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Characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential

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Abstract In convex programming, characterizations of the solution set in terms of the subdifferential have been investigated by Mangasarian. An invariance property of the subdifferential of the objective function is studied, and as a consequence, characterizations of the solution set by any solution point and any point in the relative interior of the solution set are given. In quasiconvex programming, however, characterizations of the solution set by any solution point and an invariance property of Greenberg-Pierskalla subdifferential, which is one of the well known subdifferential for quasiconvex functions, have not been studied yet as far as we know.

In this paper, we study characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential. To the purpose, we show an invariance property of Greenberg-Pierskalla subdifferential, and we introduce a necessary and sufficient optimality condition by Greenberg-Pierskalla subdifferential. Also, we compare our results with previous ones. Especially, we prove some of Mangasarian's characterizations as corollaries of our results.

Keywords quasiconvex programming · solution set · subdifferential · optimality condition

1 Introduction

In mathematical programming, optimality conditions by notions of differentials are well known. Especially, in convex programming, necessary and sufficient opti-

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mality condition by the subdifferential plays important and essential roles. In [14], characterizations of the solution set for convex programming in terms of the subdifferential have been investigated by Mangasarian. At first, it is shown that the subdifferential of the objective function is constant on the relative interior of the solution set. As a consequence, characterizations of the solution set by any solution point and any point in the relative interior of the solution set are given. Motivated by these results, various characterizations of the solution set for mathematical programming have been studied, for example, [2, 8–12, 23, 29–31].

In quasiconvex programming, various subdifferentials and optimality conditions have been investigated, see [4–7, 13, 15–22, 24–28]. Greenberg-Pierskalla subdifferential in [6] is one of the most important subdifferential for quasiconvex functions. Greenberg-Pierskalla subdifferential is a simple concept in the research of subdifferentials for quasiconvex functions, and is closely related to surrogate duality. Various result concerning with Greenberg-Pierskalla subdifferential have been investigated extensively, for example, conjugate functions, duality theorems, optimality conditions, and so on. Especially, in [20], optimality conditions in terms of Greenberg-Pierskalla subdifferential are given by Penot. Also, optimality conditions in terms of infradifferential and lower subdifferential, and invariance properties of infradifferential and lower subdifferential are studied. However, as far as we know, characterizations of the solution set by any solution point and an invariance property of Greenberg-Pierskalla subdifferential have not been studied yet.

In this paper, we study characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential. At first, we show an invariance property of Greenberg-Pierskalla subdifferential. We introduce a necessary and sufficient optimality condition by Greenberg-Pierskalla subdifferential. As a consequence, we investigate characterizations of the solution set by any solution point and any point in the relative interior of the solution set. Also, we compare our results with previous ones. Especially, we prove some of Mangasarian's characterizations as corollaries of our results.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we study an invariance property of Greenberg-Pierskalla subdifferential. In Section 4, we show an optimality condition by Greenberg-Pierskalla subdifferential. We investigate characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential. In Section 5, we compare our results with previous ones.

2 Preliminaries

Let $\langle v, x \rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Given nonempty sets A, $B \subset \mathbb{R}^n$, and $\Lambda \subset \mathbb{R}$, we define $A + B$ and ΛA as follows:

$$A + B = \{x + y \in \mathbb{R}^n \mid x \in A, y \in B\},$$
$$\Lambda A = \{\lambda x \in \mathbb{R}^n \mid \lambda \in \Lambda, x \in A\}.$$

Also, we define $A + \emptyset = \Lambda\emptyset = \emptyset A = \emptyset$. We denote the closure, and the relative interior, generated by A, by $\mathrm{cl}A$, and $\mathrm{ri}A$, respectively. The normal cone of A at

$x \in A$ is denoted by $N_A(x) = \{v \in \mathbb{R}^n \mid \forall y \in A, \langle v, y - x\rangle \leq 0\}$. The indicator function δ_A is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ \infty & otherwise. \end{cases}$$

