (5.1) is fulfilled. Then $\bar{x}$ is a minimizer of f in S.

Proof. It is clear that (vi) implies (v), (v) implies (iii), (vi) implies (iv), and (iv) implies (ii). Then we will show that (iii) implies (ii), (ii) implies (i), and (i) implies (vi).

Suppose that (iii) holds. To show (ii), assume that (5.1) is fulfilled for a linear function $f : \mathbb{R}^n \to \mathbb{R}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$. Let $k \in \mathbb{N}$ and define $\tilde{f}^k : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{f}^k(x) = \frac{1}{k}\| x - \bar{x}\|^2 + f(x) - f(\bar{x})$$

for each $x \in \mathbb{R}^n$. Since $\nabla \tilde{f}^k(\bar{x}) = \nabla f(\bar{x})$, the assumption of (iii) also holds for function $\tilde{f}^k$ which is both strictly convex and differentiable at $\bar{x}$. Therefore $\bar{x}$ is a minimizer of $\tilde{f}^k$ in S. For any $x \in S$ and $k \in \mathbb{N}$,

$$0 = \tilde{f}^k(\bar{x}) \leq \tilde{f}^k(x) = \frac{1}{k}\| x - \bar{x}\|^2 + f(x) - f(\bar{x}),$$

and taking the limit as $k \to +\infty$, we have that $f(\bar{x}) \leq f(x)$. Thus $\bar{x}$ is a minimizer of f in S, and then (ii) holds.

Next, suppose that (ii) holds. To show (i), we may assume that $I(\bar{x}) \neq \emptyset$. Let $l \in I(\bar{x})$ and define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = -\langle \nabla g_l(\bar{x}), x\rangle$$

for each $x \in \mathbb{R}^n$, and put

$$\lambda_i = \begin{cases} 1 & (i = l) \\ 0 & (i \neq l) \end{cases}$$

for each $i \in I(\bar{x})$. Then we have

$$\nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = -\nabla g_l(\bar{x}) + \nabla g_l(\bar{x}) = 0.$$

Since the assumption of (ii) holds, $\bar{x}$ is a minimizer of f in S. For each $x \in S$, we have

$$-\langle \nabla g_l(\bar{x}), \bar{x} \rangle = f(\bar{x}) \leq f(x) = -\langle \nabla g_l(\bar{x}), x \rangle,$$

so $\langle \nabla g_l(\bar{x}), x - \bar{x} \rangle \leq 0$. Since this inequality holds for any $l \in I(\bar{x})$ and $x \in S$, we obtain

$$S \subset \bar{x} + \{d \in \mathbb{R}^n : \langle \nabla g_l(\bar{x}), d \rangle \leq 0 \text{ for each } \ l \in I(\bar{x})\}$$

and hence (i) holds.

Finally, suppose that (i) holds. To show (vi), assume that (5.1) is fulfilled for a function $f : \mathbb{R}^n \to \mathbb{R}$ which is pseudoconvex at $\bar{x}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$. For any $x \in S$, $\langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \leq 0$ for each $i \in I(\bar{x})$ because (CQ2) holds. Thus it follows from (5.1) that

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle = -\sum_{i \in I(\bar{x})} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \geq 0.$$

Since f is pseudoconvexity at $\bar{x}$, we obtain that $f(x) \geq f(\bar{x})$. Then $\bar{x}$ is a minimizer of f in S and hence (vi) holds. This completes the proof. □

This theorem shows that (CQ2) is a necessary and sufficient constraint qualification for sufficient optimality conditions in differentiable programming, where the objective function is pseudoconvex at a point. Remark that Theorem 5.1 can be shown as a corollary of Theorem 5.3.

