Mem. Gra. Sci. Eng. Shimane Univ.
Series B: Mathematics
47 (2014), pp. 37–62

FLAT PARALLEL DISPLACEMENTS AND HOLONOMY GROUPS IN A TOPOLOGICAL CONNECTION THEORY

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Communicated by Takumi Yamada

(Received: October 28, 2013)

Abstract. In a topological connection theory, we introduce the notion of flat for a parallel displacement and investigate holonomy groups of flat parallel displacements. Moreover, we assert a classification theorem in a category of principal bundles with flat parallel displacements, and give a sufficient condition for the existence of an initial object in the category.

1. Introduction and the main result

In the previous paper [5] we have considered, in a topological connection theory, parallel displacements (along admissible sequences), introduced the notion of holonomy group of a parallel displacement, and clarified some fundamental properties for holonomy groups. This paper is a sequel to [5]. In this paper, we introduce the notion of *flat* for a parallel displacement and study holonomy groups of flat parallel displacements. The main purpose of this paper is to demonstrate Theorem 1.3 (below).

Let us state the main Theorem 1.3 after generalizing the notion of flat in the smooth category to that in the topological one. For smooth connections, we can assert Proposition 3.9 (see Section 3) which provides a one-to-one correspondence between flat smooth connections and special smooth slicing functions. Motivated by Proposition 3.9 we introduce the notion of flat for a continuous slicing function as follows (see Definition 3.1 also):

Definition 1.1. Let $\pi : E \to X$ be a bundle, U a subset of X^2 containing the diagonal set Δ_X, and $\mathcal{C}$ a covering of X. For an invertible (continuous) slicing function ω in π over U, we say that ω is *$\mathcal{C}$-flat* if it satisfies

$$\omega_{x,y} \circ \omega_{y,z} = \omega_{x,z}$$

for any $C \in \mathcal{C}$ and any $x, y, z \in X$ with $(x, y), (y, z), (x, z) \in U \cap C^2$.

2000 *Mathematics Subject Classification.* 53C05, 53C29, 55R15, 54A10.

Key words and phrases. slicing function, direct connection, flat parallel displacement, holonomy group, classification theorem, initial object.

This is a generalized notion of flat in the smooth category. We here establish Definition 1.1 from a viewpoint of slicing functions. In fact, Proposition 4.5 enables us to rephrase Definition 1.1 as the following definition, from a viewpoint of parallel displacements (see Section 4 for notation):

Definition 1.2. Let $\pi : E \to X$ be a bundle, $\mathfrak{S}$ an admissible sequence space over X, and $\mathcal{C}$ a covering of X. For a parallel displacement P along $\mathfrak{S}$ in π, we say that P is *$\mathcal{C}$-flat* if it satisfies

$$P_{\mathbf{x}} = P_{(p_\infty(\mathbf{x}), p_0(\mathbf{x}))}$$

for any $C \in \mathcal{C}$ and any $\mathbf{x} \in \mathfrak{S} \cap C^{\sqcup}$ with $(p_\infty(\mathbf{x}), p_0(\mathbf{x})) \in \mathfrak{S}_{(1)}$.

Now, we are in a position to state the main theorem:

Theorem 1.3. *Let $\pi : E \to X$ be a principal G-bundle, $\mathcal{C}$ a covering of X, and P a $\mathcal{C}$-flat G-compatible parallel displacement along $\langle U_{\mathcal{C}} \rangle$ in π, where $U_{\mathcal{C}} := \bigcup_{C \in \mathcal{C}} C \times C$. Suppose that $\mathcal{C}$ is* (C-N) *or $\mathcal{C} \subset \mathcal{O}_X$, and X is $\langle U_{\mathcal{C}} \rangle$-connected. Then, the following* (i) *and* (ii) *hold for $u \in E$:*

(i) *If Φ^u is the strong holonomy group, then it is a discrete group.*
(ii) *If π^u is the strong holonomy bundle, then it is a Φ^u-bundle.*

Note that Theorem 1.3 is no longer true if one removes the supposition "strong" from the statement (i), see Example 4.14. In the smooth category, the holonomy group of a flat connection is discrete. We can think of Theorem 1.3 as a generalization of this fact.

This paper is organized as follows: In Section 2 we prepare notation and some topological facts. Section 3 is devoted to recalling the definition of slicing function and proving Proposition 3.9. In Section 4 we first consider relation between parallel displacements and slicing functions, next conclude Proposition 4.5 and lastly demonstrate Theorem 1.3 (in Subsection 4.2). Finally in Section 5 we assert a classification theorem in a category of principal bundles with flat parallel displacements (see Theorem 5.2), and furthermore, we give a sufficient condition for the existence of an initial object in the category (see Theorem 5.3).

Acknowledgment. The author would like to express his sincerely gratitude to the referee for valuable suggestions and comments.

2. Preliminaries

First, let us prepare notation and some topological facts.

2.1. Let $f : X \to Y$ be a map. Take subsets $A \subset X$, $B \subset Y$ satisfying $f(A) \subset B$. Then there exists a unique map $k : A \to B$ such that $f \circ i_{A,X} = i_{B,Y} \circ k$, where $i_{A,X}$ is the inclusion. Denote by ${}_B|f|_A$, $f|_A$ and ${}_B|f$, the maps k, ${}_Y|f|_A$ and ${}_B|f|_X$, respectively. If there is no confusion, we denote by $f|_A$ or $f : A \to B$ the map ${}_B|f|_A$ as usual.

For two maps $f : X \to Y$ and $g : Z \to W$, we denote the composition $(g|_{Y \cap Z}) \circ ({}_{Y \cap Z}|f|_{f^{-1}(Y \cap Z)})$ simply by $g \circ f$. Remark here that if $Y \cap Z = \emptyset$, then $g \circ f : \emptyset \to W$.

Now, suppose that $X = Z$. Let $\delta : X \to X \times X$ be a diagonal map. We denote by $f \hat{\times} g$ the composition $(f \times g) \circ \delta : X \to Y \times W$. Explicitly,

$$(f \hat{\times} g)(x) = (f(x), g(x)) \quad \text{for } x \in X.$$

If X and Y are topological spaces and $f : X \to Y$ is a continuous map, then ${}_B|f|_A$ is also continuous with respect to the relative topologies. We call $f : X \to Y$ an *identification* if the topology of Y is $\{U \in \mathcal{P}(Y) \mid f^{-1}(U) \in \mathcal{O}_X\}$, that is, the identification topology with respect to f, where $\mathcal{P}(Y)$ is the power set of Y and $\mathcal{O}_X$ is the topology of X. Needless to say, a surjective continuous open map is an identification.

The following lemmas are frequently used in this paper:

Lemma 2.1. *A surjective map $f : X \to Y$ is an identification if and only if ${}_{f(A)}|f|_A : A \to f(A)$ is also an identification for any open* (*or closed*) *subset $A \subset X$ such that $f^{-1}(f(A)) = A$.*

Lemma 2.2. *A map $f : X \to Y$ is an open map if and only if ${}_A|f|_{f^{-1}(A)} : f^{-1}(A) \to A$ is an open map for any subset $A \subset Y$.*

2.2. We mostly follow the terminology of [4] with slight changes in notation. Thus, we are going to set up notation for bundles. For a continuous map $\pi : E \to X$, we call the map $\pi : E \to X$ itself a *bundle* while usually the triple $\xi = (E, \pi, X)$ or the total space E is referred to as a bundle. Let $\pi : E \to X$ and $\pi' : E' \to X'$ be two bundles. For continuous maps $h : E \to E'$ and $f : X \to X'$, we call $(h, f) : \pi \to \pi'$ a *bundle morphism* if $\pi' \circ h = f \circ \pi$. If $X = X'$, we call $(h, id_X) : \pi \to \pi'$ an X-*morphism* and denote it simply by h. For $Y \subset X$, put

$$E\lceil_Y := \pi^{-1}(Y), \quad \pi\lceil_Y := {}_Y|\pi|_{\pi^{-1}(Y)}.$$

We call $\pi\lceil_Y : E\lceil_Y \to Y$ the *restricted bundle* of π to Y. For a continuous map $f : Z \to X$, the *induced bundle* or *pull-back* of π is denoted by $f^*\pi : f^*E \to Z$, where

$$f^*E := Z \times_X E := \{(z, u) \in Z \times E \mid f(z) = \pi(u)\}$$

is a fiber product of $Z \xrightarrow{f} X \xleftarrow{\pi} E$. The *canonical bundle map* is denoted by $(\overline{f}, f)$. For topological spaces X and F, a bundle $\mathrm{pr}_1 : X \times F \to X$ is called a *product bundle*. If π is X-isomorphic to a product bundle, we say that π is *trivial*. We say that $\pi : E \to X$ is *locally trivial* if π is locally V-isomorphic to a product bundle $\mathrm{pr}_1 : V \times F \to V$ for some open subset $V \subset X$. A V-isomorphism $\pi\lceil_V \to \mathrm{pr}_1$ is called a *local trivialization*.

