Convergence of solutions of nonlinear systems with integrable forcing term and its applications to a biological model

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Abstract

A non-autonomous nonlinear system with a time-variational forcing term is considered. For such a system, the global attraction of the origin is discussed, whose result is suggestive to an ecological problem as well because the system is reduced to a Lotka-Volterra predator-prey model with prey receiving an environmental time-variation by an appropriate transformation. A numerical simulation is also shown to suggest a new mathematical insight beyond our intuition.

Key words: Global attraction; Predator-prey systems; Weakly integrally positive; Time-variation
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1. Introduction

Consider the nonlinear system

$$\begin{aligned} x' &= c(1-e^{-y}), \\ y' &= -\,a(1-e^{-x}), \end{aligned} \tag{1.1}$$

where the prime denotes d/dt and parameters a and c are assumed to be positive. This system has a single equilibrium point $(0,0)$, which is a center, i.e., a “neutrally stable ” equilibrium surrounded by a family of periodic orbits whose amplitudes depend on the initial data since a conserved quantity $V(x,y)$ is given as

$$V(x,y) = a(e^{-x}+x-1)+c(e^{-y}+y-1).$$

The importance of these properties is the fact that system (1.1) has relevance to a biological problem. By the transformation $x = -\log(bP/a)$ and $y = -\log(dN/c)$

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Email addresses: `jsugie@riko.shimane-u.ac.jp` (Jitsuro Sugie), `ysaito@riko.shimane-u.ac.jp` (Yasuhisa Saito) for positive constants b and d, system (1.1) is reduced to the classical Lotka-Volterra model which is well-known as the origin of theoretical study on predator-prey systems in mathematical ecology:

$$\begin{aligned} N' &= (a - bP)N, \\ P' &= (-\,c + dN)P. \end{aligned} \tag{LV}$$

Here N and P represent the prey and predator population densities, respectively. Correspondingly to the properties of system (1.1) mentioned above, system (LV) has a single interior equilibrium point $(c/d, a/b)$, which is also a center surrounded by a family of periodic orbits whose amplitudes depend on the initial population sizes. This implies that the population state once changed by an external factor cannot return to the original one. Besides, the slightest change to the structure of system (LV) typically results in qualitatively different behavior (see [3]). This structural instability is often criticized because it is desirable that models describing periodical population behavior observed in nature involve robust properties such that population states strayed away from the orbit will return to the original orbit as time passes. In fact, predator-prey systems in nature apparently persist stably (in spite of being affected by external factors). This gap suggests that our insight is not enough to understand mechanisms acting in nature which stabilize population dynamics. To resolve the gap, theoreticians and experimentalists have made a long list of such processes (see, for example, [1, 4, 9, 10, 11]).

In connection with such an ecological aspect, it is significant to consider what additional factors can stabilize the origin $(0, 0)$ for system (1.1). For example, Sugie *et al.* [17] have proved that the origin can be globally stabilized by adding $-\xi(t)(1 - e^{-y})$ with a nonnegative function $\xi(t)$ to the right-hand side of the second equation of (1.1) as follows:

$$\begin{aligned} x' &= c(1 - e^{-y}), \\ y' &= -\,a(1 - e^{-x}) - \xi(t)(1 - e^{-y}). \end{aligned} \tag{1.2}$$

In particular, mathematically surprising thing is that the global stabilization was shown to be realized even by nonnegative functions $\xi(t)$ which converge to 0, despite the fact that the limiting system of (1.2) is system (1.1). Special cases of these results also contribute to the above-mentioned ecological problem of stabilizing system (LV).

In this paper, we develop our considerations into the following system with a forcing term $p(t)$:

$$\begin{aligned} x' &= c(1 - e^{-y}), \\ y' &= -\,a(1 - e^{-x}) - \xi(t)(1 - e^{-y}) + p(t), \end{aligned} \tag{E}$$

where a, c and $\xi(t)$ are the same as the ones given above and $p(t)$ is a continuous function of t. In the next section, we present our results and prove them, where the global attraction of the origin can be realized by using a similar method of [17] under the assumption that the forcing term $p(t)$ is absolutely integrable. In Section 3, we discuss a nonautonomous Lotka-Volterra predator-prey model which is equivalent to system (E) and supply a new aspect of the above-mentioned biological problem. Many studies have been made on nonautonomous Lotka-Volterra models. For example, see [2]. However, there is little research using the concept of the weak integral positivity (see Section 2 for the definition).