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. A function f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\mathrm{dom} f$, that is, $\mathrm{dom} f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}$. The epigraph of f is defined as $\mathrm{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq r\}$, and f is said to be convex if $\mathrm{epi} f$ is convex. The subdifferential of f at x is defined as $\partial f(x) = \{v \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, f(y) \geq f(x) + \langle v, y - x\rangle\}$. Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \beta) := \{x \in \mathbb{R}^n \mid f(x) \diamond \beta\}$$

for any $\beta \in \mathbb{R}$. A function f is said to be quasiconvex if for all $\beta \in \mathbb{R}$, $L(f, \leq, \beta)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true. A function f is said to be essentially quasiconvex if f is quasiconvex and each local minimizer $x \in \mathrm{dom} f$ of f in $\mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$. Clearly, all convex functions are essentially quasiconvex. It is known that a pseudoconvex differentiable function is essentially quasiconvex, see [3, 8, 9] for more details. Also, it is shown that a real-valued continuous quasiconvex function is essentially quasiconvex if and only if it is semistrictly quasiconvex, see Theorem 3.37 in [1].

In [6], Greenberg and Pierskalla introduced the Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{GP} f(x_0) = \{v \in \mathbb{R}^n \mid \langle v, x\rangle \geq \langle v, x_0\rangle \text{ implies } f(x) \geq f(x_0)\}.$$

In this paper, we study the following quasiconvex programming problem (P):

$$(P) \begin{cases} \text{minimize } f(x), \\ \text{subject to } x \in F, \end{cases}$$

where f is a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and F is a convex subset of $\mathbb{R}^n$. Let S be the solution set of (P), that is,

$$S := \{x \in F \mid f(x) = \min_{y \in F} f(y)\}.$$

In [14], Mangasarian studied the following characterizations of the solution set for convex programming and an invariance property of the subdifferential.

Theorem 1 *[14] Let f be a real-valued convex function, F a nonempty convex subset of $\mathbb{R}^n$, $\bar{x} \in S$, and $x_0 \in \mathrm{ri} S$. Then the following statements hold:*

(i) $\partial f(\bar{x}) \supset \partial f(x_0)$,
(ii) $\partial f(x)$ *is constant on* $x \in \mathrm{ri} S$,
(iii) the following sets are equal:
- *(a)* $S = \{x \in F \mid f(x) = \min_{y \in F} f(y)\}$,
- *(b)* $\bar{S} = \{x \in F \mid \exists v \in \partial f(\bar{x}) \cap \partial f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle = 0\}$,
- *(c)* $\hat{S} = \{x \in F \mid \exists v \in \partial f(\bar{x}) \cap \partial f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle \leq 0\}$,
- *(d)* $\bar{S}_0 = \{x \in F \mid \partial f(x_0) \subset \partial f(x), \exists v \in \partial f(x_0) \text{ s.t. } \langle v, x - x_0\rangle = 0\}$,

(e) $\hat{S}_0 = \{x \in F \mid \partial f(x_0) \subset \partial f(x), \exists v \in \partial f(x_0) \text{ s.t. } \langle v, x - x_0 \rangle \leq 0\}$,
(f) $\bar{S}_1 = \{x \in F \mid \exists v \in \partial f(x) \text{ s.t. } \langle v, x - \bar{x} \rangle = 0\}$,
(g) $\hat{S}_1 = \{x \in F \mid \exists v \in \partial f(x) \text{ s.t. } \langle v, x - \bar{x} \rangle \leq 0\}$.

In [20], Penot studied optimality conditions in terms of Greenberg-Pierskalla subdifferential, infradifferential and lower subdifferential. The infradifferential of Gutiérrez [7] is defined as follows:

$$\partial^{\leq} f(x_0) = \{v \in \mathbb{R}^n \mid \forall x \in L(f, \leq, f(x_0)), \langle v, x - x_0 \rangle \leq f(x) - f(x_0)\}.$$

The lower subdifferential of Plastria [22] is defined as follows:

$$\partial^{<} f(x_0) = \{v \in \mathbb{R}^n \mid \forall x \in L(f, <, f(x_0)), \langle v, x - x_0 \rangle \leq f(x) - f(x_0)\}.$$

The same inequality $\langle v, x - x_0 \rangle \leq f(x) - f(x_0)$, which appears in the definitions of infradifferential $\partial^{\leq} f(x_0)$ and lower subdifferential $\partial^{<} f(x_0)$, also appears in the definition of the subdifferential $\partial f(x)$. This inequality plays important roles in subdifferential calculus. The following inclusions are also important:

$$\partial^{\leq} f(x_0) \subset \partial^{<} f(x_0) \subset \partial^{GP} f(x_0). \tag{1}$$

We introduce Penot's optimality conditions and invariance properties.