Example 5.2. Consider the problem:

$$\begin{array}{ll} \text{minimize} & x_2/(x_1^2+1), \\ \text{subject to} & x_1^3 - x_2 \leq 0, \\ & -x_1 \leq 0. \end{array}$$

In Example 5.1, we have already seen that (CQ2) is fulfilled at $\bar{x} = (0,0)$. Since $f(x_1, x_2) = x_2/(x_1^2+1)$ is pseudoconvex at $\bar{x}$ and (5.1) is fulfilled for $\lambda_1 = 1$ and $\lambda_2 = 0$, then we have that $\bar{x} = (0,0)$ is a minimizer of f in S by using Theorem 5.3.

5.2 Results of differentiable multiobjective programming

In this section, we observe necessary and sufficient constraint qualifications for sufficient conditions for Pareto optimality and weak Pareto optimality in differentiable multiobjective programming, where the components of the objective function or the linear combination of them is assumed some convexity condition. We consider the following multiobjective programming problem:

$$\begin{array}{ll} \text{minimize} & F(x), \\ \text{subject to} & g_i(x) \leq 0, \ i \in I, \end{array}$$

where F is a function from $\mathbb{R}^n$ to $\mathbb{R}^p$. Let C be a closed convex pointed cone in $\mathbb{R}^p$ such that C is not the whole space and the interior of C is not empty. It is known that int $C^+ \neq \emptyset$ and

$$\text{int } C^+ = \{\mu \in \mathbb{R}^p : \langle \mu, \nu \rangle > 0 \text{ for each } \nu \in C \setminus \{0\}\},$$

where C^+ is the positive polar cone of C, that is, $C^+ = \{\mu \in \mathbb{R}^p : \langle \mu, \nu \rangle \geq 0 \text{ for each } \nu \in C\}$. For a function $F : \mathbb{R}^n \to \mathbb{R}^p$, we say that $\bar{x} \in S$ is a Pareto minimizer of F in S with respect to C if $F(S) \cap (F(\bar{x}) - C) = \{F(\bar{x})\}$. Also, we say that $\bar{x} \in S$ is a weak Pareto minimizer of F in S with respect to C if $F(S) \cap (F(\bar{x}) - \text{int } C) = \emptyset$, see [9].

First, let us see results related to differentiable multiobjective programming, where the linear combination of the components of the objective function is assumed some convexity condition. For a function $F : \mathbb{R}^n \to \mathbb{R}^p$ and a vector $\mu \in \mathbb{R}^p$, the composition of F and $\langle \mu, \cdot \rangle$ is denoted by $\mu \circ F$ for convenience.

Theorem 5.4. The following statements are equivalent:

(i) (CQ2) is fulfilled.

(ii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$, assume that there exist $\mu \in C^+ \setminus \{0\}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $\mu \circ F$ is linear, and

$$\nabla(\mu \circ F)(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0. \tag{5.2}$$

Then $\bar{x}$ is a weak Pareto minimizer of F in S with respect to C.

(iii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$, assume that there exist $\mu \in C^+ \setminus \{0\}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $\mu \circ F$ is both convex and differentiable at $\bar{x}$, and (5.2) is fulfilled. Then $\bar{x}$ is a weak Pareto minimizer of F in S with respect to C.

(iv) For each $F : \mathbb{R}^n \to \mathbb{R}^p$, assume that there exist $\mu \in C^+ \setminus \{0\}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $\mu \circ F$ is pseudoconvex at $\bar{x}$, and (5.2) is fulfilled. Then $\bar{x}$ is a weak Pareto minimizer of F in S with respect to C.

Proof. It is clear that (iv) implies (iii), and (iii) implies (ii). We will show (ii) implies (i) and (i) implies (iv).

Suppose that (ii) holds. To show (i), it suffices to show that (ii) of Theorem 5.3 holds. Assume that (5.1) is fulfilled for a linear function $f : \mathbb{R}^n \to \mathbb{R}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$. Let $c \in \operatorname{int} C$ and $\mu \in C^+ \setminus \{0\}$ such that $\langle \mu, c \rangle = 1$ and define $F : \mathbb{R}^n \to \mathbb{R}^p$ by

$$F(x) = f(x)\, c$$

for each $x \in \mathbb{R}^n$. Then we have $\mu \circ F = f$ and

$$\nabla(\mu \circ F)(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = \nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0,$$

and then the assumption (ii) of the theorem holds. Therefore $\bar{x}$ is a weak Pareto minimizer of F in S with respect to C, and so $\bar{x}$ is a minimizer of f in S. Consequently, (ii) of Theorem 5.3 holds.