2. Mathematical results and proofs

The purpose of this section is to examine the asymptotic behaviour of all solutions of (E). To begin with, we show that all solution of (E) are bounded. Next, we give sufficient conditions for all solution of (E) to converge to the origin $(0,0)$ as time t increases.

Theorem 2.1. *If $\xi(t)$ is nonnegative for $t \geq 0$ and $p(t)$ is absolutely integrable on $[0,\infty)$, then all solutions of (E) are uniformly bounded.*

Proof. Define

$$V(x,y) = af(x) + cf(y),$$

where $f(z) = e^{-z} + z - 1$ for $z \in \mathbb{R}$. It is clear that

$$f(z) \geq \begin{cases} z^2/4 & \text{if } \ 0 \leq |z| \leq 2 \\ |z| - 1 & \text{if } \ |z| > 2. \end{cases}$$

Differentiate $V(x,y)$ along any solution of (E) to obtain

$$\dot{V}_{(E)}(t,x,y) = -c\,\xi(t)\big(1 - e^{-y}\big)^2 + c\,p(t)\big(1 - e^{-y}\big)$$

on $[0,\infty) \times \mathbb{R}^2$. Let

$$P(t) = \int_0^t |p(s)|ds.$$

Next, define $W(t,x,y) = e^{-2P(t)}V(x,y)$ on $[0,\infty) \times \mathbb{R}^2$, so that

$$\begin{aligned} \dot{W}_{(E)}(t,x,y) &= -\,2|p(t)|e^{-2P(t)}V(x,y) + e^{-2P(t)}\dot{V}_{(E)}(t,x,y) \\ &\leq -\,2c|p(t)|e^{-2P(t)}f(y) - c\,\xi(t)e^{-2P(t)}\big(1 - e^{-y}\big)^2 \\ &\quad + c\,p(t)e^{-2P(t)}\big(1 - e^{-y}\big) \\ &\leq -\,c|p(t)|e^{-2P(t)}\big(2f(y) - \big|1 - e^{-y}\big|\big) \end{aligned}$$

Taking into account that

$$2f(y) - \big|1 - e^{-y}\big| = 2\big(e^{-y} + y - 1\big) - \big|1 - e^{-y}\big| \geq -\frac{1}{2}$$

for $y \in \mathbb{R}$, we obtain

$$\dot{W}_{(E)}(t,x,y) \leq \frac{c}{2}|p(t)|e^{-2P(t)}$$

Let $\chi(t) = c|p(t)|e^{-2P(t)}/2$ and take

$$U(t,x,y) = W(t,x,y) + \int_t^\infty \chi(s)ds$$

on $[0,\infty) \times \mathbb{R}^2$, so that

$$\dot{U}_{(E)}(t,x,y) = \dot{W}_{(E)}(t,x,y) - \chi(t) \leq 0.$$

Since $p(t)$ is absolutely integrable on $[0,\infty)$, we can find a positive number M such that $P(t) < M$ for $t \geq 0$. Hence, we see that

$$U(t,x,y) \geq W(t,x,y) \geq e^{-2M}V(x,y) = e^{-2M}\big(af(x)+cf(y)\big)$$

and

$$\begin{aligned} U(t,x,y) &= W(t,x,y) + \frac{c}{2}\int_t^\infty |p(s)|e^{-2P(s)}ds \leq W(t,x,y) + \frac{c}{2}\int_0^\infty |p(s)|ds \\ &\leq V(x,y) + \frac{c}{2}M = af(x)+cf(y) + \frac{c}{2}M \end{aligned}$$

on $[0,\infty)\times\mathbb{R}^2$. To sum up,

(i) $U(t,x,y)$ is not less than a continuous, strictly increasing and divergent function with respect to $|x|+|y|$;

(ii) $U(t,x,y)$ is not greater than another continuous, strictly increasing function with respect to $|x|+|y|$;

(iii) $\dot{U}_{(E)}(t,x,y)$ is nonpositive.

Hence, we conclude that all solutions of (E) are uniformly bounded by using a Lyapunov-type theorem due to Yoshizawa [18] (refer also to Theorem 10.2 in [19, p. 38]). ☐

Remark 2.1. In Theorem 2.1, if $p(t)$ is nonnegative for $t \geq 0$, then we can show that all solutions of (E) are equi-bounded without assuming the absolute integrability of $p(t)$. As to the difference between the terms 'uniformly bounded' and 'equi-bounded', refer the books [8, 12, 19, 20] for example.