Theorem 2 *[20] The following statements hold:*

(i) *Let f be quasiconvex, F a convex set, and $x \in F$ which is not a local minimizer of f in $\mathbb{R}^n$. Assume that f is upper semicontinuous (usc) at each point of $L(f, \leq, f(x))$. Then, $x \in S$ if and only if $\partial^{GP} f(x) \cap (-N_F(x)) \neq \emptyset$.*
(ii) *Let f be continuous essentially quasiconvex and $\inf_{z \in F} f(z) > \inf_{z \in \mathbb{R}^n} f(z)$. Assume that f is Lipschitzian on $L(f, \leq, \inf_{z \in F} f(z))$. Then, $x \in S$ if and only if $\partial^{\leq} f(x) \cap (-N_F(x)) \neq \emptyset$ if and only if $\partial^{<} f(x) \cap (-N_F(x)) \neq \emptyset$.*
(iii) *$\partial^{\leq} f(x) \cap (-N_F(x))$ is constant on $x \in S$.*
(iv) *If f is quasiconvex, then $\partial^{\leq} f(x)$ is constant on $x \in \mathrm{ri} S$.*
(v) *If f is continuous essentially quasiconvex and $\inf_{z \in F} f(z) > \inf_{z \in \mathbb{R}^n} f(z)$, then $\partial^{<} f(x)$ is constant on $x \in \mathrm{ri} S$.*

In [19], the following relations between subdifferentials have been investigated.

Theorem 3 *[19] Let f be a convex function finite at x. If x is not a global minimizer of f in $\mathbb{R}^n$, then*

$$[1, \infty) \partial f(x_0) = \partial^{\leq} f(x_0) = \partial^{<} f(x_0),$$

where $[1, \infty) = \{t \in \mathbb{R} \mid t \geq 1\}$.

Moreover, if $\mathbb{R}_+(\mathrm{dom} f + \{-x\}) = \mathbb{R}^n$, then

$$\mathbb{R}_{++} \partial f(x_0) = (0, 1] \partial^{\leq} f(x_0) = (0, 1] \partial^{<} f(x_0) = \partial^{GP} f(x_0),$$

where $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$, $\mathbb{R}_{++} = \{t \in \mathbb{R} \mid t > 0\}$, and $(0, 1] = \{t \in \mathbb{R} \mid 0 < t \leq 1\}$.

3 Invariance properties of Greenberg-Pierskalla subdifferential

As seen in Theorem 1 (ii), the subdifferential of the convex objective function is constant on ri$S$. Characterizations of the solution set in [14] are consequences of this invariance property. Motivated by this, we study an invariance property of Greenberg-Pierskalla subdifferential. Throughout this section, let f be an essentially quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, F a nonempty convex subset of $\mathbb{R}^n$, and $S = \{x \in F \mid f(x) = \min_{y \in F} f(y)\}$. Since f is quasiconvex and F is convex, S is also convex.

At first, we observe an invariance property of $\partial^{GP} f(x) \cap (-N_S(x))$.

Theorem 4 *Let f be an essentially quasiconvex function, and F a nonempty convex subset of $\mathbb{R}^n$. Then, for each x, $y \in S$,*

$$\partial^{GP} f(x) \cap (-N_S(x)) \subset \partial^{GP} f(y).$$

Furthermore, if $\inf_{z \in F} f(z) > \inf_{z \in \mathbb{R}^n} f(z)$, then $\partial^{GP} f(x) \cap (-N_S(x))$ is constant on $x \in S$.

Proof Let $v \in \partial^{GP} f(x)$ such that $v \in -N_S(x)$. Since $y \in S$ and $v \in -N_S(x)$, $\langle -v, y - x \rangle \leq 0$, that is, $\langle v, y \rangle \geq \langle v, x \rangle$. Then, by the definition of Greenberg-Pierskalla subdifferential, for each $z \in \mathbb{R}^n$ with $\langle v, z \rangle \geq \langle v, y \rangle$,

$$f(z) \geq f(x) = f(y).$$

This shows that $v \in \partial^{GP} f(y)$.

