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1-HARMONIC FUNCTIONS ON A NETWORK

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Abstract. A minimizer of the Dirichlet norm of order 1 is called a 1-harmonic function. The aim of this paper is a research of properties of 1-harmonic functions on a network. First we consider the 1-Dirichlet space and show that every network is of 1-hyperbolic type and that the ideal boundary coincides with the 1-harmonic boundary. Next we introduce the notion of 1-harmonic functions and that of strongly 1-harmonic functions. We discuss the Dirichlet problem and the maximum principle with respect to (strongly) 1-harmonic functions.

1. Introduction

For $1 < p < \infty$ a minimizer of the Dirichlet norm of order p is called a p-harmonic function. Properties of p-harmonic functions on a network, as well as on a Euclidean space, have been deeply studied (see, for example, [4], [5]). A minimizer of the Dirichlet norm of order ∞ is called an ∞-harmonic function. Properties of ∞-harmonic functions on a network was studied in [1]. On the other hand a minimizer of the Dirichlet norm of order 1 seems not to be studied yet. The aim of this paper is a research of properties of 1-harmonic functions on a network.

In Section 2 we consider the functional space of functions with finite Dirichlet norms of order 1. In case $1 < p < \infty$ a network is classified into that of p-hyperbolic type and of p-parabolic type (see [4]); on the other hand Theorem 2.1 shows that any networks are of 1-hyperbolic type. Also Theorem 2.2 implies that all the ideal boundary points are 1-harmonic boundary points.

In Section 3 we define the notion of 1-harmonic functions as a local minimizer of the Dirichlet norm of order 1. Also we define the notion of strongly 1-harmonic functions, which is a limiting case of that of p-harmonic functions as $p \to 1$. We discuss in Theorem 3.1 the Dirichlet problem with respect to strongly 1-harmonic functions. Theorems 3.2 and 3.3 show the maximum principle for (strongly) 1-harmonic functions.

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2. Functional Spaces

Let $N = \{X, Y, K, r\}$ be an infinite network which is connected and locally finite and has no self-loop. Here, X is the set of nodes, Y is the set of arcs, K is the node-arc incidence matrix, and r is the resistance. For $y \in Y$ let $e(y) = \{x \in X; K(x, y) \neq 0\}$. For $a \in X$ let ∂a be the set of neighboring nodes, i.e., $x \in \partial a$ if and only if there exists $y \in Y$ such that $e(y) = \{a, x\}$. Let $X(a) = \partial a \cup \{a\}$. For $D \subset X$ let $\overline{D} = \bigcup_{a \in D} X(a)$ and $\partial D = \overline{D} \setminus D$. Let $P = (C_X(P), C_Y(P), p)$ be a path, where $C_X(P)$ is the series of nodes, $C_Y(P)$ is the series of arcs, and p is the path index. More precisely, let $C_X(P) = \{x_0, x_1, \ldots, x_l\}$ and $C_Y(P) = \{y_1, y_2, \ldots, y_l\}$ with $e(y_j) = \{x_{j-1}, x_j\}$. Then $p(y_j) = -K(x_{j-1}, y_j) = K(x_j, y_j)$ and $p(y) = 0$ for $y \notin C_Y(P)$. Let $\mathbf{P}_{a,b}$ be the set of paths from $a \in X$ to $b \in X$. Let $\mathbf{P}_{a,\infty}$ be the set of infinite paths from $a \in X$.

Denote by $L(X)$ the set of real valued functions on X. Let $L_0(X)$ be the set of real valued functions on X with finite supports. The sets $L(Y)$ and $L_0(Y)$ are similarly defined. For $u \in L(X)$ we let

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x, y) u(x).$$

For $w \in L(Y)$ and $P \in \mathbf{P}_{a,\infty}$ we let

$$w(P) = \sum_{y \in C_Y(P)} r(y) p(y) w(y)$$

if it converges. Let ε_A be the characteristic function for $A \subset X$. If A is a singleton $\{a\}$, then we denote by ε_a instead of $\varepsilon_{\{a\}}$. Most of notations and terminology are the same as in our preceding papers.

For $u \in L(X)$, its Dirichlet sum $D_1[u]$ of order 1 is defined by

$$D_1[u] = \sum_{y \in Y} r(y) |du(y)| = \sum_{y \in Y} \left| \sum_{x \in X} K(x, y) u(x) \right|,$$

which is a semi-norm on the space $\mathbf{D}^{(1)}(N) = \{u \in L(X); D_1[u] < \infty\}$. Notice that $D_1[u]$ does not depend on the resistance r. For an arbitrarily fixed node $a_0 \in X$, the space $\mathbf{D}^{(1)}(N)$ is a Banach space with the norm

$$\|u\|_1 = D_1[u] + |u(a_0)|.$$

Notice that choosing another $a_0 \in X$ makes an equivalent norm. Denote by $\mathbf{D}_0^{(1)}(N)$ the closure of $L_0(X)$ in the Banach space $\mathbf{D}^{(1)}(N)$.

Lemma 2.1. *The inequalities $|u(x) - u(z)| \leq D_1[u]$ and $|u(x)| \leq \|u\|_1$ hold for any $u \in \mathbf{D}^{(1)}(N)$ and for any $x, z \in X$. Especially u is bounded.*

Proof. We may assume that $x \neq z$. Let P be a path from x to z and let $C_X(P) = \{x = x_0, x_1, \ldots, x_n = z\}$. Then we have

$$|u(z) - u(x)| \leq \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| \leq D_1[u].$$

It follows that $|u(x)| \le D_1[u] + |u(a_0)| = \|u\|_1$. □

Proposition 2.1. *If $u_n, u \in \mathbf{D}^{(1)}(N)$ and $\|u_n - u\|_1 \to 0$ as $n \to \infty$, then $\{u_n(x)\}_n$ converges to $u(x)$ for each $x \in X$.*

Proof. Lemma 2.1 shows that $|u_n(x) - u(x)| \le \|u_n - u\|_1 \to 0$. This means that $\{u_n(x)\}_n$ converges to $u(x)$. □