Let us leave the boundedness of solutions of (E) and turn to the second subject; that is, the attraction of all solutions of (E). To state our result concerning the second topic, we define a family of functions. A nonnegative function ϕ is called *weakly integrally positive* if

$$\int_I \phi(t)dt = \infty$$

for every set $I = \bigcup_{n=1}^{\infty}[\tau_n,\sigma_n]$ such that $\tau_n + \delta < \sigma_n < \tau_{n+1} \leq \sigma_n + \Delta$ for some $\delta > 0$ and $\Delta > 0$. We can find the concept of the weak integral positivity in the papers [5, 6, 7, 13, 14, 16]. Even some of nonnegative and decaying functions are included in the family of weakly integrally positive functions. For example, $1/(1+t)$ and $\sin^2 t/(1+t)$ are weakly integrally positive (for the proof, see Proposition 2.1 in [15]). Any nonnegative periodic function is also weakly integrally positive. If ϕ is weakly integrally positive, then it naturally follows that

$$\lim_{t\to\infty}\int^t \phi(s)ds = \infty.$$

We are now ready to exhibit our second result.

Theorem 2.2. *In addition to the assumptions in Theorem* 2.1, *if* $\xi(t)$ *is bounded and weakly integrally positive, then all solutions of* (E) *tend to the origin* $(0,0)$ *as* $t \to \infty$.

Before proving Theorem 2.2, it is helpful to describe the properties of functions $f(z)$ and

$$g(z) \overset{\text{def}}{=} \frac{d}{dz}f(z) = 1 - e^{-z}$$

for $z \in \mathbb{R}$. Since $f(z)$ is increasing for $z \geq 0$ and decreasing for $z \leq 0$, there exists the inverse function $\hat{f}^{-1}(w)$ of $w = \hat{f}(z) \overset{\text{def}}{=} f(z)\mathrm{sgn}z$. Needless to say, $\hat{f}^{-1}(w)$ is increasing for $w \in \mathbb{R}$ and $\hat{f}^{-1}(0) = 0$. It is clear that $f(-z) \geq f(z)$ for $z \geq 0$, with equality if and only if $z = 0$. From this inequality it follows that

$$0 \leq f(z) \leq f(-\alpha) \quad \text{for } |z| \leq \alpha \tag{2.1}$$

with α positive and

$$0 < -\hat{f}^{-1}(-w) < \hat{f}^{-1}(w) \quad \text{for } w > 0.$$

It is also clear that $g(z)$ is increasing for $z \in \mathbb{R}$ with $g(0) = 0$, $\lim_{z\to\infty} g(z) = 1$, $\lim_{z\to-\infty} g(z) = -\infty$ and the inequality $-g(-z) \geq g(z)$ holds for $z \geq 0$, with equality if and only if $z = 0$. Hence, it turns out that $g(z)$ has the following properties:

$$g^2(z) \geq g^2(\alpha) > 0 \quad \text{for } |z| \geq \alpha \tag{2.2}$$

and

$$|g(z)| \leq |g(-\alpha)| \quad \text{for } |z| \leq \alpha \tag{2.3}$$

with α positive.

Proof of Theorem 2.2. Let $(x(t), y(t))$ be any solution of (E) with the initial time $t_0 \geq 0$. It follows from Theorem 2.1 that there exists a $\beta > 0$ such that

$$|x(t)| + |y(t)| < \beta \quad \text{for } t \geq t_0. \tag{2.4}$$

Let

$$v(t) = V(x(t), y(t)) = af(x(t)) + cf(y(t)). \tag{2.5}$$

Then, by (2.3) and (2.4) we have

$$v'(t) = -c\xi(t)g^2(y(t)) + cp(t)g(y(t)) \leq c|p(t)||g(y(t))| \leq c|g(-\beta)||p(t)| \tag{2.6}$$

for $t \geq t_0$. For the sake of brevity, we write

$$v'_+(t) = \max\{0, v'(t)\} \quad \text{and} \quad v'_-(t) = \max\{0, -v'(t)\}.$$

Note that $v'(t) = v'_+(t) - v'_-(t)$ and $|v'(t)| = v'_+(t) + v'_-(t)$. Since $c|g(-\beta)||p(t)| \geq 0$ for $t \geq 0$, we get

$$v'_+(t) \leq c|g(-\beta)||p(t)| \quad \text{for } t \geq t_0.$$

Integrating both sides from t_0 to ∞, we obtain

$$\int_{t_0}^{\infty} v'_+(t)dt \le c|g(-\beta)|\int_{t_0}^{\infty} |p(t)|dt < \infty.$$