2.3. Let us recall the notion of G-space. Let G be a topological group. A *right G-space* is a topological space E equipped with a continuous right action $\mu : E \times G \to E$. We often denote $\mu(u, a)$ simply by ua. For $u \in E$ and $a \in G$, one can define maps $l_u : G \to E$, $r_a : E \to E$ by

$$l_u(b) := ub, \quad r_a(v) := va,$$

respectively. A *left G-space* is defined in a similar way. Remark that by a G-space we mean a right G-space, unless otherwise mentioned. Now, let E be a G-space.

We call E a *free G-space* if the right action is free. Denote by E/G the orbit space, and by $q_G^E : E \to E/G$ the natural projection, where the topology of E/G is the identification topology (that is, the quotient topology) induced by q_G^E. Note here that q_G^E is a surjective open map. A *translation function* $T : E^* \to G$ is a (not necessarily continuous) map such that $uT(u,v) = v$ for any $(u,v) \in E^*$, where

$$E^* := \{(u, ua) \in E^2 \mid a \in G\}.$$

Suppose that E is a free G-space. Then, we get a translation function $T : E^* \to G$ by setting

$$T(u,v) := a$$

because for any $(u,v) \in E^*$ there exists a unique $a \in G$ satisfying $v = ua$. It follows that

(1) $T(u,u) = 1_G$ for any $u \in E$;
(2) $(ua, vb) \in E^*$ and $T(ua, vb) = a^{-1}T(u,v)b$ for any $(u,v) \in E^*$, $(a,b) \in G^2$;
(3) $T(u,v)T(v,w) = T(u,w)$ for any $(u,v,w) \in E^3$ with $(u,v),(v,w) \in E^*$.

We call E a *principal G-space* if T is continuous.

2.4. Let $\pi : E \to X$ be a bundle such that E is a G-space. We call π a *G-bundle* if q_G^E and π are isomorphic by (id_E, f), where f is a unique continuous map such that $f \circ q_G^E = \pi \circ id_E$. Denote by $\pi/^G$ the map f. The following lemma provides a rather practical condition for a bundle to be a G-bundle.

Lemma 2.3. *Let π be a bundle whose total space is a G-space. Then π is a G-bundle if and only if the map $\pi/^G$ is well-defined and a homeomorphism.*

Let π (resp. π') be a G (resp. G')-bundle. For a continuous group homomorphism $\rho : G \to G'$ and a bundle morphism $(h,f) : \pi \to \pi'$, we call a triple $(h,f,\rho) : (\pi, G) \to (\pi', G')$ a *homomorphism* if $h(ua) = h(u)\rho(a)$ for $(u,a) \in E \times G$. If $G = G'$, we call (h, f, id_G) a *G-morphism* and denote it simply by (h,f). We call $h : (\pi, G) \to (\pi', G)$ an *(X,G)-morphism* if it is an X-morphism and a G-morphism. We call a G-bundle $\pi : E \to X$ a *principal G-bundle* if E is a principal G-space. Let $\pi : E \to X$ be a principal G-bundle. Lemma 2.2 implies that the restricted bundle $\pi\lceil_Y$ is a principal G-bundle. The induced bundle $f^*\pi$ is a principal G-bundle in the natural way.

Let us recall the notion of associated bundle. Let G and G' be topological groups, $\rho : G \to G'$ a continuous group homomorphism, and E a G-space. It is natural that the product space $E \times G'$ is a G-space by a right action $(u,a)b := (ub, \rho(b)^{-1}a)$. We denote by E^ρ the orbit space $(E \times G')/G$. The orbit space E^ρ is a G'-space by a right action $[u,b]c := [u, bc]$. This action is continuous. We call E^ρ a *G'-space associated with E*. If E is a free G-space, then E^ρ is a free G'-space. If E is a principal G-space with the translation function T, then E^ρ is a principal G'-space with the translation function given by

$$T^\rho([u,a],[v,b]) := a^{-1}\rho(T(u,v))b \quad \text{for } ([u,a],[v,b]) \in (E^\rho)^*.$$

Lemma 2.2 assures that T^ρ is continuous. Now, let $\pi : E \to X$ be a principal G-bundle, and $\pi^\rho : E^\rho \to X$ the map such that $\pi^\rho \circ q_G^{E \times G'} = \pi \circ \mathrm{pr}_1$. In this case

π^ρ is a principal G'-bundle. We call π^ρ the *principal G'-bundle associated with* π. Define a map $\theta^\rho : E \to E^\rho$ by

$$\theta^\rho(u) := [u, 1_{G'}] \quad \text{for } u \in E.$$

This $(\theta^\rho, \rho) : (\pi, G) \to (\pi^\rho, G')$ is a homomorphism. Let $\pi' : E' \to X$ be a principal G'-bundle and $(h, \rho) : (\pi, G) \to (\pi', G')$ an X-morphism. A map $h^\rho : E^\rho \to E'$ is given by

$$h^\rho([u, a]) := h(u)a \quad \text{for } [u, a] \in E^\rho.$$

Then $h^\rho : (\pi^\rho, G') \to (\pi', G')$ is an (X, G')-morphism and $h = h^\rho \circ \theta^\rho$. From Theorem 3.2 in [4, Chapter 4] and the succeeding observation, we have

Lemma 2.4. *If π' is a principal G'-bundle, then h^ρ is an (X, G')-isomorphism.*

Let $E \xrightarrow{\pi} X$ be a G-bundle. We say that π is *locally G-trivial* or simply *locally trivial* if π is locally (V, G)-isomorphic to a product G-bundle $\mathrm{pr}_1 : V \times G \to V$ for some open subset $V \subset X$. A (V, G)-isomorphism $(\pi\lceil_V, G) \to (\mathrm{pr}_1, G)$ is called a *local trivialization.* For a local trivialization $\alpha : (\pi\lceil_V, G) \to (\mathrm{pr}_1, G)$, put $U_\alpha := V$. For local trivializations α and β, the transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is given by

$$g_{\alpha\beta}(x) := (\mathrm{pr}_2 \circ \alpha \circ \beta^{-1})(x, 1_G).$$

Note that a locally trivial G-bundle is a principal G-bundle. For a local trivialization α, let $s_\alpha : U_\alpha \to E\lceil_{U_\alpha}$ be the local section given by $s_\alpha(x) := \alpha^{-1}(x, 1_G)$. Then $T \circ (s_\alpha \hat{\times} s_\beta) = g_{\alpha\beta}$ holds. If π is a locally trivial G-bundle, then both $\pi\lceil_Y$ and $f^*\pi, \pi^\rho$ are locally trivial.

3. Slicing functions

Our aim in this section is to recall the definition of slicing function, to construct a flat slicing function by fixed local trivializations or transition functions, and to prove Proposition 3.9 which leads us to introduce the notion of flat for a continuous slicing function as Definition 1.1.

3.1. Definition of slicing for a continuous map. Let $\pi : E \to X$ be a bundle, Δ_X the diagonal set of X, and $U \subset X^2$ with $\Delta_X \subset U$. For each $i \in \{0, 1\}$, we define a map $p_i^{(1)} : X^2 \to X$ by

$$p_i^{(1)}(x_1, x_0) := x_i \quad \text{for } (x_1, x_0) \in X^2.$$

Definition 3.1 (cf. [5], [7])**.** (I) Let $\omega : (p_0^{(1)}|_U)^* E \to E$ be a continuous map, and let $\omega_{x,y} := \omega(x, y, \cdot) : E_y \to E$ for $(x, y) \in U$. We call ω a *slicing function in π over U* if it satisfies the following (1) and (2):

(1) $(\omega, p_1^{(1)}|_U) : (p_0^{(1)}|_U)^*\pi \to \pi$ is a bundle morphism;
(2) $\omega_{x,x} = id_{E_x}$ for any $x \in X$.

(II) Suppose that U is symmetric, that is, $(y, x) \in U$ for all $(x, y) \in U$. In this case, a slicing function ω is said to be *invertible* if it satisfies

$$\omega_{y,x} = \omega_{x,y}^{-1} \quad \text{for any } (x, y) \in U.$$

(III) In the case where π is a G-bundle, we say that ω is *G-compatible* if $(\omega, p_1^{(1)}|_U)$ is a G-morphism. Henceforth, we denote by $SF(\pi, U)$, $SF_{\mathrm{inv}}(\pi, U)$, and $SF(\pi, U)_G$ the sets of slicing functions, invertible slicing functions, and G-compatible slicing functions on π over U, respectively. In addition, we set

$$\begin{aligned}
&SF_{\mathrm{inv}}(\pi, U)_G := SF_{\mathrm{inv}}(\pi, U) \cap SF(\pi, U)_G,\\
&SF_{\mathcal{C}\text{-flat}}(\pi, U) := \{\omega \in SF_{\mathrm{inv}}(\pi, U) \mid \omega \text{ is } \mathcal{C}\text{-flat}\},\\
&SF_{\mathcal{C}\text{-flat}}(\pi, U)_G := SF_{\mathcal{C}\text{-flat}}(\pi, U) \cap SF_{\mathrm{inv}}(\pi, U)_G,
\end{aligned}$$

where $\mathcal{C}$ is a covering of X (recall Definition 1.1 for the word "$\mathcal{C}$-flat").