Example 2.1. Let $X = \{x_j\}_{j=0}^{\infty}$, $Y = \{y_j\}_{j=1}^{\infty}$, and $r = 1$. Define K by $K(x_n, y_n) = 1$ and $K(x_{n-1}, y_n) = -1$ for each n, and $K(x, y) = 0$ for any other pair (x, y). Let $u(x_k) = 1/(k+1)^2$. Then $u \in \mathbf{D}^{(1)}(N)$. In fact,

$$D_1[u] = \sum_{k=1}^{\infty} |u(x_k) - u(x_{k-1})| = \sum_{k=1}^{\infty} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) = 1.$$

We show that $u \in \mathbf{D}_0^{(1)}(N)$. Let $f_n(x_k) = u(x_k)$ for $k \le n$ and $f_n(x_k) = 0$ for $k \ge n+1$. Then $f_n \in L_0(X)$ and

$$\begin{aligned} \|u - f_n\|_1 = D_1[u - f_n] &= |u(x_{n+1})| + \sum_{k=n+2}^{\infty} |u(x_k) - u(x_{k-1})| \\ &= \frac{1}{(n+2)^2} + \sum_{k=n+2}^{\infty} (\frac{1}{k^2} - \frac{1}{(k+1)^2}) \\ &= \frac{2}{(n+2)^2} \to 0 \quad (n \to \infty). \end{aligned}$$

Namely, we see that $\mathbf{D}_0^{(1)}(N) \ne L_0(X)$.

A network N is said to be of p-hyperbolic type if $1 \notin \mathbf{D}_0^{(p)}(N)$; otherwise N is said to be of p-parabolic type; see [4]. On the other hand any networks are of 1-hyperbolic type. Namely

Theorem 2.1. *$1 \notin \mathbf{D}_0^{(1)}(N)$.*

Proof. Let $f \in L_0(X)$. We choose a path $P \in \mathbf{P}_{a_0,\infty}$ and let $C_X(P) = \{x_j\}_{j=0}^{\infty}$ with $x_0 = a_0$. Since $f(x_j) = 0$ for sufficiently large j, we have

$$\begin{aligned} \|1 - f\|_1 &\ge \sum_{j=1}^{\infty} |f(x_j) - f(x_{j-1})| + |1 - f(x_0)| \\ &\ge \sum_{j=1}^{\infty} (|f(x_{j-1})| - |f(x_j)|) + 1 - |f(x_0)| = 1. \end{aligned}$$

This means $1 \notin \mathbf{D}_0^{(1)}(N)$. □

Lemma 2.2. *Let $P \in \mathbf{P}_{x_0,\infty}$ with $C_X(P) = \{x_j\}_{j=0}^{\infty}$. Let $u \in L(X)$. If $du(P)$ exists, then $\lim_{n\to\infty} u(x_n)$ exists and*

$$du(P) = u(x_0) - \lim_{n\to\infty} u(x_n).$$

Proof. Let $C_Y(P) = \{y_j\}_{j=1}^{\infty}$ and p the path index of P. We have

$$\sum_{j=1}^{n} r(y_j)p(y_j)du(y_j) = \sum_{j=1}^{n}(u(x_{j-1}) - u(x_j)) = u(x_0) - u(x_n).$$

By the assumption the left-hand side converges to $du(P)$ as $n \to \infty$. We have the assertion. □

Lemma 2.3. *Let* $u \in \mathbf{D}_0^{(1)}(N)$*. Then* $du(P)$ *exists for every* $P \in \mathbf{P}_{x_0,\infty}$ *and is equal to* $u(x_0)$*.*

Proof. There exists a sequence $\{f_k\}$ in $L_0(X)$ such that $\|u - f_k\|_1 \to 0$ as $k \to \infty$. Let $w(y) = du(y)$ and $w_k(y) = df_k(y)$. Then $\lim_{k\to\infty} \sum_{y\in Y} r(y)|w_k(y) - w(y)| = 0$. It follows that $\sum_{y\in Y} r(y)|w_k(y) - w(y)| < \varepsilon/2$ for sufficiently large k. Especially $w(P)$ exists and satisfies $|w_k(P) - w(P)| < \varepsilon/2$.

Proposition 2.1 shows that $|f_k(x_0) - u(x_0)| < \varepsilon/2$ for sufficiently large k. Applying Lemma 2.2 to f_k we have $f_k(x_0) = w_k(P)$, and

$$|u(x_0) - w(P)| \leq |u(x_0) - f_k(x_0)| + |w_k(P) - w(P)| < \varepsilon.$$

Therefore $u(x_0) = w(P)$. □

The p-harmonic boundary is the set of infinite paths $P \in \mathbf{P}_{a_0,\infty}$ with $C_X(P) = \{x_n\}_n$ such that $\lim_{n\to\infty} u(x_n) = 0$ for all $u \in \mathbf{D}_0^{(p)}(N)$; see [5]. The next theorem means that 1-harmonic boundary coincides with $\mathbf{P}_{a_0,\infty}$.

Theorem 2.2. *Let* $u \in \mathbf{D}_0^{(1)}(N)$*. Then* $\lim_{n\to\infty} u(x_n) = 0$ *for every* $P \in \mathbf{P}_{x_0,\infty}$ *with* $C_X(P) = \{x_n\}_n$*.*

Proof. Lemmas 2.2 and 2.3 show the assertion. □

Lemma 2.4. *Let* $u \in \mathbf{D}_0^{(1)}(N)$*. For any* $\varepsilon > 0$*, there exists a finite subset* X' *of* X *such that* $|u(x)| < \varepsilon$ *on* $X \setminus X'$*.*

Proof. Since $D_1[u] = \sum_{y\in Y} |w(y)| < \infty$ with $w(y) = \sum_{x\in X} K(x,y)u(x)$, there exists a finite subnetwork $N' = \langle X', Y' \rangle$ of N such that $\sum_{y\in Y\setminus Y'} |w(y)| < \varepsilon$. We may assume that $X \setminus X'$ has no finite connected component. Let $x \in X \setminus X'$ and let $P \in \mathbf{P}_{x,\infty}$ be such that $C_Y(P) \cap Y' = \emptyset$. Let $C_X(P) = \{x_j\}_{j=0}^{\infty}$ with $x_0 = x$. Theorem 2.2 shows that

$$|u(x)| = \lim_{n\to\infty} |u(x_n) - u(x_0)| \leq \lim_{n\to\infty} \sum_{j=1}^{n} |u(x_j) - u(x_{j-1})|$$
$$\leq \sum_{y\in Y\setminus Y'} |w(y)| < \varepsilon.$$