Assume that $\inf_{z \in F} f(z) > \inf_{z \in \mathbb{R}^n} f(z)$. It is clear that x and y are not global minimizers of f in $\mathbb{R}^n$. Let $v \in \partial^{GP} f(x) \cap (-N_S(x))$. Since $y \in S$ and $v \in -N_S(x)$, $\langle v, y \rangle \geq \langle v, x \rangle$. If $\langle v, y \rangle > \langle v, x \rangle$, then there exists a neighborhood U of y such that $U \subset \{z \in \mathbb{R}^n \mid \langle v, z \rangle > \langle v, x \rangle\}$. Since y is not a global minimizer and f is essentially quasiconvex, there exists $z_0 \in U$ such that $f(z_0) < f(y)$. It is clear that $\langle v, z_0 \rangle > \langle v, x \rangle$. Since $v \in \partial^{GP} f(x)$ and $\langle v, z_0 \rangle > \langle v, x \rangle$,

$$f(z_0) \geq f(x) = f(y) > f(z_0).$$

This is a contradiction. Hence $\langle v, y \rangle = \langle v, x \rangle$, that is, $v \in -N_S(y)$. This shows that

$$\partial^{GP} f(x) \cap (-N_S(x)) \subset \partial^{GP} f(y) \cap (-N_S(y)).$$

Similarly, we can prove the converse inclusion. This completes the proof.

Theorem 5 *Let f be an essentially quasiconvex function, F a nonempty convex subset of $\mathbb{R}^n$, $\bar{x} \in S$ and $x_0 \in$ ri$S$. Then the following statements hold:*

(i) if $\inf_{z \in F} f(z) > \inf_{z \in \mathbb{R}^n} f(z)$, then $\partial^{GP} f(x_0) \subset -N_S(x_0)$,
(ii) $\partial^{GP} f(x_0) \subset \partial^{GP} f(\bar{x})$.

Proof (i) Let $v \in \partial^{GP} f(x_0)$ and $y \in S$. Assume that $\langle v, y - x_0 \rangle < 0$. Since $x_0 \in$ ri$S$, $z_0 = y + (1 + \varepsilon)(x_0 - y) \in S$ for sufficiently small $\varepsilon > 0$. Then,

$$\begin{aligned} \langle v, z_0 \rangle &= (1 + \varepsilon) \langle v, x_0 \rangle - \varepsilon \langle v, y \rangle \\ &= \langle v, x_0 \rangle + \varepsilon(\langle v, x_0 \rangle - \langle v, y \rangle) \\ &> \langle v, x_0 \rangle . \end{aligned}$$

Since $\inf_{z\in F} f(z) > \inf_{z\in\mathbb{R}^n} f(z)$, z_0 is not a global minimizer of f in $\mathbb{R}^n$. By the definition of essential quasiconvexity, there exists $\bar{z} \in L(f,<,f(z_0))$ such that $\langle v,\bar{z}\rangle > \langle v,x_0\rangle$. However, since $v \in \partial^{GP} f(x_0)$,

$$L(f,<,f(z_0)) = L(f,<,f(x_0)) \subset L(v,<,\langle v,x_0\rangle).$$

This is a contradiction. Hence, $v \in -N_S(x_0)$.

(ii) If $\inf_{z\in F} f(z) = \inf_{z\in\mathbb{R}^n} f(z)$, then $\partial^{GP} f(x_0) = \partial^{GP} f(\bar{x}) = \mathbb{R}^n$. Let $v \in \partial^{GP} f(x_0)$ and assume that $\inf_{z\in F} f(z) > \inf_{z\in\mathbb{R}^n} f(z)$. By the condition (i), we can see that $\partial^{GP} f(x_0) \subset -N_S(x_0)$. Also, by Theorem 4,

$$v \in \partial^{GP} f(x_0) \cap (-N_S(x_0)) \subset \partial^{GP} f(\bar{x}).$$

This completes the proof.

In the following theorem, we show an invariance property of $\partial^{GP} f(x)$ on $x \in \mathrm{ri}S$.

Theorem 6 *Let f be an essentially quasiconvex function, and F a nonempty convex subset of $\mathbb{R}^n$. Then $\partial^{GP} f(x)$ is constant on $x \in \mathrm{ri}S$.*

Proof Let x, $y \in \mathrm{ri}S$. By Theorem 5 (ii), $\partial^{GP} f(x) \subset \partial^{GP} f(y)$ and $\partial^{GP} f(x) \supset \partial^{GP} f(y)$. This completes the proof.

4 Characterizations of the solution set

In this section, we show characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential.

As seen in Theorem 2, Penot studied necessary and sufficient optimality conditions for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential, infradifferential, and lower subdifferential. We show another necessary and sufficient optimality condition for usc essentially quasiconvex objective by Greenberg-Pierskalla subdifferential.