Next, suppose that (i) holds. To show (iv), assume that (5.2) is fulfilled for a function $F : \mathbb{R}^n \to \mathbb{R}^p$, $\mu \in C^+ \setminus \{0\}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$ such that $\mu \circ F$ is pseudoconvex at $\bar{x}$. Define $f = \mu \circ F$, then f is pseudoconvex at $\bar{x}$ and (5.1) holds for function f, and then the assumption of (vi) of Theorem 5.3 holds. By using Theorem 5.3, $\bar{x}$ is a minimizer of $\mu \circ F$ in S. This implies that $\bar{x}$ is a weak Pareto minimizer of F in S with respect to C, and hence (iv) of the theorem holds. This completes the proof. □

This theorem shows that (CQ2) is a necessary and sufficient constraint qualification for sufficient conditions for weak Pareto optimality in differentiable multiobjective programming, where the linear combination of the components of the objective function is pseudoconvex at a point.

In the following theorem, we show that (CQ2) is also a necessary and sufficient constraint qualification for sufficient conditions for Pareto optimality.

Theorem 5.5. The following statements are equivalent:

(i) (CQ2) is fulfilled.

(ii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$, assume that there exist $\mu \in \operatorname{int} C^+$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $\mu \circ F$ is linear, and (5.2) is fulfilled. Then $\bar{x}$ is a Pareto minimizer of F in S with respect to C.

(iii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$, assume that there exist $\mu \in C^+ \setminus \{0\}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $\mu \circ F$ is both strictly convex and differentiable at $\bar{x}$, and (5.2) is fulfilled. Then $\bar{x}$ is a Pareto minimizer of F in S with respect to C.

(iv) For each $F : \mathbb{R}^n \to \mathbb{R}^p$, assume that there exist $\mu \in \operatorname{int} C^+$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $\mu \circ F$ is both convex and differentiable at $\bar{x}$, and (5.2) is fulfilled. Then $\bar{x}$ is a Pareto minimizer of F in S with respect to C.

(v) For each $F : \mathbb{R}^n \to \mathbb{R}^p$, assume that there exist $\mu \in C^+ \setminus \{0\}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $\mu \circ F$ is strictly pseudoconvex at $\bar{x}$, and (5.2) is fulfilled. Then $\bar{x}$ is a Pareto minimizer of F in S with respect to C.

(vi) For each $F : \mathbb{R}^n \to \mathbb{R}^p$, assume that there exist $\mu \in \operatorname{int} C^+$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $\mu \circ F$ is pseudoconvex at $\bar{x}$, and (5.2) is fulfilled. Then $\bar{x}$ is a Pareto minimizer of F in S with respect to C.

Proof. It is clear that (vi) implies (iv), (iv) implies (ii), and (v) implies (iii). We will show that (ii) implies (i), (i) implies (vi), (iii) implies (i), and (i) implies (v).

The proofs of (ii) implies (i) and (i) implies (vi) are almost same to the proofs of (ii) implies (i) and (i) implies (iv) of Theorem 5.4, respectively; the differences are the following: $c \in C \setminus \{0\}$, $\mu \in \operatorname{int} C^+$, and $\bar{x}$ is a Pareto minimizer.

Also the proof (iii) implies (i) is almost same to the proof of (ii) implies (i) of Theorem 5.4; the differences are the following: to show that (iii) of Theorem 5.3 holds, f is both strictly convex and differentiable at $\bar{x}$, and $\bar{x}$ is a Pareto minimizer of F.