On the other hand, we have

$$\int_{t_0}^{\infty} v'_-(t)dt = \lim_{t\to\infty}\left[\int_{t_0}^{t} v'_+(s)ds - \int_{t_0}^{t} v'(s)ds\right] \le \int_{t_0}^{\infty} v'_+(t)dt + v(t_0) - \liminf_{t\to\infty} v(t).$$

Since $v(t) \ge 0$ for $t \ge t_0$, we see that

$$\int_{t_0}^{\infty} v'_-(t)dt \le \int_{t_0}^{\infty} v'_+(t)dt + v(t_0) < \infty.$$

We therefore conclude that

$$\int_{t_0}^{\infty} |v'(t)|dt = \int_{t_0}^{\infty} v'_+(t)dt + \int_{t_0}^{\infty} v'_-(t)dt < \infty,$$

namely, $v'(t)$ is absolutely integrable. From this fact it follows that $v(t)$ has a nonnegative limit v_0.

If $v_0 = 0$, then from (2.5) we see that both $x(t)$ and $y(t)$ converge to zero as $t \to \infty$, and therefore, the solution $(x(t), y(t))$ of (E) tends to $(0, 0)$ as time t increases. This completes the proof. Thus, we have only to consider the case in which $v_0 > 0$. To tell the truth, this case does not happen. We will show this fact hereafter.

Since $|y(t)|$ is bounded, there exist the lower limit and the upper limit of $|y(t)|$. First, we shall show that the lower limit of $|y(t)|$ is zero, and we shall then show that the upper limit of $|y(t)|$ is also zero.

Suppose that $\liminf_{t\to\infty} |y(t)| > 0$. Then, we can find a $\gamma > 0$ and a $T_1 \ge t_0$ satisfying

$$|y(t)| > \gamma \quad \text{for} \ \ t \ge T_1.$$

From (2.2), we see that

$$g^2(y(t)) \ge g^2(\gamma) \quad \text{for} \ \ t \ge T_1,$$

and therefore, by (2.3) and (2.4) again,

$$v'(t) = -c\,\xi(t)g^2(y(t)) + c\,p(t)g(y(t)) \le -cg^2(\gamma)\xi(t) + c|g(-\beta)||p(t)|$$

for $t \ge T_1$. Hence, using (2.6), we obtain

$$-v(t_0) = v(t) - v(t_0) = \int_{t_0}^{t} v'(s)ds \le -cg^2(\gamma)\int_{T_1}^{t} h(s)ds + c|g(-\beta)|\int_{t_0}^{t} |p(s)|ds,$$

which tends to $-\infty$ as $t \to \infty$ because $h(t)$ is weakly integrally positive and $p(t)$ is absolutely integrable. This is a contradiction. Thus, it turns out that $\liminf_{t\to\infty} |y(t)| = 0$.

By way of contradiction, we suppose that $\lambda \overset{\text{def}}{=} \limsup_{t\to\infty} |y(t)| > 0$. Since $\xi(t)$ is bounded, we can choose a $\rho > 0$ satisfying

$$|\xi(t)| \le \rho \quad \text{for} \;\; t \ge 0. \tag{2.7}$$

Recall that $v(t)$ is assumed to tend to a positive value v_0 as $t \to \infty$. Hence, there exists a $T_2 \ge t_0$ such that

$$0 < \frac{1}{2}v_0 < v(t) < \frac{3}{2}v_0 \quad \text{for} \;\; t \ge T_2. \tag{2.8}$$

Let ε be so small that

$$0 < \varepsilon < -\hat{f}^{-1}\Bigl(-\frac{v_0}{2c}\Bigr), \tag{2.9}$$

$$\frac{\rho}{a}|g(-\varepsilon)| < 1 - \exp\biggl(-\hat{f}^{-1}\biggl(\frac{v_0 - cf(-\varepsilon)}{a}\biggr)\biggr) \tag{2.10}$$

and

$$\frac{\rho}{a}|g(-\varepsilon)| < \exp\biggl(-\hat{f}^{-1}\biggl(\frac{cf(-\varepsilon) - v_0}{a}\biggr)\biggr) - 1. \tag{2.11}$$

We can find such a positive number ε because $g(-\varepsilon)$ approaches zero and the right-hand sides of (2.10) and (2.11) approach positive numbers as $\varepsilon \to 0$. We also assume that $\varepsilon < \lambda/8$. Since $p(t)$ is absolutely integrable on $[0,\infty)$, it follows that

$$\int_{T_3}^{\infty} |p(t)|dt < \varepsilon \tag{2.12}$$

for some $T_3 \ge T_2$.