□

3. 1-Harmonic Functions and Strongly 1-Harmonic Functions

For a finite set of real numbers $S = \{a_1, \dots, a_n\}$ and a real number α we denote by $S+\alpha = \{a_1+\alpha, \dots, a_n+\alpha\}$, $\alpha S = \{\alpha a_1, \dots, \alpha a_n\}$, and $-S = \{-a_1, \dots, -a_n\}$. We renumber S as $a_1 \le a_2 \le \cdots \le a_n$ and define $M_-(S)$ and $M_+(S)$ by

$$M_-(S) = M_+(S) = a_{m+1} \quad \text{in case } n = 2m+1;$$
$$M_-(S) = a_m, \quad M_+(S) = a_{m+1} \quad \text{in case } n = 2m.$$

It is easy to see the following:

Lemma 3.1. (1) *$M_-(S) \le M_+(S)$;*
(2) *$M_-(S+\alpha) = M_-(S) + \alpha$ and $M_+(S+\alpha) = M_+(S) + \alpha$;*
(3) *If $\alpha > 0$, then $M_-(\alpha S) = \alpha M_-(S)$ and $M_+(\alpha S) = \alpha M_+(S)$;*
(4) *$M_-(-S) = -M_+(S)$ and $M_+(-S) = -M_-(S)$;*
(5) *Let $S_\nu = \{a_1^{(\nu)}, \dots, a_n^{(\nu)}\}$ for $\nu = 1, 2, \dots$ and $S = \{a_1, \dots, a_n\}$. If $a_i^{(\nu)} \to a_i$ as $\nu \to \infty$ for each i, then $M_-(S_\nu) \to M_-(S)$ and $M_+(S_\nu) \to M_+(S)$ as $\nu \to \infty$.*

Lemma 3.2. *Let $S = \{a_1, \dots, a_n\}$ be a finite set of real numbers. Let $f_S(t) = \sum_{i=1}^n |t - a_i|$ and $g_S(t) = \sum_{i=1}^n \operatorname{sgn}(t - a_i)$. Let $t_0 \in \mathbb{R}$.*

(1) *The following are equivalent:*
  (a) *$f_S(t)$ is nondecreasing for $t > t_0$;*
  (b) *$M_-(S) \le t_0$;*
  (c) *$g_S(t) \ge 0$ for all $t > t_0$;*
  (d) *$(t - t_0)g_S(t) \ge 0$ for all $t > t_0$.*
(2) *The following are equivalent:*
  (a) *$f_S(t)$ is nonincreasing for $t < t_0$;*
  (b) *$t_0 \le M_+(S)$;*
  (c) *$g_S(t) \le 0$ for all $t < t_0$;*
  (d) *$(t - t_0)g_S(t) \ge 0$ for all $t < t_0$.*
(3) *The following are equivalent:*
  (a) *$f_S(t)$ attains its minimum at $t = t_0$;*
  (b) *$M_-(S) \le t_0 \le M_+(S)$;*
  (c) *$g_S(t_1) \le 0 \le g_S(t_2)$ if $t_1 < t_0 < t_2$;*
  (d) *$(t - t_0)g_S(t) \ge 0$ for all $t \in \mathbb{R}$.*

Proof. We shall show (1). It is easy to see that f_S is piece-wise linear and is continuous and that its slope is $g_S(t)$ except at $t = a_i$ for some i. Since $g_S(t)$ is nondecreasing, it follows that (1a) is equivalent to (1c).

It is obvious that (1c) is equivalent to (1d).

To show the equivalence of (1b) and (1c) it suffices to prove that $M_-(S) = \inf\{t \in \mathbb{R}; g_S(t) \ge 0\}$. First assume that $n = 2m+1$ and

$$a_1 \le \cdots \le a_{i_0 - 1} < a_{i_0} = \cdots = a_{m+1} = \cdots = a_{j_0} < a_{j_0+1} \le \cdots \le a_n,$$

where $i_0 \leq m+1 \leq j_0$. For $t < a_{i_0}$

$$\begin{aligned} g_S(t) &\leq (i_0-1)-(n-i_0+1) = 2i_0-n-2 \\ &\leq 2(m+1)-(2m+1)-2 = -1; \end{aligned}$$

for $t > a_{j_0}$

$$g_S(t) \geq j_0-(n-j_0) = 2j_0-n \geq 2(m+1)-(2m+1) = 1.$$

Thus $\inf\{t \in \mathbb{R}; g_S(t) \geq 0\} = a_{m+1} = M_-(S)$.

Next assume that $n = 2m$ and

$$a_1 \leq \cdots \leq a_{i_0-1} < a_{i_0} = \cdots = a_m = a_{m+1} = \cdots = a_{j_0} < a_{j_0+1} \leq \cdots \leq a_n,$$

where $i_0 \leq m < m+1 \leq j_0$. For $t < a_{i_0}$

$$g_S(t) \leq (i_0-1)-(n-i_0+1) = 2i_0-n-2 \leq 2m-2m-2 = -2;$$

for $t > a_{j_0}$

$$g_S(t) \geq j_0-(n-j_0) = 2j_0-n \geq 2(m+1)-2m = 2.$$

Thus $\inf\{t \in \mathbb{R}; g_S(t) \geq 0\} = a_m = M_-(S)$.

Last assume that $n = 2m$ and

$$a_1 \leq \cdots \leq a_{i_0-1} < a_{i_0} = \cdots = a_m < a_{m+1} = \cdots = a_{j_0} < a_{j_0+1} \leq \cdots \leq a_n,$$

where $i_0 \leq m < m+1 \leq j_0$. For $t = a_m$

$$g_S(t) = (i_0-1)-(n-m) = i_0-n+m-1 \leq m-2m+m-1 = -1;$$

for $a_m < t < a_{m+1}$

$$g_S(t) = m-(n-m) = 2m-n = 0.$$

Thus $\inf\{t \in \mathbb{R}; g_S(t) \geq 0\} = a_m = M_-(S)$.