Suppose that (i) holds. To show (v), assume that (5.2) is fulfilled for a function $F : \mathbb{R}^n \to \mathbb{R}^p$, $\mu \in C^+ \setminus \{0\}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$ such that $\mu \circ F$ is strictly pseudoconvex at $\bar{x}$. If $\bar{x}$ is not a Pareto minimizer of F in S with respect to C, there exists $x_0 \in S$ such that $F(x_0) \in F(\bar{x}) - C$ and $F(x_0) \neq F(\bar{x})$. From (i), $x_0 - \bar{x} \in C_S(\bar{x})$, that is, $\langle \nabla g_i(\bar{x}), x_0 - \bar{x} \rangle \leq 0$ for all $i \in I(\bar{x})$. From (5.2),

$$\langle \nabla(\mu \circ F)(\bar{x}), x_0 - \bar{x} \rangle = -\left\langle \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}), x_0 - \bar{x} \right\rangle \geq 0,$$

then $\langle \nabla(\mu \circ F)(\bar{x}), x_0 \rangle \geq \langle \nabla(\mu \circ F)(\bar{x}), \bar{x} \rangle$. Since $\mu \circ F$ is strictly pseudoconvex at $\bar{x}$ and $x_0 \neq \bar{x}$, $\mu \circ F(x_0) > \mu \circ F(\bar{x})$, that is, $\langle \mu, F(x_0) - F(\bar{x}) \rangle > 0$. This contradicts to $\mu \in C^+$ and $F(x_0) - F(\bar{x}) \in -C$. Therefore $\bar{x}$ is a Pareto minimizer of F in S with respect to C and consequently (v) holds. This completes the proof. $\square$

Example 5.3. Consider the problem:

$$\begin{array}{ll} \text{minimize} & (x_1, x_2), \\ \text{subject to} & x_1^3 - x_2 \leq 0, \\ & -x_1 \leq 0. \end{array}$$

In Example 5.1, we have already seen that (CQ2) is fulfilled at $\bar{x} = (0, 0)$. Let $C = \mathbb{R}^2_+$, and $F(x_1, x_2) = (F_1(x_1, x_2), F_2(x_1, x_2)) = (x_1, x_2)$. Clearly F_1, F_2 and $\mu_1 F_1 + \mu_2 F_2$ are linear at $\bar{x}$. Put $\mu_1 = \mu_2 = \lambda_1 = \lambda_2 = 1$, then (5.2) is fulfilled. Hence, $\bar{x} = (0, 0)$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$ by Theorem 5.5.

In the rest of the chapter, we consider the special case when $C = \mathbb{R}^p_+$. Clearly, $C^+ = C$, $\operatorname{int} C^+ = \{(\mu_1, \ldots, \mu_p) : \mu_j > 0 \text{ for all } j \in J\}$, and $\mu \circ F = \sum_{j=1}^p \mu_j F_j$. In Theorems 5.4 and 5.5, it is required that some linear combination of the components of the objective function $\sum_{j=1}^p \mu_j F_j$ holds linear, convex, strictly convex, pseudoconvex, or strictly pseudoconvex at a point. If all F_j are linear, convex, or strictly convex at a point, and $(\mu_1, \ldots, \mu_p) \in \mathbb{R}^p_+ \setminus \{0\}$, then $\sum_{j=1}^p \mu_j F_j$ is also linear, convex, or strictly convex at the point, respectively; We have seen the situation in Example 5.3. However, even if all F_j are pseudoconvex at a point, $\sum_{j=1}^p \mu_j F_j$ is not pseudoconvex at the point in general. Therefore, we give results when the components of the objective function are assumed some convexity condition.

Theorem 5.6. The following statements are equivalent:

(i) (CQ2) is fulfilled.

(ii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$ such that F_j is linear for all $j \in J$, assume that there exist $\mu_j \geq 0$, $j \in J$, and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $(\mu_1, \mu_2, \ldots, \mu_p) \neq (0, 0, \ldots, 0)$ and

$$\sum_{j \in J} \mu_j \nabla F_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0. \tag{5.3}$$

is fulfilled. Then $\bar{x}$ is a weak Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$.