Since $\liminf_{t\to\infty} |y(t)| = 0$, we can choose two intervals $[\tau_n, \sigma_n]$ and $[t_n, s_n]$ with $[t_n, s_n] \subset [\tau_n, \sigma_n]$, $T_3 < \tau_n$ and $\tau_n \to \infty$ as $n \to \infty$ such that $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$, $|y(t_n)| = \lambda/2$, $|y(s_n)| = 3\lambda/4$ and

$$|y(t)| \ge \varepsilon \quad \text{for} \;\; \tau_n < t < \sigma_n, \tag{2.13}$$

$$|y(t)| \le \varepsilon \quad \text{for} \;\; \sigma_n < t < \tau_{n+1}, \tag{2.14}$$

$$\frac{1}{2}\lambda < |y(t)| < \frac{3}{4}\lambda \quad \text{for} \;\; t_n < t < s_n. \tag{2.15}$$

Using (2.1), (2.5), (2.8) and (2.14), we obtain

$$af(x(t)) = v(t) - cf(y(t)) > \frac{1}{2}v_0 - cf(-\varepsilon)$$

for $\sigma_n \le t \le \tau_{n+1}$. For the sake of brevity, let

$$w_0 = \frac{1}{2}v_0 - cf(-\varepsilon).$$

Then, it follows from (2.9) that w_0 is positive. Since $f(x(t)) > w_0/a$ for $\sigma_n \le t \le \tau_{n+1}$, there are two cases to consider: (i) $x(t) > \hat{f}^{-1}(w_0/a) > 0$; (ii) $x(t) < \hat{f}^{-1}(-w_0/a) < 0$.

In case (i), from (2.3), (2.7) and (2.14) and the second equation in system (E), we can estimate that

$$\begin{aligned} y'(t) &= -ag(x(t)) - \xi(t)g(y(t)) + p(t) \\ &\le -ag(x(t)) + \rho|g(y(t)| + p(t) \\ &\le -a\Big(1-\exp\Big(-\hat{f}^{-1}(w_0/a)\Big)\Big) + \rho|g(-\varepsilon)| + p(t) \end{aligned}$$

for $\sigma_n \le t \le \tau_{n+1}$. Let

$$\mu_1 = a\Big(1-\exp\Big(-\hat{f}^{-1}(w_0/a)\Big)\Big) - \rho|g(-\varepsilon)|.$$

Then, from (2.10), we see that μ_1 is a positive number that does not depend on n. Similarly, in case (ii), we obtain

$$\begin{aligned} y'(t) &\ge -ag(x(t)) - \rho|g(y(t)| + p(t) \\ &\ge a\Big(\exp\Big(-\hat{f}^{-1}(-w_0/a)\Big)-1\Big) - \rho|g(-\varepsilon)| + p(t) \end{aligned}$$

for $\sigma_n \le t \le \tau_{n+1}$. Let

$$\mu_2 = a\Big(\exp\Big(-\hat{f}^{-1}(-w_0/a)\Big)-1\Big) - \rho|g(-\varepsilon)|.$$

Then, from (2.11), we see that μ_2 is also positive and independent of n. In either case, we get

$$|y'(t) - p(t)| \ge \mu \quad \text{for} \;\; \sigma_n \le t \le \tau_{n+1},$$

where $\mu = \min\{\mu_1, \mu_2\} > 0$. Integrate this inequality from σ_n to τ_{n+1} to obtain

$$\begin{aligned} |y(\tau_{n+1})| + |y(\sigma_n)| + \int_{\sigma_n}^{\tau_{n+1}} |p(t)|dt &\ge \left| y(\tau_{n+1}) - y(\sigma_n) - \int_{\sigma_n}^{\tau_{n+1}} p(t)dt \right| \\ &= \left| \int_{\sigma_n}^{\tau_{n+1}} (y'(t) - p(t))dt \right| \\ &= \int_{\sigma_n}^{\tau_{n+1}} |y'(t) - p(t)|dt \ge \mu(\tau_{n+1} - \sigma_n). \end{aligned}$$

From (2.4) and (2.12), we see that $2\beta + \varepsilon > \mu(\tau_{n+1} - \sigma_n)$, or

$$\tau_{n+1} < \sigma_n + \Delta \quad \text{for} \;\; n \in \mathbb{N}, \tag{2.16}$$

where $\Delta = (2\beta + \varepsilon)/\mu > 0$.