We can similarly show (2). Combining (1) and (2) we have (3). □

For $a \in X$ and $u \in L(X(a))$, let

$$S_a(u) = \{K(a,y)r(y)du(y); y \in Y \text{ with } a \in e(y)\} = \{u(x)-u(a); x \in \partial a\}.$$

We say that u is 1-*superharmonic* (1-*subharmonic*, resp.) *at* a if

$$M_-(S_a(u)) \leq 0 \quad (M_+(S_a(u)) \geq 0, \text{ resp.}).$$

In case that u is both 1-superharmonic and 1-subharmonic at a, we say that u is 1-*harmonic at* a. For a subset D of X we say that $u \in L(\overline{D})$ is 1-*harmonic* (1-*superharmonic*, 1-*subharmonic*, resp.) *in* D if u is 1-harmonic (1-superharmonic, 1-subharmonic, resp.) at every node in D.

Let $a \in X$ and $u \in L(X(a))$. We define the 1-*Laplacian* Δ_1 as

$$\Delta_1 u(a) = \sum_{y \in Y} \operatorname{sgn}(K(a,y)du(y)) = \sum_{y \in Y} K(a,y)\operatorname{sgn}(du(y)).$$

Let $D \subset X$. A function $u \in L(\overline{D})$ is said to be *strongly* 1-*superharmonic* (*strongly* 1-*subharmonic*, resp.) *in* D if

$$t \sum_{x \in A} \Delta_1(u - t\varepsilon_A)(x) \geq 0 \qquad \text{for all } t < 0 \text{ (for all } t > 0\text{, resp.)}$$

for all finite subset $A \subset D$. A function $u \in L(\overline{D})$ is said to be *strongly* 1-*harmonic in* D if u is both strongly 1-superharmonic and strongly 1-subharmonic in D. In case $D = \{a\}$, we replace the terminology "in D" by "at a". It is easy to see that every constant function is 1-harmonic and strongly 1-harmonic in X.

For $A, B \subset X$ we let

$$A \ominus B = \{y \in Y; e(y) \cap A \neq \emptyset, e(y) \cap B \neq \emptyset\}.$$

For $A \subset X$ and $y \in Y$ we denote by $\mathrm{n}_A(y)$ a node in $e(y) \cap A$ if $e(y) \cap A$ consists of exactly one node; otherwise $\mathrm{n}_A(y)$ is undefined. Note that

$$\sum_{x \in A} \Delta_1 u(x) = \sum_{y \in A \ominus (X \setminus A)} K(\mathrm{n}_A(y), y) \operatorname{sgn}(du(y)).$$

Also note that, for $y \in A \ominus (X \setminus A)$, we have

$$d\varepsilon_A(y) = -r(y)^{-1} K(\mathrm{n}_A(y), y),$$

and

$$\begin{aligned} \sum_{x \in A} \Delta_1(u - t\varepsilon_A)(x) &= \sum_{y \in A \ominus (X \setminus A)} \operatorname{sgn}\big(K(\mathrm{n}_A(y), y) du(y) + t r(y)^{-1}\big) \\ &= \sum_{y \in A \ominus (X \setminus A)} \operatorname{sgn}\big(K(\mathrm{n}_A(y), y) r(y) du(y) + t\big). \end{aligned}$$

This implies that $\sum_{x \in A} \Delta_1(u - t\varepsilon_A)(x)$ is nondecreasing function of t. We use these relations repeatedly.

Remark 3.1. For $1 < p < \infty$ the p-Laplacian Δ_p is defined as

$$\Delta_p u(a) = \sum_{y \in Y} \varphi_p(K(a, y) du(y)),$$

where $\varphi_p(t) = |t|^{p-1} \operatorname{sgn}(t)$. A function u is p-harmonic in $D \subset X$ if $\Delta_p u = 0$ in D, which is equivalent to

$$t \sum_{x \in A} \Delta_p(u - t\varepsilon_A)(x) \geq 0 \qquad \text{for all } t \in \mathbb{R}$$

for all finite subset $A \subset D$. Therefore the definition of the strong 1-harmonicity is a limiting case of the p-harmonicity.

Proposition 3.1. *Let $a \in X$. A function u is strongly 1-harmonic at a if and only if u is 1-harmonic at a.*

Proof. Suppose that u is strongly 1-harmonic at a. Then $t\Delta_1(u - t\varepsilon_a)(a) \geq 0$ for all $t \in \mathbb{R}$. This is equivalent to

$$t \sum_{\substack{y \in Y \\ a \in e(y)}} \operatorname{sgn}\bigl(K(a, y)r(y)du(y) + t\bigr) \geq 0$$

for all $t \in \mathbb{R}$. Lemma 3.2 implies that $M_-(-S_a(u)) \leq 0 \leq M_+(-S_a(u))$, or $M_-(S_a(u)) \leq 0 \leq M_+(S_a(u))$. This means that u is 1-harmonic at a.

Conversely, suppose that u is 1-harmonic at a. We follow the previous implication in reverse order and obtain that u is strongly 1-harmonic at a. □

Proposition 3.2. *Let $D \subset X$. If u is a strongly 1-harmonic function in D, then u is 1-harmonic in D.*

Proof. Suppose that u is strongly 1-harmonic in D. It is obvious that u is strongly 1-harmonic at each node in D. Proposition 3.1 shows that u is 1-harmonic at each node in D. This implies that u is 1-harmonic in D. □

The converse of Proposition 3.2 is not true; see Example 3.1.