(iii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$ such that F_j is both convex and differentiable at $\bar{x}$ for all $j \in J$, assume that there exist $\mu_j \geq 0$, $j \in J$, and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $(\mu_1, \mu_2, \ldots, \mu_p) \neq (0, 0, \ldots, 0)$ and (5.3) is fulfilled. Then $\bar{x}$ is a weak Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$.

(iv) For each $F : \mathbb{R}^n \to \mathbb{R}^p$ such that F_j is pseudoconvex at $\bar{x}$ for all $j \in J$, assume that there exist $\mu_j \geq 0$, $j \in J$, and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $(\mu_1, \mu_2, \ldots, \mu_p) \neq (0, 0, \ldots, 0)$ and (5.3) is fulfilled. Then $\bar{x}$ is a weak Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$.

Proof. It is clear that (iv) implies (iii), and (iii) implies (ii). Then we may show that (ii) implies (i) and (i) implies (iv).

Suppose that (ii) holds. To show (i), it suffices to show that (ii) of Theorem 5.3 holds. Assume that (5.1) is fulfilled for a linear function $f : \mathbb{R}^n \to \mathbb{R}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$. Define $F : \mathbb{R}^n \to \mathbb{R}^p$ by

$$F_j(x) = f(x)$$

for each $x \in \mathbb{R}^n$, where $F(x) = (F_1(x), \ldots, F_p(x))$, and put $\mu = (1/p, \ldots, 1/p)$. Then we have

$$\sum_{j \in J} \mu_j \nabla F_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = \nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0.$$

By the assumption (ii) of the theorem, $\bar{x}$ is a weak Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$. This implies that $\bar{x}$ is a minimizer of f in S, and hence (ii) of Theorem 5.3 holds.

Suppose that (i) holds. To show (iv), assume that (5.3) is fulfilled for a function $F : \mathbb{R}^n \to \mathbb{R}^p$, $\mu_j \geq 0$, $j \in J$, $(\mu_1, \dots, \mu_p) \neq (0, \dots, 0)$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, where $F = (F_1, \dots, F_p)$, all F_j, $j \in J$, are pseudoconvex at $\bar{x}$. Define $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}^p$ by

$$\widetilde{F}_j(x) = \langle \nabla F_j(\bar{x}), x \rangle$$

for each $x \in \mathbb{R}^n$, where $\widetilde{F}(x) = (\widetilde{F}_1(x), \dots, \widetilde{F}_p(x))$. Since $\sum_{j \in J} \mu_j \widetilde{F}_j$ is linear and

$$\nabla \Big(\sum_{j \in J} \mu_j \widetilde{F}_j \Big)(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = \sum_{j \in J} \mu_j \nabla F_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0,$$

the assumption of (ii) of Theorem 5.4 holds. By using Theorem 5.4, $\bar{x}$ is a weak Pareto minimizer of $\widetilde{F}$ in S with respect to $\mathbb{R}^p_+$. This implies that $\bar{x}$ is a weak Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$. If not, there exists $x_0 \in S$ such that $F_j(x_0) < F_j(\bar{x})$ for all $j \in J$. Since all F_j are pseudoconvex at $\bar{x}$, $\langle \nabla F_j(\bar{x}), x_0 - \bar{x} \rangle < 0$, that is $\widetilde{F}_j(x_0) < \widetilde{F}_j(\bar{x})$ for all $j \in J$. This shows $\bar{x}$ is not a weak Pareto minimizer of $\widetilde{F}$ in S with respect to $\mathbb{R}^p_+$, and this is a contradiction. Hence (iv) holds. This completes the proof. □

This theorem shows that (CQ2) is a necessary and sufficient constraint qualification for sufficient conditions for weak Pareto optimality in differentiable multiobjective programming, where the components of the objective function are pseudoconvex at a point.