Let $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$. Then, it follows from (2.2) and (2.13) that

$$g^2(y(t)) \ge g^2(\varepsilon) > 0 \quad \text{for} \;\; t \in I.$$

Using this inequality and (2.6), we obtain

$$\int_{t_0}^{\infty} v'(t)dt \le -cg^2(\varepsilon)\int_I \xi(t)dt + c|g(-\beta)|\int_{t_0}^{\infty} |p(t)|dt.$$

On the other hand,

$$\int_{t_0}^{\infty} v'(t)dt = \lim_{t\to\infty} v(t) - v(t_0) = v_0 - v(t_0) > -\infty.$$

Hence, taking into account that $p(t)$ is absolutely integrable, we see that

$$\int_I \xi(t)dt < \infty. \tag{2.17}$$

Suppose that there exists a $\delta > 0$ such that $\sigma_n - \tau_n > \delta$ for all $n \in \mathbb{N}$. Then, from (2.16) and the assumption that $h(t)$ is weakly integrally positive, it follows that

$$\int_I \xi(t)dt = \infty.$$

This contradicts (2.17). Thus, there is no such $\delta > 0$, namely, $\liminf_{n\to\infty}(\sigma_n - \tau_n) = 0$. Since $[t_n, s_n] \subset [\tau_n, \sigma_n]$, it turns out that

$$\liminf_{n\to\infty}(s_n - t_n) = 0. \tag{2.18}$$

From (2.3), (2.4) and (2.15), we see that

$$|g(x(t))| \le |g(-\beta)| \quad \text{for } \ t \ge t_0$$

and

$$|g(y(t))| \le |g(-3\lambda/4)| \quad \text{for } \ t_n \le t \le s_n.$$

Hence, we obtain

$$\begin{aligned}|y'(t)| &\le a|g(x(t))| + \rho|g(y(t))| + |p(t)|\\ &\le a|g(-\beta)| + \rho|g(-3\lambda/4)| + |p(t)|\\ &\overset{\text{def}}{=} \nu + |p(t)|\end{aligned}$$

for $t_n \le t \le s_n$. It is clear that ν is positive. Integrating this inequality from t_n to s_n, we get

$$\begin{aligned}\frac{1}{4}\lambda &= |y(s_n)| - |y(t_n)| \le |y(s_n) - y(t_n)|\\ &= \left|\int_{t_n}^{s_n} y'(t)dt\right| \le \int_{t_n}^{s_n} |y'(t)|dt \le \nu(s_n - t_n) + \int_{t_n}^{s_n} |p(t)|dt.\end{aligned}$$

Since $t_n > T_3$ and $\varepsilon < \lambda/8$, it follows from (2.12) that

$$\frac{1}{4}\lambda < \nu(s_n - t_n) + \varepsilon < \nu(s_n - t_n) + \frac{1}{8}\lambda.$$

This contradicts (2.18). We therefore conclude that $\limsup_{t\to\infty} |y(t)| = \lambda = 0$.

As proved above, $y(t)$ tends to zero as $t \to \infty$. Hence, there exists a $T_4 \geq T_3$ such that

$$|y(t)| < \varepsilon \quad \text{for} \;\; t \geq T_4.$$

Using this estimation instead of (2.14) and repeating the same process as in the above argument once more, we obtain

$$|y'(t) - p(t)| \geq \mu \quad \text{for} \;\; t \geq T_4.$$

This inequality yields

$$\begin{aligned} 2\beta + \varepsilon > |y(t) - y(T_4)| + \int_{T_4}^{t} |p(s)|ds &\geq \left| \int_{T_4}^{t} (y'(s) - p(s))ds \right| \\ = \int_{T_4}^{t} |y'(s) - p(s)|ds &\geq \mu(t - T_4) \end{aligned}$$

for $t \geq T_4$. This is a contradiction. Thus, the case of $v_0 > 0$ does not occur.