Theorem 7 *Let f be a usc essentially quasiconvex function, F a nonempty convex subset of $\mathbb{R}^n$, and $x \in F$. Then, the following statements are equivalent:*

(i) $f(x) = \min_{y\in F} f(y)$,

(ii) $0 \in \partial^{GP} f(x) + N_F(x)$.

Proof Assume that $\inf_{z\in F} f(z) = \inf_{z\in\mathbb{R}^n} f(z)$. If x is a minimizer of f in F, then we can prove easily that $\partial^{GP} f(x) = \mathbb{R}^n$. Hence the condition (ii) holds. Conversely, if the condition (ii) holds, then there exists $v \in \partial^{GP} f(x)$ such that $v \in -N_F(x)$. Since $v \in -N_F(x)$, for each $y \in F$, $\langle v,y\rangle \geq \langle v,x\rangle$. Also since $v \in \partial^{GP} f(x)$, $f(y) \geq f(x)$. This shows that the condition (i) holds.

Assume that $\inf_{z\in F} f(z) > \inf_{z\in\mathbb{R}^n} f(z)$. Then it is clear that x is not a global minimizer of f in $\mathbb{R}^n$. By the definition of essential quasiconvexity, x is not a local minimizer of f in $\mathbb{R}^n$. Hence by Theorem 2 (i), the conditions (i) and (ii) are equivalent.

In Theorem 7, upper semicontinuity of f is needed. We show the following example.

Example 1 Let $F = [0,1] \times [0,1]$, and f a real-valued function on $\mathbb{R}$ as follows:

$$f(x_1, x_2) = \begin{cases} x_1 & x_1 > 0, \\ 0 & (x_1, x_2) = (0,0), \\ \frac{x_2}{\sqrt{x_1^2 + x_2^2}} - 1 & \textit{otherwise}. \end{cases}$$

Then, F is convex, f is essentially quasiconvex, $x_0 = (0,0)$ is a global minimizer of f in F, and $f(x_0) = 0$. However, f is not usc since $f(0,-1) = -2$ and $f(\frac{1}{k}, -1) = \frac{1}{k}$ converges to 0.

We show that $\partial^{GP} f(x_0)$ is empty. Assume that $v = (v_1, v_2) \in \partial^{GP} f(x_0)$, then for each $x \in \mathbb{R}^n$, $\langle v, x \rangle \geq \langle v, x_0 \rangle = 0$ implies $f(x) \geq f(x_0) = 0$.

(i) If $v_1 \leq 0$, then $\langle v, (-1,0) \rangle = -v_1 \geq 0$, and $f(-1,0) = -1 < 0$. This is a contradiction.

(ii) If $v_1 > 0$ and $v_2 > 0$, then $\left\langle v, \left(-\frac{v_2}{v_1}, 1\right)\right\rangle = -v_2 + v_2 = 0$, and

$$f\left(-\frac{v_2}{v_1}, 1\right) = \frac{1}{\sqrt{\left(\frac{v_2}{v_1}\right)^2 + 1}} - 1 = \frac{v_1}{\sqrt{v_1^2 + v_2^2}} - 1 < 0.$$

This is a contradiction.

(iii) If $v_1 > 0$ and $v_2 \leq 0$, then $\left\langle v, \left(\frac{v_2}{v_1}, -1\right)\right\rangle = v_2 - v_2 = 0$, and

$$f\left(\frac{v_2}{v_1}, -1\right) = \frac{-1}{\sqrt{\left(\frac{v_2}{v_1}\right)^2 + 1}} - 1 = \frac{-v_1}{\sqrt{v_1^2 + v_2^2}} - 1 < 0.$$

This is a contradiction.

Hence, $\partial^{GP} f(x_0)$ is empty, that is, $0 \notin \partial^{GP} f(x_0) + N_F(x_0)$.

In the following theorem, we introduce characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential.