In the following two theorems, we show that (CQ2) is a necessary and sufficient constraint qualification for sufficient conditions for Pareto optimality.

Theorem 5.7. The following statements are equivalent:

(i) (CQ2) is fulfilled.

(ii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$ such that F_j is both strictly convex and differentiable at $\bar{x}$ for all $j \in J$, assume that there exist $\mu_j \geq 0$, $j \in J$, and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $(\mu_1, \mu_2, \dots, \mu_p) \neq (0, 0, \dots, 0)$ and (5.3) is fulfilled. Then $\bar{x}$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$.

(iii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$ such that F_j is strictly pseudoconvex at $\bar{x}$ for all $j \in J$, assume that there exist $\mu_j \geq 0$, $j \in J$, and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $(\mu_1, \mu_2, \dots, \mu_p) \neq (0, 0, \dots, 0)$ and (5.3) is fulfilled. Then $\bar{x}$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$.

Proof. It is clear that (iii) implies (ii). We will show that (ii) implies (i) and (i) implies (iii).

Suppose that (ii) holds. To show (i), it suffices to show that (iii) of Theorem 5.3 holds. Assume that (5.1) is fulfilled for a function $f : \mathbb{R}^n \to \mathbb{R}$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, where f is both strictly convex and differentiable at $\bar{x}$. Define $F : \mathbb{R}^n \to \mathbb{R}^p$ by

$$F_j(x) = f(x)$$

for each $x \in \mathbb{R}^n$, where $F = (F_1, \ldots, F_p)$, and put $\mu_j = 1/p$. Then we have

$$\sum_{j \in J} \mu_j \nabla F_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = \nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0.$$

By the assumption (ii) of the theorem, $\bar{x}$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$. This implies that $\bar{x}$ is a minimizer of f in S, and hence (iii) of Theorem 5.3 holds.

Next suppose that (i). To show (iii), assume that (5.3) is fulfilled for a function $F : \mathbb{R}^n \to \mathbb{R}^p$, $\mu_j \geq 0$, $j \in J$, $(\mu_1, \ldots, \mu_p) \neq (0, \ldots, 0)$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, where $F = (F_1, \ldots, F_p)$, all F_j, $j \in J$, are strictly pseudoconvex at $\bar{x}$. Define $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}^p$ by

$$\widetilde{F}_j(x) = \langle \nabla F_j(\bar{x}), x \rangle$$

for each $x \in \mathbb{R}^n$, where $\widetilde{F}(x) = (\widetilde{F}_1(x), \ldots, \widetilde{F}_p(x))$. Since $\sum_{j \in J} \mu_j \widetilde{F}_j$ is linear and

$$\nabla\Big(\sum_{j \in J} \mu_j \widetilde{F}_j\Big)(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = \sum_{j \in J} \mu_j \nabla F_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0,$$

the assumption of (ii) of Theorem 5.4 for the function $\widetilde{F}$ holds. By using Theorem 5.4, $\bar{x}$ is a weak Pareto minimizer of $\widetilde{F}$ in S with respect to $\mathbb{R}^p_+$. This implies $\bar{x}$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$. If not, there exists $x_0 \in S$ such that $F(x_0) - F(\bar{x}) \in -\mathbb{R}^p_+$ and $F(x_0) \neq F(\bar{x})$. For all $j \in J$, therefore, $F_j(x_0) \leq F_j(\bar{x})$ and then $\langle \nabla F_j(\bar{x}), x_0 - \bar{x} \rangle < 0$ because F_j is strictly pseudoconvex at $\bar{x}$ and $x_0 \neq \bar{x}$. From the definition of $\widetilde{F}_j$,

$$\widetilde{F}_j(x_0) < \widetilde{F}_j(\bar{x})$$

for all $j \in J$. This shows $\bar{x}$ is not a weak Pareto minimizer of $\widetilde{F}$ in S with respect to $\mathbb{R}^p_+$, and this is a contradiction. Therefore $\bar{x}$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$. Hence (iii) of the theorem holds. This completes the proof. □

Theorem 5.8. The following statements are equivalent:

(i) (CQ2) is fulfilled.