Lemma 3.3. *Let $D \subset X$. If $u \in L(\overline{D})$ satisfies that $D_1[u] \leq D_1[u - t\varepsilon_A] < \infty$ for all $t \in \mathbb{R}$ and for each finite subset A of D, then u is strongly 1-harmonic in D.*

Proof. Since $D_1[u] \leq D_1[u - t\varepsilon_A]$ and $d\varepsilon_A(y) = 0$ for $y \notin A \ominus (X \setminus A)$, we have

$$\sum_{y \in A \ominus (X \setminus A)} r(y)|du(y)| \leq \sum_{y \in A \ominus (X \setminus A)} r(y)|du(y) - td\varepsilon_A(y)|,$$

so that

$$\sum_{y \in A \ominus (X \setminus A)} |K(\mathrm{n}_A(y), y)r(y)du(y)| \leq \sum_{y \in A \ominus (X \setminus A)} |K(\mathrm{n}_A(y), y)r(y)du(y) + t|.$$

This means that the right-hand side attains its minimum at $t = 0$. Lemma 3.2 shows that $t \sum_{y \in A \ominus (X \setminus A)} \operatorname{sgn}\bigl(K(\mathrm{n}_A(y), y)r(y)du(y) + t\bigr) \geq 0$ for all $t \in \mathbb{R}$, which is equivalent to $t \sum_{x \in A} \Delta_1(u - t\varepsilon_A)(x) \geq 0$. □

Theorem 3.1. *Let $D \subsetneq X$ and f a function on $X \setminus D$. We consider the extremal problem*

$$\alpha := \inf\{D_1[u]; u = f \text{ on } X \setminus D\}. \tag{1}$$

If $\alpha < \infty$, then there exists an optimal solution u to the problem (1). Moreover each optimal solution is strongly 1-harmonic in D.

Proof. We take a minimizing sequence $\{u_n\}_n$ of feasible solutions. Fix a node $a \in X \setminus D$ and let $x \in D$. Lemma 2.1 shows that $\{u_n(x)\}_n$ is bounded for each x. By taking a subsequence, we may assume that $u(x) := \lim_{n\to\infty} u_n(x)$ exists for each x. Then u is a feasible solution to (1). Fatou's lemma implies

$$\alpha \leq D_1[u] \leq \liminf_{n\to\infty} D_1[u_n] = \alpha.$$

This means that u is an optimal solution to (1).

Let u be an optimal solution and let A be a finite subset of D. Since $u - t\varepsilon_A$ is a feasible solution for each $t \in \mathbb{R}$, we have $D_1[u] \le D_1[u - t\varepsilon_A]$. Lemma 3.3 implies that u is strongly 1-harmonic in D. □

Theorem 3.2 (the maximum principle for strongly 1-subharmonic functions)**.** *Let D be a finite subset of X. If u is a strongly 1-subharmonic function in D, then $\max_D u \le \max_{\partial D} u$.*

Proof. Suppose that $A := \{x \in D; u(x) > \max_{\partial D} u\} \neq \emptyset$. For $y \in A \ominus (X \setminus A)$ we have

$$K(\mathrm{n}_A(y), y)r(y)du(y) = -u(\mathrm{n}_A(y)) + u(\mathrm{n}_{X\setminus A}(y)) < 0.$$

Taking a small $t > 0$ with $K(\mathrm{n}_A(y), y)r(y)du(y) + t < 0$ for all $y \in A \ominus (X \setminus A)$, we have

$$\sum_{y \in A \ominus (X\setminus A)} \mathrm{sgn}\big(K(\mathrm{n}_A(y), y)r(y)du(y) + t\big) < 0,$$

which contradicts the strong 1-subharmonicity of u. □

Theorem 3.3 (the maximum principle for 1-subharmonic functions)**.** *Assume that $N = \{X, Y, K, r\}$ is an infinite tree such that $\deg(x) \ge 3$ for all $x \in X$. Let D be a finite subset of X. If u is 1-subharmonic in D, then $\max_D u \le \max_{\partial D} u$.*

Proof. Suppose that $\alpha := \max_D u > \max_{\partial D} u$. Let $A = \{x \in D; u(x) = \alpha\}$. Since A is a finite set, there exists $a \in A$ such that $\partial a \cap A$ consists of at most one node. For $y \in \{a\} \ominus (X \setminus \{a\})$

$$K(a, y)r(y)du(y) = -u(a) + u(\mathrm{n}_{X\setminus\{a\}}(y)).$$

If $\mathrm{n}_{X\setminus\{a\}}(y) \in A$, then $K(a, y)r(y)du(y) = 0$; otherwise $K(a, y)r(y)du(y) < 0$. Since a has at least three neighbors, we have $M_+(S_a(u)) < 0$, which contradicts the 1-subharmonicity of u. □

The maximum principle for 1-harmonic functions does not hold in general; see Example 3.1.

Lemma 3.4. *Let $D \subset X$. Let $\{u_n\}_n \subset L(\overline{D})$ and $u \in L(\overline{D})$ be such that $\lim_{n\to\infty} u_n(x) = u(x)$ for each $x \in \overline{D}$.*

1. *If u_n is 1-harmonic (1-subharmonic, 1-superharmonic, resp.) in D for all n, then u is 1-harmonic (1-subharmonic, 1-superharmonic, resp.) in D.*
2. *If u_n is strongly 1-harmonic (strongly 1-subharmonic, strongly 1-superharmonic, resp.) in D for all n, then u is strongly 1-harmonic (strongly 1-subharmonic, strongly 1-superharmonic, resp.) in D.*

Proof. (1) Suppose that u_n is 1-superharmonic in D for all n. Let $a \in D$. Since $M_-(S_a(u_n)) \le 0$ for all n, Lemma 3.1 shows that $M_-(S_a(u)) \le 0$. This means that u is 1-superharmonic in D. The other statements follows similarly.