Let us give an example of $\mathcal{C}$-flat slicing function.

Example 3.2. Let S^1 be the unit circle in $\mathbb{C}$ and $\pi' : \mathbb{R} \to S^1$ be a bundle (universal covering space) given by $\pi'(t) := e^{2\pi i t}$ for $t \in \mathbb{R}$. We give an atlas of S^1 by using π'. Put

$$\begin{aligned}
&U_1^+ := \pi'((-\frac{1}{4}, \frac{1}{4})),\ U_1^- := \pi'((\frac{1}{4}, \frac{3}{4})),\ U_2^+ := \pi'((0, \frac{1}{2})),\ U_2^- := \pi'((\frac{1}{2}, 1)),\\
&\varphi_1^+ := ({}_{U_1^+}|\pi'|_{(-\frac{1}{4},\frac{1}{4})})^{-1},\ \varphi_1^- := ({}_{U_1^-}|\pi'|_{(\frac{1}{4},\frac{3}{4})})^{-1},\\
&\varphi_2^+ := ({}_{U_2^+}|\pi'|_{(0,\frac{1}{2})})^{-1},\ \varphi_2^- := ({}_{U_2^-}|\pi'|_{(\frac{1}{2},1)})^{-1}.
\end{aligned}$$

Then $\{\varphi_j^{\pm} \mid j \in \{1, 2\}\}$ is an atlas of S^1. For $j \in \{1, 2\}$, maps $\theta_j^{\pm} : U_j^{\pm} \times U_j^{\pm} \to (-\frac{1}{2}, \frac{1}{2})$ are given by $\theta_j^{\pm}(x, y) := \varphi_j^{\pm}(x) - \varphi_j^{\pm}(y)$. Put $U := \bigcup_{j \in \{1,2\}}(U_j^+ \times U_j^+) \cup (U_j^- \times U_j^-)$ and let $\theta : U \to (-\frac{1}{2}, \frac{1}{2})$ be a map defined by $\theta(x, y) := \theta_j^{\pm}(x, y)$ if $(x, y) \in U_j^{\pm} \times U_j^{\pm}$. Note that θ is well-defined and smooth. Let $T^2 := S^1 \times S^1$ be the torus and $\mathrm{pr}_1 : T^2 \to S^1$ the product bundle. For $\alpha \in \mathbb{R}$, put

$$\omega_\alpha(x, y, (y, a)) := (x, ae^{2\pi i \alpha \theta(x,y)})$$

for $(x, y, (y, a)) \in (p_0^{(1)}|_U)^* T^2$ and $\mathcal{C} := \{U_j^{\pm} \mid j \in \{1, 2\}\}$. Then we can see that $\omega_\alpha \in SF_{\mathcal{C}\text{-flat}}(\mathrm{pr}_1, U)_{S^1}$.

In [5] we confirmed that slicing functions induced Asada's connections. The following example implies that flat slicing functions induce Asada's connections which have a property of flatness:

Example 3.3 (cf. [1, 2, 3], [5])**.** Let π be a G-bundle, and let $C^1(\pi, U)_G$ denote the set of continuous maps $s : E^2\lceil_U \to G$ such that

(1) $s(u, u) = 1_G$ for $u \in E$,
(2) $s(ua, vb) = a^{-1}s(u, v)b$ for $(u, v) \in E^2\lceil_U$ and $a, b \in G$.

Considering elements of the inductive limit $\varinjlim_U C^1(\pi, U)_G$ over all neighborhoods U of Δ_X in X^2 as connections in π, Asada [1, 2, 3] has constructed a connection theory in a category of topological fiber bundles. We denote by $C^1_{\mathrm{inv}}(\pi, U)_G$ the set of $s \in C^1(\pi, U)_G$ such that

$$s(u, v) = s(v, u)^{-1} \text{ for } (u, v) \in E^2\lceil_U.$$

Suppose that π is a principal G-bundle. Then, $\omega \in SF(\pi, U)_G$ corresponds bijectively to $s^\omega \in C^1(\pi, U)_G$ with

$$s^\omega(u, v) := T(u, \omega(\pi(u), \pi(v), v)).$$

Note here that $SF_{\mathrm{inv}}(\pi, U)_G$ corresponds to $C^1_{\mathrm{inv}}(\pi, U)_G$. For a covering $\mathcal{C}$ of X, we denote by $C^1_{\mathcal{C}\text{-flat}}(\pi, U)_G$ the set of $s \in C^1_{\mathrm{inv}}(\pi, U)_G$ such that

$$s(u, v)s(v, w) = s(u, w)$$

for any $C \in \mathcal{C}$ and any $u, v, w \in E$ with $(u, v), (v, w), (w, u) \in E^2\lceil_{U \cap C^2}$. Then, $SF_{\mathcal{C}\text{-flat}}(\pi, U)_G$ corresponds to $C^1_{\mathcal{C}\text{-flat}}(\pi, U)_G$.

In the previous paper [5] we gave several examples of slicing functions. Let us pick up an example from them.

Example 3.4 (cf. [7])**.** Let X be a polyhedron of a countable connected simplicial complex K in the weak topology. Put $U_K := \bigcup_{\tau \in K} |\tau| \times |\tau|$, $X^\sqcup := \bigcup_{n \geq 0} X^{n+1}$ (topological sum), and

$$\mathfrak{S}_K := \{(x_n, \ldots, x_0) \in X^\sqcup \mid (x_i, x_{i-1}) \in U_K \text{ for all } i \in \{1, \ldots, n\} \text{ when } n \geq 1\}.$$

An equivalence relation in $\mathfrak{S}_K$ is generated by the relations

$$(x_n, \ldots, x_i, \ldots, x_0) \sim (x_n, \ldots, \hat{x}_i, \ldots, x_0)$$

whenever either $x_i = x_{i-1}$ or $x_{i+1} = x_{i-1}$, where the symbol $\hat{x}$ denotes deletion. We denote by $[x_n, \ldots, x_0]$ the equivalence class of $(x_n, \ldots, x_0)$. Fix a vertex v_0 of K. Put

$$\begin{aligned}
\tilde{\mathfrak{S}}_K &:= \mathfrak{S}_K / \sim, \\
\tilde{E}_K &:= \{[x_n, \ldots, x_1, x_0] \in \tilde{\mathfrak{S}}_K \mid x_0 = v_0\}, \\
\tilde{G}_K &:= \{[x_n, \ldots, x_1, v_0] \in \tilde{E}_K \mid x_n = v_0\},
\end{aligned}$$

where a topology of $\tilde{\mathfrak{S}}_K$ is the quotient topology, and we consider $\tilde{E}_K$ and $\tilde{G}_K$ as subspaces, respectively. A unary operation $\cdot^-$ on $\tilde{\mathfrak{S}}_K$ is defined by

$$[x_n, \ldots, x_1, x_0]^- := [x_0, x_1, \ldots, x_n]$$

for $[x_n, \ldots, x_1, x_0] \in \tilde{\mathfrak{S}}_K$. A partial binary operation on $\tilde{\mathfrak{S}}_K$ is defined by

$$[x_n, \ldots, x_0][y_m, \ldots, y_0] := [x_n, \ldots, x_0, y_m, \ldots, y_0]$$

for $([x_n, \ldots, x_0], [y_m, \ldots, y_0]) \in \tilde{\mathfrak{S}}_K \times \tilde{\mathfrak{S}}_K$ such that $x_0 = y_m$. Remark here that $\tilde{G}_K$ is a topological group with respect to these operations. A bundle $\tilde{\pi}_K : \tilde{E}_K \to X$ is defined by

$$\tilde{\pi}_K([x_n, \ldots, x_1, v_0]) := x_n.$$

We can see that $\tilde{\pi}_K$ is a locally trivial principal $\tilde{G}_K$-bundle and a universal bundle, that is, $\tilde{E}_K$ is ∞-connected. For $(x, y_m, [y_m, \ldots, y_1, v_0]) \in (p_0^{(1)}|_{U_K})^* \tilde{E}_K$, put

$$\tilde{\omega}_K(x, y_m, [y_m, \ldots, y_1, v_0]) := [x, y_m][y_m, \ldots, y_1, v_0].$$

Then $\tilde{\omega}_K \in SF_{\mathrm{inv}}(\tilde{\pi}_K, U_K)_{\tilde{G}_K}$.

3.2. Flat slicing functions, local trivializations and transition functions. We want to show that one can construct a flat slicing function by fixed local trivializations or transition functions.

Let $\pi : E \to X$ be a locally trivial G-bundle and A an atlas (a system of local trivializations). Fix any $\omega \in SF(\pi, U)_G$. Then, we get a family $(\overline{g}_{\alpha\beta})_{(\alpha,\beta)\in A^2}$ of continuous maps $\overline{g}_{\alpha\beta} : (U_\alpha \times U_\beta) \cap U \to G$ by setting

$$\overline{g}_{\alpha\beta}(x, y) := (\mathrm{pr}_2 \circ \alpha \circ \omega_{x,y} \circ \beta^{-1})(y, 1_G)$$

for $(x, y) \in (U_\alpha \times U_\beta) \cap U$ (see [7]). This family satisfies the following condition:

(A) $\overline{g}_{\alpha\beta}(x, y) = g_{\alpha\alpha'}(x)\overline{g}_{\alpha'\beta'}(x, y)g_{\beta'\beta}(y)$ for any $\alpha, \beta, \alpha', \beta' \in A$ and $(x, y) \in (U_\alpha \times U_\beta) \cap (U_{\alpha'} \times U_{\beta'}) \cap U$.