Theorem 8 *Let f be a usc essentially quasiconvex function, F a nonempty convex subset of $\mathbb{R}^n$, $\bar{x} \in S$ and $x_0 \in \mathrm{ri}S$. Then, the following sets are equal:*

(i) $S = \{x \in F \mid f(x) = \min_{y \in F} f(y)\}$,
(ii) $S_1 = \{x \in F \mid \exists v \in \partial^{GP} f(\bar{x}) \cap \partial^{GP} f(x) \ \textit{s.t.} \ \langle v, x - \bar{x} \rangle = 0\}$,
(iii) $S_2 = \{x \in F \mid \exists v \in \partial^{GP} f(\bar{x}) \cap \partial^{GP} f(x) \ \textit{s.t.} \ \langle v, x - \bar{x} \rangle \leq 0\}$,
(iv) $S_3 = \{x \in F \mid \partial^{GP} f(x_0) \subset \partial^{GP} f(x), \exists v \in \partial^{GP} f(x_0) \ \textit{s.t.} \ \langle v, x - x_0 \rangle = 0\}$,
(v) $S_4 = \{x \in F \mid \partial^{GP} f(x_0) \subset \partial^{GP} f(x), \exists v \in \partial^{GP} f(x_0) \ \textit{s.t.} \ \langle v, x - x_0 \rangle \leq 0\}$,
(vi) $S_5 = \{x \in F \mid \exists v \in \partial^{GP} f(x) \ \textit{s.t.} \ \langle v, x - \bar{x} \rangle = 0\}$,
(vii) $S_6 = \{x \in F \mid \exists v \in \partial^{GP} f(x) \ \textit{s.t.} \ \langle v, x - \bar{x} \rangle \leq 0\}$.

Proof It is clear that

$$S_1 \subset S_2 \subset S_6, \ S_1 \subset S_5 \subset S_6, \ \text{and} \ S_3 \subset S_4.$$

We need to show that $S_6 \subset S \subset S_1$ and $S_4 \subset S \subset S_3$.

Let $x \in S_6$. Then there exists $v \in \partial^{GP} f(x)$ such that $\langle v, x - \bar{x} \rangle \leq 0$. Since $\langle v, \bar{x} \rangle \geq \langle v, x \rangle$ and $v \in \partial^{GP} f(x)$, $f(\bar{x}) \geq f(x)$. This shows that $x \in S$.

Let $x \in S$ and $y = \dfrac{x + \bar{x}}{2}$. Since f is quasiconvex and F is convex, $y \in S$. By Theorem 7, $0 \in \partial^{GP} f(y) + N_F(y)$. Hence, there exists $v \in \partial^{GP} f(y)$ such that $-v \in N_F(y)$. Since $-v \in N_F(y)$, $\langle v, y - x \rangle \leq 0$ and $\langle v, y - \bar{x} \rangle \leq 0$. We can prove easily that $\langle v, y \rangle = \langle v, x \rangle = \langle v, \bar{x} \rangle$. Also, since $v \in \partial^{GP} f(y)$, for each $z \in \mathbb{R}^n$ with $\langle v, z \rangle \geq \langle v, y \rangle$,

$$f(z) \geq f(y) = f(x) = f(\bar{x}).$$

Hence, $v \in \partial^{GP} f(x) \cap \partial^{GP} f(\bar{x})$. This shows that $x \in S_1$.

Let $x \in S_4$, then $\partial^{GP} f(x_0) \subset \partial^{GP} f(x)$, and there exists $v \in \partial^{GP} f(x_0)$ such that $\langle v, x - x_0 \rangle \leq 0$. Hence, $v \in \partial^{GP} f(x)$, and $f(x) \leq f(x_0)$. This shows that $x \in S$.

Let $x \in S$. By Theorem 5, $\partial^{GP} f(x_0) \subset \partial^{GP} f(x)$. Since $x_0 \in \mathrm{ri} S$, $z_0 = x + (1 + \varepsilon)(x_0 - x) \in S$ for sufficiently small $\varepsilon > 0$. By Theorem 7, $0 \in \partial^{GP} f(x_0) + N_F(x_0)$. Hence, there exists $v \in \partial^{GP} f(x_0)$ such that $-v \in N_F(x_0)$. Since x, $z_0 \in F$, $\langle v, x - x_0 \rangle \geq 0$ and $\langle v, z_0 - x_0 \rangle \geq 0$. This means that $\langle v, x - x_0 \rangle = 0$, and hence $x \in S_3$. This completes the proof.

In the last of this section, we show the following example in order to illustrate our results.