(2) Suppose that u_n is strongly 1-superharmonic in D for all n. Let A be a finite subset of D. Let $t < 0$ and $\varepsilon > 0$ be such that $t + \varepsilon < 0$. We take n so large that

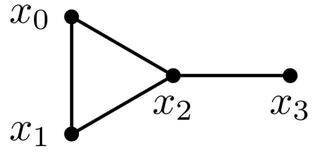

FIGURE 1

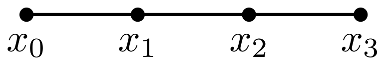

FIGURE 2

$|du(y) - du_n(y)| \le \varepsilon r(y)^{-1}$ for all $y \in A \ominus (X \setminus A)$. Then

$$K(\mathrm{n}_A(y), y)r(y)du(y) \le K(\mathrm{n}_A(y), y)r(y)du_n(y) + \varepsilon,$$

so that

$$\sum_{y \in A \ominus (X \setminus A)} \mathrm{sgn}\big(K(\mathrm{n}_A(y), y)r(y)du(y) + t\big) \le \sum_{y \in A \ominus (X \setminus A)} \mathrm{sgn}\big(K(\mathrm{n}_A(y), y)r(y)du_n(y) + t + \varepsilon\big).$$

Since u_n is strongly 1-superharmonic in D and $t + \varepsilon < 0$, it follows that the right-hand side is non-positive, and so is the left-hand side. This implies that u is strongly 1-superharmonic in D. The other statements follows similarly. □

Example 3.1. Consider the network shown in figure 1. Let $u(x_0) = u(x_1) = u(x_2) = 1$ and $u(x_3) = 0$. Let $D = \{x_0, x_1, x_2\}$. We easily verify the following:

(1) u is strongly 1-harmonic at x_2 and $\Delta_1 u(x_2) = -1$, i.e., the 1-Laplacian does not vanish for a strongly 1-harmonic function;
(2) u is strongly 1-harmonic at each node in D and not strongly 1-harmonic in D, i.e., the strong 1-harmonicity is not a local property;
(3) u is 1-harmonic in D and is not strongly 1-harmonic in D;
(4) $\max_D u > \max_{\partial D} u$, i.e., a 1-harmonic function does not satisfy the maximum principle.

Example 3.2. Let us consider the network shown in Figure 2. Let $u(x_0) = 0$, $u(x_1) = \alpha$, $u(x_2) = \beta$, and $u(x_3) = 1$. It is easily seen that u is strongly 1-harmonic in $\{x_1, x_2\}$ for any α and β with $0 \le \alpha \le \beta \le 1$. This shows that there are many solutions to the Dirichlet problem with respect to strongly 1-harmonic functions. Also this implies that Harnack's inequality does not hold for strongly 1-harmonic functions.

More precisely, we show the following:

Proposition 3.3. *Let $X = \{x_n\}_{n=-\infty}^{\infty}$ and $Y = \{y_n\}_{n=-\infty}^{\infty}$. Let $K(x_{n-1}, y_n) = -1$, $K(x_n, y_n) = 1$ and $K(x, y) = 0$ for any other pairs. A function $u \in L(X)$ is strongly 1-harmonic in X if and only if $\{u(x_n)\}_n$ is either non-decreasing or non-increasing.*

Proof. First assume that u is strongly 1-harmonic in X. The maximum principle (Theorem3.2) shows that $\{u(x_n)\}_n$ is either non-decreasing or non-increasing.

Conversely, we assume that $\{u(x_n)\}_n$ is non-decreasing. Let $A \subset X$ be a finite set. We need to show that

$$t \sum_{x \in A} \sum_{y \in Y} K(x,y) \operatorname{sgn}(du(y) - td\varepsilon_A(y)) \geq 0$$

for all $t \in \mathbb{R}$. Notice that $\sum_{x \in A} K(x,y) \operatorname{sgn}(du(y) - td\varepsilon_A(y)) = 0$ for $y \in Y$ with $e(y) \subset A$. Let

$$\{y \in Y \mid e(y) \cap A \text{ consists of exactly one node}\} = \{y_{l_1}, y_{r_1}, y_{l_2}, y_{r_2}, \dots, y_{l_k}, y_{r_k}\}$$

with $l_1 < r_1 < l_2 < r_2 < \cdots < l_k < r_k$. It suffices to show that

(2)
$$t \sum_{x \in A} \Big(K(x, y_{l_j}) \operatorname{sgn}(du(y_{l_j}) - td\varepsilon_A(y_{l_j})) + K(x, y_{r_j}) \operatorname{sgn}(du(y_{r_j}) - td\varepsilon_A(y_{r_j})) \Big) \geq 0$$

for each j and for all $t \in \mathbb{R}$.

We know that $e(y_{l_j}) \cap A = x_{l_j}$ and $e(y_{r_j}) \cap A = x_{r_j - 1}$. The left-hand side of (2) equals to

$$\begin{aligned} & t\Big(K(x_{l_j}, y_{l_j}) \operatorname{sgn}(du(y_{l_j}) - td\varepsilon_A(y_{l_j})) + K(x_{r_j-1}, y_{r_j}) \operatorname{sgn}(du(y_{r_j}) - td\varepsilon_A(y_{r_j})) \Big) \\ &= t\Big(\operatorname{sgn}(-\frac{u(x_{l_j}) - u(x_{l_j - 1})}{r(y_{l_j})} + \frac{t}{r(y_{l_j})}) - \operatorname{sgn}(-\frac{u(x_{r_j}) - u(x_{r_j-1})}{r(y_{r_j})} - \frac{t}{r(y_{r_j})}) \Big) \end{aligned}$$

(3)
$$= t \left(\operatorname{sgn}(t - u(x_{l_j}) + u(x_{l_j-1})) + \operatorname{sgn}(t + u(x_{r_j}) - u(x_{r_j-1})) \right).$$

Note that $u(x_{l_j}) - u(x_{l_j-1}) \geq 0$ and $u(x_{r_j}) - u(x_{r_j-1}) \geq 0$. Also note that $t(\operatorname{sgn}(t - \alpha) + \operatorname{sgn}(t + \beta)) \geq 0$ for $t \in \mathbb{R}$ and for nonnegative numbers α and β. Thus (3) is nonnegative, and that (2) holds. □

A similar argument shows the following:

Proposition 3.4. *Let $X = \{x_{m,n}\}_{m \in \mathbb{Z}, n \in \mathbb{Z}}$ and $Y = \{y_{m,n}, y'_{m,n}\}_{m \in \mathbb{Z}, n \in \mathbb{Z}}$. Let $K(x_{m-1,n}, y_{m,n}) = -1$, $K(x_{m,n}, y_{m,n}) = 1$, $K(x_{m,n-1}, y'_{m,n}) = -1$, $K(x_{m,n}, y'_{m,n}) = 1$ and $K(x,y) = 0$ for any other pairs. If $u \in L(X)$ satisfies that both $\{u(x_{m,i})\}_i$ and $\{u(x_{j,n})\}_j$ are non-decreasing for each m and n, then u is strongly 1-harmonic in X.*

Proposition 3.5. *Let $u \in \mathbf{D}^{(1)}(N)$ and $\tilde{v} \in \mathbf{D}_0^{(1)}(N)$ satisfy the relation*

$$D_1[u - \tilde{v}] = \min\{D_1[u - v]; v \in \mathbf{D}_0^{(1)}(N)\}.$$

Then $u - \tilde{v}$ is strongly 1-harmonic in X.