Conversely, if a family $(\overline{g}_{\alpha\beta})_{(\alpha,\beta)\in A^2}$ of continuous maps $\overline{g}_{\alpha\beta} : (U_\alpha \times U_\beta) \cap U \to G$ satisfies the condition (A), then a G-compatible slicing function ω is defined by

$$\omega(x, y, u) := \alpha^{-1}(x, \overline{g}_{\alpha\beta}(x, y)(\mathrm{pr}_2 \circ \beta)(u))$$

for $(x, y, u) \in (p_0^{(1)}|_U)^*E$ with $(x, y) \in (U_\alpha \times U_\beta) \cap U$. Here, we can assert that G-compatible slicing functions ω over U correspond bijectively to families $(\overline{g}_{\alpha\beta})_{(\alpha,\beta)\in A^2}$ satisfying the condition (A). Moreover, $\mathcal{C}$-flat G-compatible slicing functions ω over U correspond bijectively to families $(\overline{g}_{\alpha\beta})_{(\alpha,\beta)\in A^2}$ satisfying, in addition to (A), the condition:

(B) $\overline{g}_{\alpha\beta}(x, y)\overline{g}_{\beta\gamma}(y, z) = \overline{g}_{\alpha\gamma}(x, z)$ for $x, y, z \in X$ such that $(x, y) \in (U_\alpha \times U_\beta) \cap U \cap C^2$, $(y, z) \in (U_\beta \times U_\gamma) \cap U \cap C^2$, $(z, x) \in (U_\gamma \times U_\alpha) \cap U \cap C^2$.

Mishchenko and Teleman [9] have constructed an almost flat quasi-connection by almost flat transition functions on a continuous vector bundle over simplicial space. In our context, assuming that G is a discrete group, we can construct a flat slicing function by fixed local trivializations or transition functions.

Proposition 3.5 (cf. [9]). *Let X be a polyhedron of a simplicial complex K in the weak topology, $\pi : E \to X$ a locally trivial G-bundle, and $\{V_K(x) \mid x \in X\}$ the set of all open star neighborhoods. For each $x \in X$, since $V_K(x)$ is contractible, there exists a local trivialization $\varphi_x : E\lceil_{V_K(x)} \to V_K(x)\times G$. Then, if G is a discrete group, there exists a $\mathcal{C}_K$-flat G-slicing function ω_K in π over U_K. Moreover, ω_K is unique for A. Here $A := \{\varphi_x \mid x \in X\}$, $\mathcal{C}_K := \{|\tau| \mid \tau \in K\}$, and $U_K := \bigcup_{\tau\in K} |\tau| \times |\tau|$.*

Proof. For any $z, w \in X$, since $V_K(z) \cap V_K(w)$ is connected and G is a discrete group, the transition function $g_{zw} : V_K(z) \cap V_K(w) \to G$ is constant. Thus, we can uniquely continuously extend g_{zw} on $\mathrm{st}_K(z) \cap \mathrm{st}_K(w)$, where $\mathrm{st}_K(z) = \bigcup\{|\tau| \mid z \in |\tau|, \tau \in K\}$ is the star neighborhood of z. Denote by g'_{zw} the extension of g_{zw}. For $z, w \in X$, a map $\overline{g}_{zw} : (V_K(z) \times V_K(w)) \cap U_K \to G$ is defined as follows. Let $(x, y) \in (V_K(z) \times V_K(w)) \cap U_K$. Since $(x, y) \in U_K$, there exists $\tau \in K$ such that $x, y \in |\tau|$. Since $(x, y) \in V_K(z) \times V_K(w)$, there exist $\sigma, \sigma' \in K$ such that $z \in |\sigma|$ and $x \in \mathrm{Int}\sigma$, and $w \in |\sigma'|$ and $y \in \mathrm{Int}\sigma'$. Then, we have $x \in \mathrm{Int}\sigma \cap |\tau|$. Since any two simplexes do not intersect with each other at the interior of the other, we get $\sigma < \tau$. Similarly, we have $\sigma' < \tau$. Thus, we get $x, y, z, w \in |\tau|$. Let $v \in |\tau|$ be an arbitrary point. Then, put

$$\overline{g}_{zw}(x,y) := g'_{zv}(x)g'_{vw}(y).$$

Note that $x \in \mathrm{st}_K(z) \cap \mathrm{st}_K(v)$ and $y \in \mathrm{st}_K(v) \cap \mathrm{st}_K(w)$. This definition does not depend on the choice of $v \in |\tau|$. Indeed, for another point $v' \in |\tau|$, we have

$$\begin{aligned} g'_{zv'}(x)g'_{v'w}(y) &= g'_{zv}(x)g'_{vv'}(x)g'_{v'v}(y)g'_{vw}(y) \\ &= g'_{zv}(x)g'_{vv'}(x)g'_{v'v}(x)g'_{vw}(y) = g'_{zv}(x)g'_{vw}(y). \end{aligned}$$

To show that the condition (A) holds, let $z, z', w, w' \in X$ and $(x,y) \in (V_K(z) \times V_K(w)) \cap (V_K(z') \times V_K(w')) \cap U_K$. From the same argument above, there exists $\tau \in K$ such that $x, y, z, z', w, w' \in |\tau|$. Let $v \in |\tau|$. Then,

$$\begin{aligned} \overline{g}_{zw}(x,y) &= g'_{zv}(x)g'_{vw}(y) = g'_{zz'}(x)g'_{z'v}(x)g'_{vw'}(y)g'_{w'w}(y) \\ &= g'_{zz'}(x)\overline{g}_{z'w'}(x,y)g'_{w'w}(y) = g_{zz'}(x)\overline{g}_{z'w'}(x,y)g_{w'w}(y). \end{aligned}$$

To show that the condition (B) holds, let $|\tau| \in \mathcal{C}_K,\ z, w, x \in X$, and $(y_2, y_1, y_0) \in (V_K(z) \times V_K(w) \times V_K(x)) \cap |\tau|^3$. Then, we have

$$\begin{aligned} \overline{g}_{zw}(y_2,y_1)\overline{g}_{wx}(y_1,y_0) &= g'_{zv}(y_2)g'_{vw}(y_1)g'_{wv}(y_1)g'_{vx}(y_0) \\ &= g'_{zv}(y_2)g'_{vx}(y_0) = \overline{g}_{zx}(y_2,y_0). \end{aligned}$$

Therefore, a $\mathcal{C}_K$-flat G-compatible slicing function ω_A can be defined from the family $(\overline{g}_{zw})_{(z,w)\in X^2}$. Next, we will show that $(\overline{g}_{zw})_{(z,w)\in X^2}$ is unique for A. To this end, let $(\tilde{g}_{zw})_{(z,w)\in X^2}$ be another family satisfying the conditions (A) and (B). Let $z \in X$ and $(x,y) \in V_K(z)^2 \cap U_K$. Then, there exists $\tau \in K$ such that $x, y \in |\tau|$. Since G is a discrete group and $V_K(z) \cap |\tau|$ is connected, $\tilde{g}_{zz}(x,y) = \tilde{g}_{zz}(x,x) = 1$. Therefore, we get

$$\begin{aligned} \tilde{g}_{zw}(x,y) &= g_{zv}(x)\tilde{g}_{vv}(x,y)g_{vw}(y) \\ &= g_{zv}(x)g_{vw}(y) = g'_{zv}(x)g'_{vw}(y) = \overline{g}_{zw}(x,y). \end{aligned}$$

This completes the proof. □

If a covering $\mathcal{C}$ of the base space of a locally trivial principal bundle with discrete structure group satisfies appropriate condition, there exists a $\mathcal{C}$-flat slicing function.