The proof of Theorem 2.2 is now complete. ☐

3. Application

Let

$$\xi(t) = \frac{c}{d}h(t) \quad \text{and} \quad p(t) = \xi(t) - k(t),$$

where c and d are positive constants, $h(t)$ is a nonnegative and continuous function, and $k(t)$ is a continuous function for $t \geq 0$. Then, by the transformation

$$x = -\log(bP/a) \quad \text{and} \quad y = -\log(dN/c),$$

system (E) is reduced to a predator-prey system of the form:

$$\begin{aligned} N' &= (a + k(t) - h(t)N - bP)N, \\ P' &= (-c + dN)P. \end{aligned} \tag{3.1}$$

This transformation is a one-to-one correspondence from the first quadrant $Q \overset{\text{def}}{=} \{(N, P) : N > 0 \text{ and } P > 0\}$ to the whole real plane $\{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$. The interior point $(c/d, a/b) \in Q$ corresponds to the origin $(0, 0) \in \mathbb{R}^2$. System (3.1) is a Lotka-Volterra model with a time-variational component of the environment, where prey are assumed to have a carrying capacity and to receive the environmental time-variation more effectively than their predators (the component was ignored in system (LV)). By virtue of Theorem 2.2, we have the following result.

Theorem 3.1. *Suppose that* $h(t)$ *is nonnegative for* $t \geq 0$ *and it is bounded and weakly integrally positive. Then, if* $ch(t)/d - k(t)$ *is absolutely integrable on* $[0, \infty)$*, then all trajectories of* (3.1) *go to the point* $(c/d, a/b)$ *as* $t \to \infty$*.*

Sugie *et al.* [17] have considered system (3.1) under the assumption that $ch(t)/d \equiv k(t)$. In this special case, system (3.1) has a unique interior equilibrium point $(c/d, a/b)$ even though it is nonautonomous. It was surprisingly proved that the equilibrium can be globally stabilized even by a nonnegative function $h(t)$ (or $k(t)$) that makes the limiting system which is equivalent to the structurally unstable model (LV). Our results here develop such a technical setting into a model that provides different time-variational functions on per capita birth and mortality rates of the prey, which is a more biologically practical scenario. In Theorem 3.1, the global attraction of the interior point $(c/d, a/b)$ additionally requires only the absolute integrability of $ch(t)/d - k(t)$, which is a trivial condition when $ch(t)/d \equiv k(t)$, but the surprising fact remains that all the solutions can be attracted to the interior point even by nonnegative functions $h(t)$ and $k(t)$ that make the limiting system equivalent to the structurally unstable model (LV).

When the function $ch(t)/d - k(t)$ is not absolutely integrable on $[0, \infty)$, we can no longer show that all solutions of (3.1) converge to $(c/d, a/b)$ even if

$$\frac{c}{d}h(t) - k(t) \to 0 \quad \text{as} \ \ t \to \infty. \tag{3.2}$$

For example, consider system (3.1) with

$$a = b = c = d = 1, \quad h(t) = \frac{1}{1+t} \quad \text{and} \quad k(t) = \frac{1}{1+t} - \frac{1}{\log(2+t)}. \tag{3.3}$$

Then, it is clear that $0 \leq h(t) \leq 1$ for $t \geq 0$, $h(t)$ is weakly integrally positive and