Proof. For any finite subset $A \subset X$ and $t \in \mathbb{R}$, we have $\tilde{v} + t\varepsilon_A \in \mathbf{D}_0^{(1)}(N)$, so that $D_1[u - \tilde{v}] \leq D_1[u - \tilde{v} - t\varepsilon_A]$. Lemma 3.3 shows that $u - \tilde{v}$ is strongly 1-harmonic in X. □

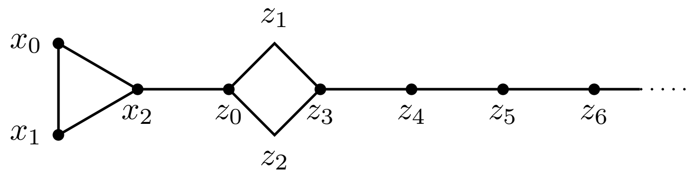

Figure 3

Question 3.1. For $u \in \mathbf{D}^{(1)}(N)$ does there exist $\tilde{v} \in \mathbf{D}_0^{(1)}(N)$ satisfying the relation in Proposition 3.5?

Proposition 3.6. *Denote by $\tilde{\mathbf{HD}}^{(1)}(N)$ the set of $u \in \mathbf{D}^{(1)}(N)$ such that u is strongly 1-harmonic in X. Then $\mathbf{D}_0^{(1)}(N) \cap \tilde{\mathbf{HD}}^{(1)}(N) = \{0\}$.*

Proof. Let $u \in \mathbf{D}_0^{(1)}(N) \cap \tilde{\mathbf{HD}}^{(1)}(N)$. Lemma 2.4 implies that, for any $\varepsilon > 0$, there exists a finite subset $D \subset X$ such that $|u(x)| < \varepsilon$ for $x \in X \setminus D$. Theorem 3.2 shows that $-\varepsilon < \min_{\partial D} u \le \min_D u \le \max_D u \le \max_{\partial D} u < \varepsilon$. This implies that $|u| \le \varepsilon$ in X. Since ε is arbitrary, it follows that $u \equiv 0$. □

Example 3.3. Let $\mathbf{HD}^{(1)}(N)$ be the set of $u \in \mathbf{D}^{(1)}(N)$ such that u is 1-harmonic in X. We consider the network N shown in Figure 3. Let $u(x_j) = 1$ for $j = 0, 1, 2$ and $u(z_j) = 0$ for $j \ge 0$. Then $u \in L_0(X) \cap \mathbf{HD}^{(1)}(N) \subset \mathbf{D}_0^{(1)}(N) \cap \mathbf{HD}^{(1)}(N)$. Namely $\mathbf{D}_0^{(1)}(N) \cap \mathbf{HD}^{(1)}(N) \ne \{0\}$.

Let A and B be mutually disjoint nonempty subsets of X and consider the following extremal problems:

$$d_1(A, B) = \inf\{D_1[u]; u \in \mathbf{D}^{(1)}(N), u = 1 \text{ on } A, u = 0 \text{ on } B\},$$
$$d_1(A, \infty) = \inf\{D_1[u]; u \in L_0(X), u = 1 \text{ on } A\}.$$

Theorem 3.4. *Assume that $d_1(A, B) < \infty$. Then there exists an optimal solution v to the problem $d_1(A, B)$ such that $0 \le v \le 1$ in X and v is strongly 1-harmonic in $X \setminus (A \cup B)$.*

Proof. Let $D = X \setminus (A \cup B)$. Let $f = 1$ on A and $f = 0$ on B. Applying Theorem 3.1 we find an optimal solution u to $d_1(A, B)$. Let $v = \min(\max(u, 0), 1)$. Since v is also a feasible solution and $D_1[v] \le D_1[u]$, we see that v is also an optimal solution. Theorem 3.1 implies that v is strongly 1-harmonic in $X \setminus (A \cup B)$. □

Lemma 3.5. *Let $\{N_n = \langle X_n, Y_n \rangle\}$ be an exhaustion of N with $A \subset X_1$. Then $\lim_{n\to\infty} d_1(A, X \setminus X_n) = d_1(A, \infty)$.*

Proof. It is easy to see that $d_1(A, X \setminus X_n) \ge d_1(A, X \setminus X_{n+1}) \ge d_1(A, \infty)$. Conversely, for $\varepsilon > 0$, we find $u \in L_0(X)$ such that $D_1[u] \le d_1(A, \infty) + \varepsilon$ and $u = 1$ on A. We take n so large that $u = 0$ on $X \setminus X_n$. Then $d_1(A, X \setminus X_n) \le D_1[u] \le d_1(A, \infty) + \varepsilon$. Since ε is arbitrary, we have $\lim_{n\to\infty} d_1(A, X \setminus X_n) \le d_1(A, \infty)$. □

Theorem 3.5. *For $a \in X$ there exists a function $u \in L_0(X)$ such that u is strongly 1-superharmonic in X, strongly 1-harmonic in $X \setminus \{a\}$, $u(a) = 1$, and $0 \le u \le 1$ on X.*