Proposition 3.6 (cf. [5]). *Let $\pi : E \to X$ be a locally trivial G-bundle and A a bundle atlas (a system of local trivializations). Suppose that for any $(\alpha, \beta) \in A^2$, $U_\alpha \cap U_\beta$ is connected. Then, if G is a discrete group, there exists a $\mathcal{C}_A$-flat G-compatible slicing function ω_A in π over U_A. Moreover, ω_A is unique for A. Here $U_A := \bigcup_{\alpha\in A} U_\alpha \times U_\alpha$ and $\mathcal{C}_A := \{U_\alpha \mid \alpha \in A\}$.*

Proof. For $(x,y,u) \in (p_0^{(1)}|_{U_A})^*E$ with $(x,y) \in U_\alpha \times U_\alpha$, put

$$\omega_A(x,y,u) := \alpha^{-1}(x, (\mathrm{pr}_2 \circ \alpha)(u)).$$

This definition does not depend on the choice of α. Indeed, for any $\beta \in A$ and $(x,y,u) \in (p_0^{(1)}|_{U_A})^*E$, we have

$$\begin{aligned} &\beta^{-1}(x, (\mathrm{pr}_2 \circ \beta)(u)) = \alpha^{-1}(x, (\mathrm{pr}_2 \circ \alpha \circ \beta^{-1})(x, (\mathrm{pr}_2 \circ \beta \circ \alpha^{-1})(y, (\mathrm{pr}_2 \circ \alpha)(u)))) \\ &= \alpha^{-1}(x, g_{\alpha\beta}(x)g_{\beta\alpha}(y)(\mathrm{pr}_2 \circ \alpha)(u)))) = \alpha^{-1}(x, (\mathrm{pr}_2 \circ \alpha)(u)). \end{aligned}$$

We can see that $\omega_A \in SF_{\mathrm{inv}}(\pi, U_A)_G$. By the definition, we have

$$(\omega_A)_{x,y} \circ (\omega_A)_{y,z} = (\omega_A)_{x,z}$$

for any $\alpha \in A$ and $x, y, z \in U_\alpha$. Namely, ω_A is $\mathcal{C}_A$-flat. We show that $\mathcal{C}_A$-flat slicing function is unique for A. To this end, let $\omega' \in SF_{\mathrm{inv}}(\pi, U_A)_G$ and $(x, y, u) \in (p_0^{(1)}|_{U_A})^*E$ with $(x, y) \in U_\alpha \times U_\alpha$. Since G is a discrete group and U_α is connected, $\overline{g}_{\alpha\alpha}(x, y) = \overline{g}_{\alpha\alpha}(x, x) = 1_G$. Thus, we have

$$\begin{aligned}\omega'(x, y, u) &= \alpha^{-1}(x, \overline{g}_{\alpha\alpha}(x, y)(\mathrm{pr}_2 \circ \alpha)(u)) \\ &= \alpha^{-1}(x, (\mathrm{pr}_2 \circ \alpha)(u)) = \omega_A(x, y, u).\end{aligned}$$

This completes the proof. □

3.3. Smooth category. We want to explain that a (continuous) slicing function is a generalization of the connection in the smooth category.

Let $\pi : E \to X$ be a smooth principal G-bundle, where G is a Lie group. A *connection* (*invariant horizontal subbundle*) H *in* π is a smooth subbundle of TE such that

(1) $T_uE = \mathrm{Ker}\pi_{*u} \oplus H_u$ for $u \in E$,
(2) $r_{a*u}(H_u) = H_{ua}$ for $(u, a) \in E \times G$,

where π_* is the differential of π (e.g. [6]) and $r_a : E \to E$ is given by $r_a(v) := va$ (see Section 2). Let H be a connection and $c : [0, 1] \to X$ a piecewise smooth curve. For each $u \in E_{c(0)}$, there exists a unique curve $\tilde{c} : I \to E$ such that $\tilde{c}(0) = u$, $\pi \circ \tilde{c} = c$, and $d\tilde{c}/dt(t) \in H_{\tilde{c}(t)}$ for $t \in I$, that is, $\tilde{c}$ is the horizontal lift of c starting from u. From now on, let us assume that X is a Riemannian manifold. A subset V in X is said to be *strongly convex* if for any $(x, y) \in V \times V$ there exists a unique geodesic $\gamma(t)$ in V joining y to x such that the length of γ is equal to the distance $d(x, y)$, where $\gamma(t) = \exp_y tv$ and $\gamma(1) = x$. Now, fix an open covering $\mathcal{V}$ of X which consists of strongly convex sets. Put $U_{\mathcal{V}} := \bigcup_{V \in \mathcal{V}} V \times V$. Let $(x, y, u) \in (p_0^{(1)}|_{U_{\mathcal{V}}})^*E$ and γ the geodesic in some V joining y to x. Let $\tilde{\gamma}$ be the horizontal lift of γ starting from u. Put

$$\omega^H(x, y, u) := \tilde{\gamma}(1). \tag{3.3.1}$$

Then $\omega^H \in SF_{\mathrm{inv}}(\pi, U_{\mathcal{V}})_G$ and it is smooth. This implies that one can obtain many invertible G-compatible slicing functions from a smooth connection H in $\pi : E \to X$, where X is a Riemannian manifold.

The following comes from (3.3.1):

Proposition 3.7. *Let $\pi : E \to X$ be a smooth principal G-bundle over a Riemannian manifold X, where G is a Lie group, and H a connection in π. Let $\omega^H \in SF_{inv}(\pi, U_{\mathcal{V}})$ be the smooth slicing function constructed as in* (3.3.1)*, where $\mathcal{V}$ is an open covering of X consisting of strongly convex sets. Suppose that H is flat. Then ω^H is $\mathcal{V}$-flat.*

Let $\pi : E \to X$ be a smooth principal G-bundle, where G is a Lie group. A *connection* 1*-form* θ is a 1-form with values in the Lie algebra $\mathcal{G}$ of G satisfying the following conditions:

(1) $\theta_u(A^*_u) = A$ for $A \in \mathcal{G}$,
(2) $\theta_{ua} \circ r_{a*u} = Ad_{a^{-1}*1_G} \circ \theta_u$ for $(u, a) \in E \times G$,

where A^* is the fundamental vector field corresponding to A, and for $b \in G$, Ad_b is the inner automorphism of G given by $Ad_b(c) := bcb^{-1}$ for $c \in G$. Note that the invariant horizontal subbundles correspond bijectively to connection 1-forms (e.g. [6]). Let $s \in C^1(\pi, U)_G$ and suppose that it is smooth. Then a connection 1-form θ^s is given by for $u \in E$ and $W \in T_uE$,

$$\theta^s_u(W) := -s(\cdot, u)_{*u}(W).$$

If s is invertible, then $\theta^s_u(W) = s(u, \cdot)_{*u}(W)$. The vertical projection v^{θ^s} and the horizontal projection h^{θ^s} are given by

$$v^{\theta^s}_u(W) = -l_{u*1_G}(s(\cdot, u)_{*u}(W)), \quad h^{\theta^s}_u(W) = W - v^{\theta^s}_u(W)$$

for $u \in E$ and $W \in T_uE$, respectively.

The following proposition assures that smooth flat slicing functions (smooth flat Asada's connections) induce flat connections in the smooth category:

Proposition 3.8. *Let $\mathcal{C}$ be an open covering of X. If s is $\mathcal{C}$-flat, then θ^s is flat, that is,*

$$d\theta^s + \frac{1}{2}[\theta^s, \theta^s] = 0.$$

Proof. Let $\Theta^s := d\theta^s + (1/2)[\theta^s, \theta^s]$. Note that $\Theta^s(W, Y) = -\theta^s([h^{\theta^s}(W), h^{\theta^s}(Y)])$ for all $W, Y \in \Gamma(TE)$. Let $u \in E$ and $f \in C^\infty(G)$. Then, we have

$$\begin{aligned}
&- \theta^s_u([h^{\theta^s}(W), h^{\theta^s}(Y)]_u)(f) \\
&= s(\cdot, u)_{*u}([h^{\theta^s}(W), h^{\theta^s}(Y)]_u)(f) = [h^{\theta^s}(W), h^{\theta^s}(Y)]_u(f \circ s(\cdot, u)) \\
&= h^{\theta^s}(W)_u(h^{\theta^s}(Y)(f \circ s(\cdot, u))) - h^{\theta^s}(Y)_u(h^{\theta^s}(W)(f \circ s(\cdot, u))).
\end{aligned}$$

Let $w \in E$ and $C \in \mathcal{C}$ with $(u, w) \in E^2|_{U \cap C^2}$. Then, for $v \in E$ with $(v, u), (v, w) \in E^2|_{U \cap C^2}$, we have $s(ws(w, v), u) = s(v, w)s(w, u) = s(v, u)$. Thus, we get $s(\cdot, u) \circ l_w \circ s(w, \cdot) = s(\cdot, u)$ and $s(\cdot, u)_{*w} \circ v^{\theta^s}_w = s(\cdot, u)_{*w}$. Then, we have

$$v^{\theta^s}(Y)_w(f \circ s(\cdot, u)) = s(\cdot, u)_{*w}(v^{\theta^s}_w(Y_w))(f) = s(\cdot, u)_{*w}(Y_w)(f) = Y_w(f \circ s(\cdot, u)).$$

Thus, we get $h^{\theta^s}(Y)(f \circ s(\cdot, u)) = (Y - v^{\theta^s}(Y))(f \circ s(\cdot, u)) = 0$. Similarly, we have $h^{\theta^s}(W)(f \circ s(\cdot, u)) = 0$. Therefore, $\Theta^s(W, Y) = 0$. □

In Proposition 3.1 [5], for a smooth slicing function $\omega \in SF(\pi, U)_G$ we defined an invariant horizontal subbundle H^ω as follows:

$$H^\omega_u = \{\omega(\cdot, \pi(u), u)_{*\pi(u)}(v) \mid v \in T_{\pi(u)}X\}$$

for $u \in E$, and we have shown that $H^{\omega^H} = H$ for a given connection H (see (3.3.1) for ω^H). The following proposition implies that $\omega^{H^\omega} = \omega$ if ω is flat:

Proposition 3.9. *Let X be a Riemannian manifold and $\mathcal{V}$ an open covering of X consisting of strongly convex sets. If ω is $\mathcal{V}$-flat, then $\omega^{H^\omega} = \omega$.*

Proof. Let $V \in \mathcal{V}$, $(x,y) \in V^2$, $u \in E$, and $\gamma : [0,1] \to V$ be the geodesic joining y to x. Put $\tilde{\gamma}(t) := \omega(\gamma(t), y, u)$ for $t \in [0,1]$. For $z \in V$, since ω is $\mathcal{V}$-flat, we have

$$\omega(z,y,u) = (\omega_{(z,\gamma(t))} \circ \omega_{(\gamma(t),y)})(u) = \omega(z, \gamma(t), \tilde{\gamma}(t)).$$

Thus, $\omega(\cdot, y, u) = \omega(\cdot, \gamma(t), \tilde{\gamma}(t))$. Then, we get

$$\frac{d\tilde{\gamma}}{dt}(t) = \omega(\cdot, y, u)_{*\gamma(t)}(\frac{d\gamma}{dt}(t)) = \omega(\cdot, \gamma(t), \tilde{\gamma}(t))_{*\gamma(t)}(\frac{d\gamma}{dt}(t)) \in H^{\omega}_{\tilde{\gamma}(t)}.$$

Thus, $\tilde{\gamma} : [0,1] \to V$ is the horizontal lift of γ. Therefore, we get

$$\omega^{H^{\omega}}(x,y,u) = \tilde{\gamma}(1) = \omega(\gamma(1), y, u) = \omega(x,y,u).$$

$\square$

Proposition 3.1 [5], together with the above Propositions 3.8 and 3.9, allows us to conclude

Proposition 3.10. *Let X be a Riemannian manifold. Then, the map*

$$\{H \mid H \text{ is a flat connection in } \pi\} \to \varinjlim_{\mathcal{V}} SF_{\mathcal{V}\text{-}flat}(\pi, U_{\mathcal{V}})_G$$

given by assigning $[\omega^H]$ to H is bijective, where the codomain is the inductive limit over all open coverings $\mathcal{V}$ of X consisting of strongly convex sets.

4. Parallel displacements and holonomy groups

The main purpose in this section is to state Proposition 4.5 and demonstrate Theorem 1.3 (See Subsection 4.2).

4.1. Parallel displacements. First, let us recall the definitions of admissible sequence space and parallel displacement. Let X be a topological space and $X^{\sqcup} := \bigcup_{n\geq 0} X^{n+1}$ the topological sum. Two maps $p_0, p_\infty : X^{\sqcup} \to X$ are defined by

$$p_0(x_n, \dots, x_0) := x_0, \quad p_\infty(x_n, \dots, x_0) := x_n$$

for $(x_n, \dots, x_0) \in X^{\sqcup}$. A binary operation $\bullet$ on $X^{\sqcup}$ is defined by

$$\bullet(\mathbf{x}, \mathbf{y}) := \mathbf{x} \bullet \mathbf{y} := (x_n, \dots, x_1, y_m, \dots, y_0)$$

for $(\mathbf{x}, \mathbf{y}) = ((x_n, \dots, x_0), (y_m, \dots, y_0)) \in X^{\sqcup} \times X^{\sqcup}$. Let $X^{\sqcup} \times_X X^{\sqcup}$ be a fiber product of $X^{\sqcup} \xrightarrow{p_0} X \xleftarrow{p_\infty} X^{\sqcup}$. Hereafter, we denote by the same symbol $\bullet$ the restriction of the binary operation $\bullet$ to $X^{\sqcup} \times_X X^{\sqcup}$, which is a partial binary operation on $X^{\sqcup}$. A unary operation $\cdot^-$ on $X^{\sqcup}$ is defined by

$$\mathbf{x}^- := (x_0, x_1, \dots, x_n) \quad \text{for } \mathbf{x} = (x_n, \dots, x_1, x_0).$$

For $\mathbf{x} \in X^{\sqcup}$, we say that the *length* of $\mathbf{x}$ is n if $\mathbf{x} \in X^{n+1}$. For any subset $\mathfrak{S} \subset X^{\sqcup}$ and $n \geq 0$, put

$$\mathfrak{S}_{(n)} := \mathfrak{S} \cap X^{n+1}.$$

Note that for $\mathfrak{S} \subset X^{\sqcup}$, we have $\mathfrak{S} = \bigcup_{n\geq 0} \mathfrak{S}_{(n)}$. We can see that $(X^{\sqcup}, \bullet)$ is associative and generated by $X \cup X^2$, and maps p_0, p_∞, $\bullet$, and $\cdot^-$ are all continuous.

Definition 4.1 (cf. [5])**.** (I) We call a subspace $\mathfrak{S} \subset X^{\sqcup}$ an *admissible sequence space over* X if it satisfies the following conditions:

(a) $\bullet(\mathfrak{S}^2 \cap (X^{\sqcup} \times_X X^{\sqcup})) \subset \mathfrak{S}$;
(b) $X \cup \Delta_X \subset \mathfrak{S}$;
(c) $\{\mathbf{x}^- \mid \mathbf{x} \in \mathfrak{S}\} \subset \mathfrak{S}$.

(II) Let $\mathfrak{S}$ be an admissible sequence space over X. We say that X is $\mathfrak{S}$-*connected* if $(p_\infty \hat{\times} p_0)(\mathfrak{S}) = X^2$, that is, for any $(x, y) \in X^2$, there exists $\mathbf{x} \in \mathfrak{S}$ satisfying $p_0(\mathbf{x}) = y$ and $p_\infty(\mathbf{x}) = x$. Henceforth we use the following notation:

$\mathcal{AS}(X)$: the set of admissible sequence spaces over X,
$\mathfrak{S}_A := (p_\infty|_{\mathfrak{S}} \hat{\times} p_0|_{\mathfrak{S}})^{-1}(A)$ for a subset $A \subset X^2$,
$\mathfrak{S}_{x,y} := \mathfrak{S}_{\{(x,y)\}}$ for $(x, y) \in X^2$, $\mathfrak{S}_x := \mathfrak{S}_{x,x}$,
$\langle U \rangle := \{(x_n, \ldots, x_0) \in X^{\sqcup} \,|\, (x_i, x_{i-1}) \in U$ for any $i \in \{1, \ldots, n\}$ if $n \geq 1\}$ for a symmetric subspace $U \subset X^2$ with $\Delta_X \subset U$,
$\mathcal{S}_{X^2}(\Delta_X) := \left\{ U \subset X^2 \,\middle|\, \begin{array}{r} \text{For any } x \in X, \text{ there exists } V \in \mathcal{O}_X(x) \\ \text{such that } V \times \{x\} \subset U \end{array} \right\}$, where $\mathcal{O}_X(x)$ is the set of all open neighborhoods of x.

Note that for $x \in X$, $\mathfrak{S}_{X \times \{x\}} = (p_0|_{\mathfrak{S}})^{-1}(\{x\})$ and $\mathfrak{S}_{\{x\} \times X} = (p_\infty|_{\mathfrak{S}})^{-1}(\{x\})$, and that the above condition (a) implies $\langle \mathfrak{S}_{(1)} \rangle \subset \mathfrak{S}$.

Definition 4.2 (cf. [5])**.** (I) Let $\pi : E \to X$ be a bundle, $\mathfrak{S} \in \mathcal{AS}(X)$ and $P : (p_0|_{\mathfrak{S}})^* E \to E$ a continuous map. Put

$$P_{\mathbf{x}} := P(\mathbf{x}, \cdot) : E_{p_0(\mathbf{x})} \to E \quad \text{for } \mathbf{x} \in \mathfrak{S}.$$

We call P a *parallel displacement along* $\mathfrak{S}$ *in* π if it satisfies the following:

(1) $(P, p_\infty|_{\mathfrak{S}}) : (p_0|_{\mathfrak{S}})^* \pi \to \pi$ is a bundle morphism;
(2) $P_{(x,x)} = id_{E_x}$ for any $(x, x) \in \mathfrak{S}_{(1)}$;
(3) $P_{\mathbf{x} \bullet \mathbf{y}} = P_{\mathbf{x}} \circ P_{\mathbf{y}}$ for any $(\mathbf{x}, \mathbf{y}) \in \mathfrak{S}^2 \cap (X^{\sqcup} \times_X X^{\sqcup})$;
(4) $P_{\mathbf{x}^-} = P_{\mathbf{x}}^{-1}$ for any $\mathbf{x} \in \mathfrak{S}$.