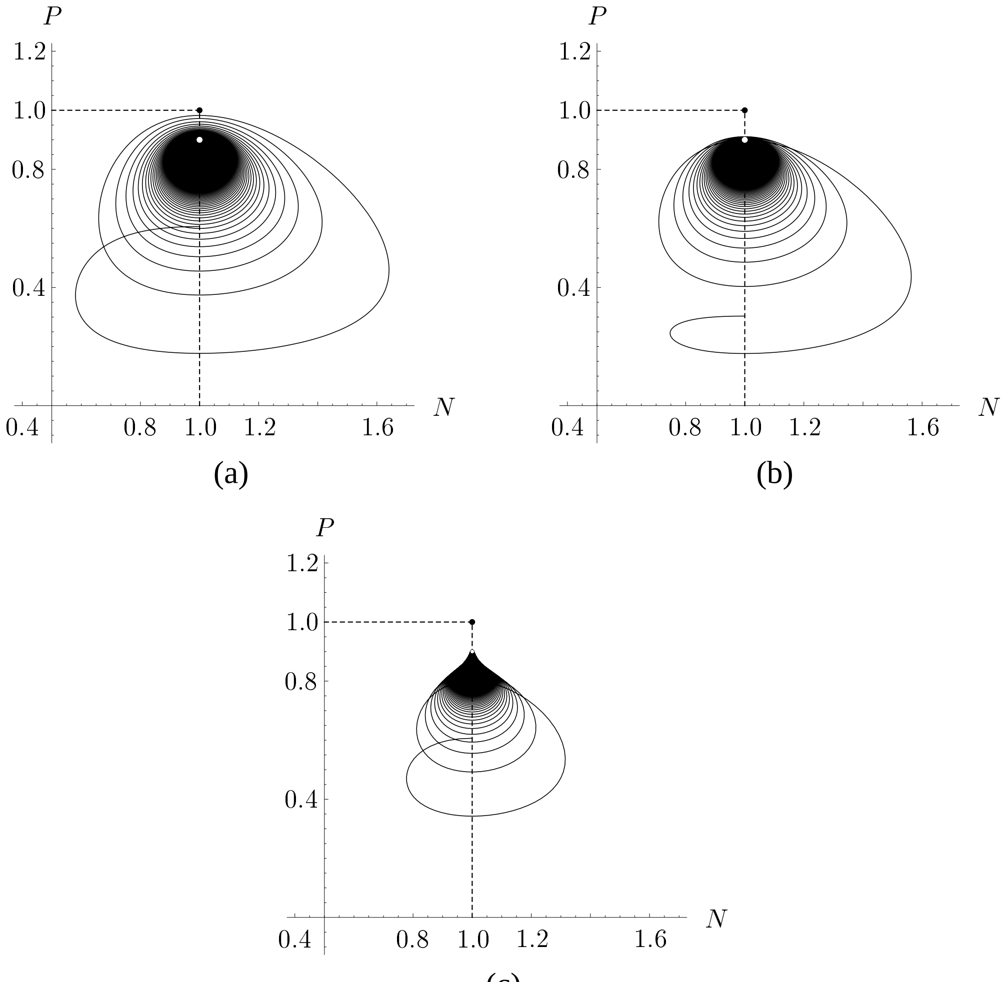
$$\frac{c}{d}h(t) - k(t) = \frac{1}{\log(2+t)} \to 0 \quad \text{as} \ \ t \to \infty.$$

However, $ch(t)/d - k(t)$ is not absolutely integrable, namely,

$$\int_0^\infty \left| \frac{c}{d}h(t) - k(t) \right| dt = \infty.$$

In Figure 3.1 (a), we sketch the trajectory of (3.1) with (3.3) starting from the point $(1, 1/\sqrt{e})$ at the initial time $t_0 = 0$. This trajectory rotates in a counterclockwise direction about a point $q \in Q$ infinitely many times and approaches q ultimately. What is important is that q does not coincide with $(c/d, a/b) = (1, 1)$. Notice that the coordinates of q are observed to be almost $(1, 0.9)$. This contradicts the conclusion of Theorem 3.1. Hence, in Theorem 3.1, we cannot replace the absolute integrability of $ch(t)/d - k(t)$ by condition (3.2). We numerically show other two trajectories of (3.1) with (3.3) which approach the point q in Figure 3.1 (b) and (c). Any trajectory of (3.1) which starts from a point other than $(1, 1/\sqrt{e})$ at the initial time $t_0 = 0$ will approach the point q ultimately (see Figure 3.1 (b)). In general, all trajectories of nonautonomous systems have various shapes according to the initial time even if they start from the same point. However, any trajectory of (3.1) converges to the point q ultimately (see Figure 3.1 (c)).

What is really the point q? This numerical result may be beyond the intuition that the possible point to whom the solution converges as $t \to \infty$ is $(c/d, a/b)$ since the limiting system of (3.1) is system (1.1) when $h(t)$ and $k(t)$ tend to 0 as $t \to \infty$. We ran additional simulations for the numerical work, with various combinations of parameters and functions (data not shown). From these results, there might be such an extraintuitional factor present in ecological problems regarding predator-prey interactions, which should be clarified both mathematically and biologically for future work.



(a)

(b)

(c)

Fig. 3.1. Illustrations that trajectories of (3.1) do not approach the point $(c/d, a/b)$ as $t \to \infty$ if $a = b = c = d = 1$, $h(t) = 1/(1+t)$ and $k(t) = 1/(1+t) - 1/(\log(2+t))$. A black dot and a white dot represent the point $(c/d, a/b) = (1, 1)$ and the point q and the point to whom trajectories approach ultimately, respectively. These two points are clearly observed to be distinct each other.

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