Example 2 Let $F = [0, 1] \times [0, 1]$, and f a real-valued function on $\mathbb{R}$ as follows:

$$f(x_1, x_2) = \begin{cases} x_1 & x_1 \geq 0, \\ \frac{x_2}{\sqrt{x_1^2 + x_2^2}} - 1 & x_1 < 0. \end{cases}$$

This function coincides the function in Example 1 on $\mathbb{R}^2 \setminus (\{0\} \times (-\infty, 0))$. We can check that F is convex, and f is usc essentially quasiconvex.

Let $\bar{x} = (0, 0)$. Then, we can easily show that $N_F(\bar{x}) = \{(v_1, v_2) \mid v_1 \leq 0, v_2 \leq 0\}$. Let $v_0 = (1, 0)$, then for each $x \in \mathbb{R}^2$ with $\langle v_0, x \rangle \geq \langle v_0, \bar{x} \rangle$, $f(x) = x_1 = \langle v_0, x \rangle \geq \langle v_0, \bar{x} \rangle = 0 = f(\bar{x})$. Hence $v_0 \in \partial^{GP} f(\bar{x})$, that is,

$$0 = v_0 - v_0 \in \partial^{GP} f(\bar{x}) + N_F(\bar{x}).$$

By Theorem 7, $\bar{x}$ is a global minimizer of f in F.

By Theorem 8,

$$S = S_5 = \{x \in F \mid \exists v \in \partial^{GP} f(x) \text{ s.t. } \langle v, x - \bar{x} \rangle = 0\}.$$

Let $x \in F$, then $L(f, <, f(x)) = \{(y_1, y_2) \mid y_1 < x_1\}$. This shows that $\partial^{GP} f(x) = \{(\lambda, 0) \mid \lambda > 0\}$. Hence,

$$\begin{aligned} S &= S_5 \\ &= \{x \in F \mid \exists v \in \partial^{GP} f(x) \text{ s.t. } \langle v, x - \bar{x} \rangle = 0\} \\ &= \{x \in F \mid \exists v \in \{(\lambda, 0) \mid \lambda > 0\} \text{ s.t. } \langle v, x - \bar{x} \rangle = 0\} \\ &= \{x \in F \mid \exists \lambda > 0 \text{ s.t. } \lambda(x_1 - \bar{x}_1) = 0\} \\ &= \{x \in F \mid x_1 = \bar{x}_1\} \\ &= \{0\} \times [0, 1]. \end{aligned}$$

Actually, for each $x \in S_5$, $f(x) = 0 = f(x_0)$.

5 Comparisons

In this section, we compare our results with previous ones. At first we show some of Mangasarian's characterizations in Theorem 1 as corollaries of Theorem 8.

Corollary 1 *Let f be a real-valued convex function on $\mathbb{R}^n$, F a nonempty convex subset of $\mathbb{R}^n$, and $\bar{x} \in S$. Assume that $\inf_{z\in F} f(z) > \inf_{z\in\mathbb{R}^n} f(z)$. Then, the following sets are equal:*

(i) $S = \{x \in F \mid f(x) = \min_{y\in F} f(y)\}$,
(ii) $\bar{S}_1 = \{x \in F \mid \exists v \in \partial f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle = 0\}$,
(iii) $\hat{S}_1 = \{x \in F \mid \exists v \in \partial f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle \le 0\}$.

Proof Let $x \in F$. Since $\inf_{z\in F} f(z) > \inf_{z\in\mathbb{R}^n} f(z)$, x is not a global minimizer of f in $\mathbb{R}^n$. Also since f is a real-valued function, $\mathbb{R}_+(\mathrm{dom} f + \{-x\}) = \mathbb{R}^n$. Hence by Theorem 3, $\mathbb{R}_{++}\partial f(x) = \partial^{GP} f(x)$.

By Theorem 8,

$$\begin{aligned} S &= S_5 \\ &= \{x \in F \mid \exists v \in \partial^{GP} f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle = 0\} \\ &= \{x \in F \mid \exists v \in \partial f(x), \exists r > 0 \text{ s.t. } \langle rv, x - \bar{x}\rangle = 0\} \\ &= \{x \in F \mid \exists v \in \partial f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle = 0\} \\ &= \bar{S}_1 \end{aligned}$$

The proof of $S = S_6 = \hat{S}_1$ is similar.