(II) Let G be a topological group. Suppose that π is a G-bundle. Then, P is said to be G-*compatible* if $(P, p_\infty|_{\mathfrak{S}})$ is a G-morphism. Hereafter, we denote by $\mathcal{PD}(\pi, \mathfrak{S})$ and $\mathcal{PD}(\pi, \mathfrak{S})_G$ the set of all parallel displacements and G-compatible parallel displacements along $\mathfrak{S}$ in π, respectively. In addition,

$$\mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi, \mathfrak{S}) := \{P \in \mathcal{PD}(\pi, \mathfrak{S}) \mid P\text{: } \mathcal{C}\text{-flat}\},$$
$$\mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi, \mathfrak{S})_G := \mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi, \mathfrak{S}) \cap \mathcal{PD}(\pi, \mathfrak{S})_G,$$

where $\mathcal{C}$ is a covering of X (refer to Definition 1.2 for the word "$\mathcal{C}$-flat").

A parallel displacement is induced by a given invertible slicing function in a natural manner as follows.

Proposition 4.3 (cf. [5], [7, p. 283])**.** *Let* $\omega \in SF_{inv}(\pi, U)$ *and* $\mathfrak{S} = \langle U \rangle$*. Put*

$$P^\omega((x_n, \ldots, x_0), u) := (\omega_{x_n, x_{n-1}} \circ \cdots \circ \omega_{x_1, x_0})(u)$$

for $((x_n, \ldots, x_0), u) \in (p_0|_{\mathfrak{S}})^* E$*. Then* $P^\omega \in \mathcal{PD}(\pi, \mathfrak{S})$*. If* ω *is* G*-compatible, so is* P^ω*.*

Example 4.4 (cf. [5], [7]). Let $\tilde{\pi}_K$ be the universal bundle which we reviewed in Example 3.4. A map $\tilde{P}_K : (p_0|_{\mathfrak{S}_K})^*\tilde{E}_K \to \tilde{E}_K$ is defined by

$$\tilde{P}_K((x_n,\ldots,x_1,y_m),[y_m,\ldots,y_1,v_0]) := [x_n,\ldots,x_1,y_m][y_m,\ldots,y_1,v_0]$$

for $((x_n,\ldots,x_1,y_m),[y_m,\ldots,y_1,v_0]) \in (p_0|_{\mathfrak{S}_K})^*\tilde{E}_K$. Then $\tilde{P}_K \in \mathcal{PD}(\tilde{\pi}_K,\mathfrak{S}_K)_{\tilde{G}_K}$. Since

$$\begin{aligned}
&\tilde{P}_K((x_n,\ldots,x_1,y_m),[y_m,\ldots,y_1,v_0]) = [x_n,\ldots,x_1,y_m][y_m,\ldots,y_1,v_0]\\
&= [x_n,x_{n-1}]\cdots[x_1,y_m][y_m,\ldots,y_1,v_0]\\
&= ((\tilde{\omega}_K)_{x_n,x_{n-1}}\circ\cdots\circ(\tilde{\omega}_K)_{x_1,y_m})([y_m,\ldots,y_1,v_0])\\
&= P^{\tilde{\omega}_K}((x_n,\ldots,x_1,y_m),[y_m,\ldots,y_1,v_0])
\end{aligned}$$

for $((x_n,\ldots,x_1,y_m),[y_m,\ldots,y_1,v_0]) \in (p_0|_{\mathfrak{S}_K})^*\tilde{E}_K$, we have $\tilde{P}_K = P^{\tilde{\omega}_K}$.

Now, we are in a position to state

Proposition 4.5. *A bijection from $SF_{\mathcal{C}\text{-flat}}(\pi,U_{\mathcal{C}})$ onto $\mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi,\langle U_{\mathcal{C}}\rangle)$ is given by $\omega \mapsto P^{\omega}$, where $U_{\mathcal{C}} := \bigcup_{C\in\mathcal{C}} C\times C$.*

All $P\lceil_Y$, f^*P, and P^{ρ} inherit the flatness from a flat parallel displacement P:

Proposition 4.6 (cf. [5]). *The following items* (i) *and* (ii) *hold*:

(i) *Suppose that $P \in \mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi,\mathfrak{S})$. Then, $P\lceil_Y \in \mathcal{PD}_{\mathcal{C}\lceil_Y\text{-flat}}(\pi\lceil_Y,\mathfrak{S}\lceil_Y)$ for all $Y \subset X$, and $f^*P \in \mathcal{PD}_{f^*(\mathcal{C})\text{-flat}}(f^*\pi,f^*\mathfrak{S})$ for any continuous map $f : X' \to X$. Here $\mathcal{C}\lceil_Y := \{C\cap Y \mid C\in\mathcal{C}\}$ and $f^*(\mathcal{C}) := \{f^{-1}(C) \mid C\in\mathcal{C}\}$.*

(ii) *If $P \in \mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi,\mathfrak{S})_G$, then $P^{\rho} \in \mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi^{\rho},\mathfrak{S})_G$, where $\rho : G \to G'$ is a continuous group homomorphism.*

Flat parallel displacements have the following fundamental property:

Proposition 4.7. *Let $P \in \mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi,\langle U_{\mathcal{C}}\rangle)$ and $(x_n,\ldots,x_0) \in \langle U_{\mathcal{C}}\rangle$. If there exist $C \in \mathcal{C}$ and $i \in \{1,\ldots,n\}$ such that $x_{i+1},x_i,x_{i-1} \in C$, then $P_{(x_n,\ldots,x_i,\ldots,x_0)} = P_{(x_n,\ldots,\hat{x}_i,\ldots,x_0)}$ holds. Here the symbol $\hat{x}_i$ denotes deletion.*

Proof. We have

$$\begin{aligned}
P_{(x_n,\ldots,x_i,\ldots,x_0)} &= P_{(x_n,\ldots,x_{i+1})}\circ P_{(x_{i+1},x_i,x_{i-1})}\circ P_{(x_{i-1},\ldots,x_0)}\\
&= P_{(x_n,\ldots,x_{i+1})}\circ P_{(x_{i+1},x_{i-1})}\circ P_{(x_{i-1},\ldots,x_0)}\\
&= P_{(x_n,\ldots,\hat{x}_i,\ldots,x_0)}.
\end{aligned}$$

□

We end this subsection with giving an example of flat parallel displacement:

Example 4.8. Let $\mathrm{pr}_1 : T^2 = S^1\times S^1 \to S^1$ be the product bundle and $\omega_\alpha \in SF_{\mathcal{C}\text{-flat}}(\mathrm{pr}_1,U)_{S^1}$ as in Example 3.2. For $((x_n,\ldots,x_0),(x_0,a)) \in (p_0|_{\langle U\rangle})^*T^2$, we have $P^{\omega_\alpha}((x_n,\ldots,x_0),(x_0,a)) = (x_n, ae^{2\pi i\alpha\sum_{k=1}^{n}\theta(x_k,x_{k-1})})$. For example, for

$(x_n, \ldots, x_0) \in \langle U \rangle \cap (U_1^+)^{\sqcup}$, since

$$\begin{aligned}\sum_{k=1}^{n} \theta(x_k, x_{k-1}) &= \theta_1^+(x_n, x_{n-1}) + \cdots + \theta_1^+(x_1, x_0) \\ &= (\varphi_1^+(x_n) - \varphi_1^+(x_{n-1})) + \cdots + (\varphi_1^+(x_1) - \varphi_1^+(x_0)) \\ &= \varphi_1^+(x_n) - \varphi_1^+(x_0) = \theta(x_n, x_0),\end{aligned}$$

we get

$$P^{\omega_\alpha}((x_n, \ldots, x_0), (x_0, a)) = (x_n, ae^{2\pi i\alpha\theta(x_n, x_0)}).$$

Thus, $P^{\omega_\alpha} \in \mathcal{PD}_{\mathcal{C}\text{-flat}}(\mathrm{pr}_1, \langle U \rangle)_{S^1}$.

4.2. Holonomy groups and Proof of Theorem 1.3.

Definition 4.9. (I) Let $\mathcal{C}$ be a covering of X, $x \in X$, and $A \subset X$ such that $x \in A$. We say that (A, x) is $\mathcal{C}$-*compatible* if it satisfies the following conditions:

(C1) $A \times \{x\} \subset U_{\mathcal{C}}$;
(C2) $x \in C \cap C'$, for any $C, C' \in \mathcal{C}$ with $A \cap C \cap C' \neq \emptyset$.

(II) We say that $\mathcal{C}$ is (C-N) if for any $x \in X$, there exists an open neighborhood V of x in X such that (V, x) is $\mathcal{C}$-compatible.

The following example gives us a (C-N) covering:

Example 4.10. Let X be a polyhedron of a simplicial complex K in the weak topology and put $U_K := \bigcup_{\tau \in K} |\tau| \times |\tau|$. For $x \in X$, let $V_K(x)$ be the open star neighborhood of x in X. Then, $(V_K(x), x)$ satisfies $V_K(x) \times \{x\} \subset U_K$. For $\tau, \sigma \in K$, suppose that $V_K(x) \cap |\tau| \cap |\sigma| \neq \emptyset$. If $y \in V_K(x) \cap |\tau| \cap |\sigma|$, there exists $\rho \in K$ such that $y \in \mathrm{Int}\rho$ and $x \in |\rho|$. Since any two simplexes do not intersect with each other at the interior of the other, $y \in \mathrm{Int}\rho \cap |\tau| \cap |\sigma|$ implies $\rho < \tau$ and $\rho < \sigma$. Thus, $x \in |\tau| \cap |\sigma|$ holds. Therefore, $(V_K(x), x)$ is $\mathcal{C}_K := \{|\tau| \mid \tau \in K\}$-compatible and $\mathcal{C}_K$ is (C-N).