(ii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$ such that F_j is both convex and differentiable at $\bar{x}$ for all $j \in J \setminus \{j_0\}$, and F_{j_0} is both strictly convex and differentiable at $\bar{x}$ for some $j_0 \in J$, assume that there exist $\mu_j > 0$, $j \in J$, and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that (5.3) is fulfilled. Then $\bar{x}$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$.

(iii) For each $F : \mathbb{R}^n \to \mathbb{R}^p$ such that F_j is both quasiconvex and differentiable at $\bar{x}$ for all $j \in J \setminus \{j_0\}$, and F_{j_0} is strictly pseudoconvex at $\bar{x}$ for some $j_0 \in J$, assume that there exist $\mu_j > 0$, $j \in J$, and $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that (5.3) is fulfilled. Then $\bar{x}$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$.

Proof. It is clear that (iii) implies (ii), and the proof of (ii) implies (i) is almost same to the proof of (ii) implies (i) of Theorem 5.7.

Suppose that (i) holds. To show (iii), assume that (5.3) is fulfilled for a function $F : \mathbb{R}^n \to \mathbb{R}^p$, $\mu_j > 0$, $j \in J$, and $\lambda_i \geq 0$, $i \in I(\bar{x})$, where $F = (F_1, \ldots, F_p)$, F_j is both quasiconvex and differentiable at $\bar{x}$ for all $j \in J \setminus \{j_0\}$, and F_{j_0} is strictly pseudoconvex at $\bar{x}$ for some $j_0 \in J$. Define $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}^p$ by

$$\widetilde{F}_j(x) = \langle \nabla F_j(\bar{x}), x \rangle$$

for each $x \in \mathbb{R}^n$, where $\widetilde{F}(x) = (\widetilde{F}_1(x), \ldots, \widetilde{F}_p(x))$. Since $\sum_{j \in J} \mu_j \widetilde{F}_j$ is linear and

$$\nabla\Big(\sum_{j \in J} \mu_j \widetilde{F}_j\Big)(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = \sum_{j \in J} \mu_j \nabla F_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0,$$

the assumption of (ii) of Theorem 5.5 for the function $\widetilde{F}$ holds. By using Theorem 5.5, $\bar{x}$ is a Pareto minimizer of $\widetilde{F}$ in S with respect to $\mathbb{R}^p_+$. This implies $\bar{x}$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$. If not, there exists $x_0 \in S$ such that $F(x_0) - F(\bar{x}) \in -\mathbb{R}^p_+$ and $F(x_0) \neq F(\bar{x})$. For each $j \in J \setminus \{j_0\}$, since $F_j(x_0) \leq F_j(\bar{x})$ and F_j are both quasiconvex and differentiable at $\bar{x}$, $\langle \nabla F_j(\bar{x}), x_0 - \bar{x} \rangle \leq 0$ holds from Proposition 1.1. Also $F_{j_0}(x_0) \leq F_{j_0}(\bar{x})$ and F_{j_0} is strictly pseudoconvex at $\bar{x}$, $\langle \nabla F_{j_0}(\bar{x}), x_0 - \bar{x} \rangle < 0$ holds. Therefore we have

$$\widetilde{F}(x_0) \leq \widetilde{F}(\bar{x}) \quad \text{and} \quad \widetilde{F}(x_0) \neq \widetilde{F}(\bar{x}),$$

and this shows $\bar{x}$ is not a Pareto minimizer of $\widetilde{F}$ in S with respect to $\mathbb{R}^p_+$. Therefore $\bar{x}$ is a Pareto minimizer of F in S with respect to $\mathbb{R}^p_+$, and hence (iii) holds. This completes the proof. □

There are trade-off relationships between conditions of F_j and μ_j in Theorems 5.7 and 5.8.

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