We show (f) and (g) in Theorem 1 as direct consequences of (vi) and (vii) in Theorem 8. However, (b), (c), (d), and (e) in Theorem 1 are not direct consequences of our characterizations in Theorem 8. Assume that $v \in \partial^{GP} f(\bar{x}) \cap \partial^{GP} f(x)$, then there exists $v_1 \in \partial f(\bar{x})$, $v_2 \in \partial f(x)$, and r_1, $r_2 > 0$ such that $v = r_1 v_1 = r_2 v_2$. Unfortunately, we can not prove $r_1 = r_2$ by only our results in general. Hence we can not show (b) and (c) in Theorem 1 by (ii) and (iii) in Theorem 8. Also, even if $\partial^{GP} f(x_0) \subset \partial^{GP} f(x)$, we can not prove $\partial f(x_0) \subset \partial f(x)$ by only our results.

In [2], Burke and Ferris also introduced characterizations of the solution set for convex programming. In the following corollary, we show a similar characterization.

Corollary 2 *Let f be a usc essentially quasiconvex function, F a nonempty convex subset of $\mathbb{R}^n$, and $\bar{x} \in S$. Assume that $\inf_{z\in F} f(z) > \inf_{z\in\mathbb{R}^n} f(z)$. Then,*

$$S = \{x \in F \mid \partial^{GP} f(x) \cap (-N_S(x)) = \partial^{GP} f(\bar{x}) \cap (-N_S(\bar{x}))\}.$$

Proof Let $x \in S$. By Theorem 4, $\partial^{GP} f(x) \cap (-N_S(x)) = \partial^{GP} f(\bar{x}) \cap (-N_S(\bar{x}))$.

Let $x \in F$ and assume that $\partial^{GP} f(x) \cap (-N_S(x)) = \partial^{GP} f(\bar{x}) \cap (-N_S(\bar{x}))$. By Theorem 7, $0 \in \partial^{GP} f(\bar{x}) + N_F(\bar{x}) \subset \partial^{GP} f(\bar{x}) + N_S(\bar{x})$. This shows that $0 \in \partial^{GP} f(x) + N_S(x)$. Hence, there exists $v \in \partial f(x)$ such that $v \in -N_S(x)$. Since $\bar{x} \in S$ and $v \in -N_S(x)$, $\langle v, \bar{x}\rangle \ge \langle v, x\rangle$. Also since $v \in \partial f(x)$, $f(\bar{x}) \ge f(x)$, that is, $x \in S$.

In general, the characterization in [2] is not a direct consequence of Corollary 2. Actually, it is difficult to show the invariance property of $\partial f(x) \cap (-N_S(x))$ by using the invariance property of $\partial^{GP} f(x) \cap (-N_S(x))$.

Finally, we compare our results with Penot's characterization in Theorem 2. Theorem 2 (i) is similar to Theorem 7. In Theorem 2 (i), essential quasiconvexity is not necessary, but $x \in F$ is not a local minimizer of f in $\mathbb{R}^n$. On the other hand, in Theorem 7, $x \in F$ can be a local minimizer as long as f is essentially quasiconvex. It is clear that Theorem 2 (i) is not a direct consequence of Theorem 7. However, as seen in the proof of Theorem 7, we can prove Theorem 7 by using Theorem 2 (i). In the proof of Theorem 2 (i), the separation theorem between a convex set F and an open convex set $L(f, <, f(x))$ plays a central role. It should be appreciated that Theorem 7 can be shown directly by the similar separation theorem in the proof of Theorem 2 (i).

Theorem 2 (ii) is also a necessary and sufficient optimality condition for quasiconvex programming. We can prove that the condition in Theorem 7 (ii) is a necessary optimality condition by using the equation (1) and Theorem 2 (ii).

The invariance properties of infradifferential and lower subdifferential are proved by using the inequality which also appears in the subdifferential. The invariance property of Greenberg-Pierskalla subdifferential have not been investigated yet in [20] and other papers as far as we know. In Theorem 6, we show an invariance property of Greenberg-Pierskalla subdifferential.

6 Conclusion

In this paper, we study characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential. At first, we show that $\partial^{GP} f(x) \cap (-N_S(x))$ are constant on $x \in S$. By using this results, we show an invariance property of Greenberg-Pierskalla subdifferential on ri$S$. We introduce a necessary and sufficient optimality condition by Greenberg-Pierskalla subdifferential. As a consequence, we show characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential. Also, we compare our results with Mangasarian's characterization, Burke and Ferris' characterization, and Penot's characterization. Especially, we prove some of Mangasarian's characterizations as corollaries of our results.

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