Proof. Let A and B be mutually disjoint nonempty subsets of X. Let

$$EL_1(A,B)^{-1} = \inf\{H_1[w];\ \begin{array}{l} w \in L_1^+(Y), \\ \sum_{y \in C_Y(P)} r(y)w(y) \geq 1 \text{ for all path } P \in \mathbf{P}_{A,B} \end{array} \},$$

$$EL_1(A,\infty)^{-1} = \inf\{H_1[w];\ \begin{array}{l} w \in L_1^+(Y), \\ \sum_{y \in C_Y(P)} r(y)w(y) \geq 1 \text{ for all path } P \in \mathbf{P}_{A,\infty} \end{array} \},$$

$$EW_\infty(A,\infty)^{-1} = \inf\{H_1[w];\ \begin{array}{l} w \in L_\infty^+(Y), \\ \sum_{y \in Q} w(y) \geq 1 \text{ for all cut } Q \in \mathbf{Q}_{A,\infty} \end{array} \}.$$

For nonempty finite subsets A and B of X with $A \cap B = \emptyset$ and for an exhaustion $\{N_n = \langle X_n, Y_n \rangle\}$, Nakamura and Yamasaki [2, Theorem 2.1] and [3, Theorem 3.2, Theorem 3.4, Corollary 2] showed that

$$\begin{aligned} d_1(A,B) &= EL_1(A,B)^{-1}, \\ \lim_{n\to\infty} EL_1(A, X \setminus X_n) &= EL_1(A,\infty), \\ EL_1(A,\infty)^{-1} &= EW_\infty(A,\infty), \\ EW_\infty(A,\infty) &= \inf\{\sum_{y \in Q} 1; Q \in \mathbf{Q}_{A,\infty}\}. \end{aligned}$$

Lemma 3.5 implies that

$$d_1(A,\infty) = \inf\{\sum_{y \in Q} 1; Q \in \mathbf{Q}_{A,\infty}\} = \inf\{\#Q; Q \in \mathbf{Q}_{A,\infty}\}.$$

Now let $A = \{a\}$. Since $d_1(\{a\},\infty) < \infty$, it follows that $d_1(\{a\},\infty) = \#Q$ for some cut $Q \in \mathbf{Q}_{A,\infty}$. Let A be a finite subset of X such that $Q = A \ominus (X \setminus A)$. Let u be a function such that $u = 1$ on A and $u = 0$ on $X \setminus A$. Then $D_1[u] = d_1(\{a\},\infty)$. We take n so large that $A \subset X_n$. Then u is an optimal solution to the problem $d_1(\{a\}, X \setminus X_n)$. Theorem 3.1 shows that u is strongly 1-harmonic in $X \setminus (\{a\} \cup X_n)$. Since n is arbitrary, u is strongly 1-harmonic in $X \setminus \{a\}$.

Next we shall prove that u is strongly 1-superharmonic in X. Let D be a finite subset of X. Let $D' = D \setminus A$. Let

$$I(t) = \sum_{x \in D} \Delta_1(u - t\varepsilon_D)(x), \qquad I'(t) = \sum_{x \in D'} \Delta_1(u - t\varepsilon_{D'})(x).$$

We have

$$\begin{aligned} I(t) &= \sum_{y \in D \ominus (X \setminus D)} \operatorname{sgn}\big(K(\mathrm{n}_D(y), y) r(y) du(y) + t\big) \\ &= \sum_{y \in D' \ominus (X \setminus D)} \operatorname{sgn}\big(K(\mathrm{n}_{D'}(y), y) r(y) du(y) + t\big) \\ &\quad + \sum_{y \in (D \cap A) \ominus (X \setminus D)} \operatorname{sgn}\big(K(\mathrm{n}_{D \cap A}(y), y) r(y) du(y) + t\big) \end{aligned}$$

and

$$I'(t) = \sum_{y \in D' \ominus (X \setminus D')} \operatorname{sgn}\bigl(K(\mathrm{n}_{D'}(y), y) r(y) du(y) + t\bigr)$$
$$= \sum_{y \in D' \ominus (X \setminus D)} \operatorname{sgn}\bigl(K(\mathrm{n}_{D'}(y), y) r(y) du(y) + t\bigr)$$
$$+ \sum_{y \in D' \ominus (D \cap A)} \operatorname{sgn}\bigl(K(\mathrm{n}_{D'}(y), y) r(y) du(y) + t\bigr),$$

and obtain

$$I(t) - I'(t) = \sum_{y \in (D \cap A) \ominus (X \setminus D)} \operatorname{sgn}\bigl(K(\mathrm{n}_{D \cap A}(y), y) r(y) du(y) + t\bigr)$$
$$- \sum_{y \in D' \ominus (D \cap A)} \operatorname{sgn}\bigl(K(\mathrm{n}_{D'}(y), y) r(y) du(y) + t\bigr).$$

For $y \in (D \cap A) \ominus (X \setminus D)$

$$K(\mathrm{n}_{D \cap A}(y), y) r(y) du(y) = -u(\mathrm{n}_{D \cap A}(y)) + u(\mathrm{n}_{X \setminus D}(y)) \leq 0,$$

and

$$\sum_{y \in (D \cap A) \ominus (X \setminus D)} \operatorname{sgn}\bigl(K(\mathrm{n}_{D \cap A}(y), y) r(y) du(y) + t\bigr) \leq 0$$

for all $t \leq 0$. For $y \in D' \ominus (D \cap A)$

$$K(\mathrm{n}_{D'}(y), y) r(y) du(y) = -u(\mathrm{n}_{D'}(y)) + u(\mathrm{n}_{D \cap A}(y)) = 1.$$

For $t \geq -1$ we have

$$\sum_{y \in D' \ominus (D \cap A)} \operatorname{sgn}\bigl(K(\mathrm{n}_{D'}(y), y) r(y) du(y) + t\bigr) \geq 0.$$

Therefore $I(t) - I'(t) \leq 0$ for $-1 \leq t \leq 0$. Since u is strongly 1-harmonic in $X \setminus \{a\}$, we know that $I'(t) \leq 0$ for $t < 0$, so that $I(t) \leq 0$ for $-1 \leq t < 0$. Since $I(t)$ is nondecreasing function of t, it follows that $I(t) \leq 0$ for $t < 0$. This means that u is strongly 1-superharmonic in X. □

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