We review the definitions of strong holonomy group and strong holonomy bundle.

Definition 4.11 (cf. [5])**.** (I) Let $\pi : E \to X$ be a principal G-bundle, $P \in \mathcal{PD}(\pi, \mathfrak{S})_G$, and $u \in E$. We call the subgroup $\Phi^u = P^u(\mathfrak{S}_{\pi(u)})$ of G the *strong holonomy group of* P *with reference point* u if it is endowed with the identification topology induced by $P^u : \mathfrak{S}_{\pi(u)} \to \Phi^u$.

(II) We call the subbundle $\pi^u : E^u \to X$ the *strong holonomy bundle through* u if the topology of E^u is the identification topology induced by $P(\cdot, u) : \mathfrak{S}_{X \times \{\pi(u)\}} \to E^u$.

We will prove the main Theorem 1.3 after preparing the following lemmas:

Lemma 4.12 (cf. [5, (ii) in Proposition 4.2])**.** *Let* $\mathfrak{S} \in \mathcal{AS}(X)$*. If* X *is* $\mathfrak{S}$*-connected and* $\mathfrak{S}_{(1)} \in \mathcal{S}_{X^2}(\Delta_X)$*, then* $p_\infty|_{\mathfrak{S}_{X \times \{x\}}} : \mathfrak{S}_{X \times \{x\}} \to X$ *is an identification for every* $x \in X$*.*

Lemma 4.13. *Let X be a topological space and $\mathcal{C}$ a covering of X. Let X' be a set and $q : \langle U_{\mathcal{C}} \rangle \to X'$ a map such that for any $(x_n, \dots, x_0) \in \langle U_{\mathcal{C}} \rangle$ and $i \in \{1, \dots, n\}$, if there exists $C \in \mathcal{C}$ such that $x_{i+1}, x_i, x_{i-1} \in C$, then $q(x_n, \dots, x_i, \dots, x_0) = q(x_n, \dots, \hat{x}_i, \dots, x_0)$, where $\hat{}$ means a deletion. Then, the following items hold:*

(i) *Let $(x_n, \dots, x_0) \in \langle U_{\mathcal{C}} \rangle$ and $(A_n, \dots, A_0) \in \mathcal{P}(X)^{n+1}$ such that (A_i, x_i) is $\mathcal{C}$-compatible for any $i \in \{0, \dots, n\}$. Then $q(\mathbf{y}) = q(y_n, x_{n-1}, \dots, x_1, x_0)$ for any $\mathbf{y} = (y_n, y_{n-1}, \dots, y_1, x_0) \in (A_n \times \cdots \times A_0) \cap \langle U_{\mathcal{C}} \rangle$, and if $y_n = x_n$, then $q(\mathbf{y}) = q(\mathbf{x})$.*

(ii) *Suppose that $\mathcal{C}$ is* (C-N) *or $\mathcal{C} \subset \mathcal{O}_X$. Then for any $\mathbf{x} = (x_n, \dots, x_0) \in \langle U_{\mathcal{C}} \rangle$, there exists an open neighborhood W of $\mathbf{x}$ in $\langle U_{\mathcal{C}} \rangle$ such that $W \subset \langle U_{\mathcal{C}} \rangle_{(n)}$, $q(\mathbf{y}) = q(y_n, x_{n-1}, \dots, x_1, x_0)$ for any $\mathbf{y} = (y_n, y_{n-1}, \dots, y_1, x_0) \in W$, and if $y_n = x_n$, then $q(\mathbf{y}) = q(\mathbf{x})$.*

Proof. First, we will show (i). Let $\mathbf{y} = (y_n, y_{n-1}, \dots, y_1, x_0) \in (A_n \times \cdots \times A_0) \cap \langle U_{\mathcal{C}} \rangle$. Since $(y_1, x_0) \in U_{\mathcal{C}}$, there exists $C \in \mathcal{C}$ such that $y_1, x_0 \in C$. On the other hand, since $(y_1, x_1) \in A_1 \times \{x_1\} \subset U_{\mathcal{C}}$, there exists $C' \in \mathcal{C}$ such that $y_1, x_1 \in C'$. Then, since $y_1 \in A_1 \cap C \cap C'$, (C2) implies $x_1 \in C \cap C'$. Thus, $y_1, x_1, x_0 \in C$ holds. Therefore, $q(\mathbf{y}) = q(y_n, \dots, y_1, x_1, x_0)$. Since $(y_2, y_1) \in U_{\mathcal{C}}$, there exists $C'' \in \mathcal{C}$ such that $y_2, y_1 \in C''$. Then, since $y_1 \in A_1 \cap C' \cap C''$, (C2) implies $x_1 \in C' \cap C''$. Thus, $y_2, y_1, x_1 \in C''$ holds. Therefore, $q(y_n, \dots, y_2, y_1, x_1, x_0) = q(y_n, \dots, y_2, x_1, x_0)$ holds. Repeating the same argument, we get $q(\mathbf{y}) = q(y_n, x_{n-1}, \dots, x_1, x_0)$. As a result, if $y_n = x_n$, $q(\mathbf{y}) = q(\mathbf{x})$ holds.

Next, we will show (ii). Suppose that $\mathcal{C}$ is (C-N). Let $\mathbf{x} = (x_n, \dots, x_0) \in \langle U_{\mathcal{C}} \rangle$. For $i \in \{0, \dots, n\}$, fix an open neighborhood V_i of x_i in X such that (V_i, x_i) is $\mathcal{C}$-compatible. Put

$$W := (V_n \times \cdots \times V_0) \cap \langle U_{\mathcal{C}} \rangle.$$

Then, W is an open neighborhood of x in $\langle U_{\mathcal{C}} \rangle$ and the conclusion is followed from (i). Now, suppose that $\mathcal{C} \subset \mathcal{O}_X$. Let $\mathbf{x} = (x_n, \dots, x_0) \in \langle U_{\mathcal{C}} \rangle$. For any $i \in \{1, \dots, n\}$, there exists $U_i \in \mathcal{C}$ such that $x_i, x_{i-1} \in U_i$. Put

$$W' := (U_n \times (U_n \cap U_{n-1}) \times \cdots \times (U_2 \cap U_1) \times U_1) \cap \langle U_{\mathcal{C}} \rangle.$$

Let $\mathbf{y} = (y_n, y_{n-1}, \dots, y_1, x_0) \in W'$. Then, since $y_1, x_1, x_0 \in U_1$ and $y_2, y_1, x_1 \in U_2$, we have

$$q(\mathbf{y}) = q(y_n, \dots, y_1, x_1, x_0) = q(y_n, \dots, y_2, x_1, x_0).$$

Repeating this operation, we get $q(\mathbf{y}) = q(y_n, x_{n-1}, \dots, x_1, x_0)$ and $q(\mathbf{y}) = q(\mathbf{x})$ if $y_n = x_n$. □

From now on, let us demonstrate Theorem 1.3.

Proof of Theorem 1.3. First, we will show (i). Note that the topology of Φ^u is the identification topology induced by P^u from $\langle U_{\mathcal{C}} \rangle_{\pi(u)}$. We will show that $(P^u)^{-1}(\{a\})$ is an open set in $\langle U_{\mathcal{C}} \rangle_{\pi(u)}$. Let $\mathbf{x} \in (P^u)^{-1}(\{a\})$, where $p_0(\mathbf{x}) = p_\infty(\mathbf{x}) = \pi(u)$. Since $\mathcal{C}$ is (C-N) or $\mathcal{C} \subset \mathcal{O}_X$, from Proposition 4.7 and (ii) of Lemma 4.13, there exists an open neighborhood W of $\mathbf{x}$ in $\langle U_{\mathcal{C}} \rangle_{\pi(u)}$ such that for any $\mathbf{y} \in W$, $P^u(\mathbf{y}) = P^u(\mathbf{x}) = a$. Thus, we have $W \subset (P^u)^{-1}(\{a\})$.

Next, we will show (ii). Let $\mu : E \times G \to E$ be the continuous right action. At first, we show that E^u is a Φ^u-space, that is, ${}_{E^u}|\mu|_{E^u\times\Phi^u} : E^u \times \Phi^u \to E^u$ is continuous. Since Φ^u is a discrete group, it is sufficient to show that $({}_{E^u}|\mu|_{E^u\times\Phi^u})(\cdot, a) : E^u \to E^u$ is continuous for any $a = P^u(\mathbf{x}) \in \Phi^u$. Indeed, since

$$({}_{E^u}|\mu|_{E^u\times\Phi^u})(\cdot, a) \circ P(\cdot, u) = P(\cdot, u) \circ \bullet(\cdot, \mathbf{x})$$

and $P(\cdot, u)$ is an identification, $({}_{E^u}|\mu|_{E^u\times\Phi^u})(\cdot, a)$ is continuous. Next, we show that π^u is a Φ^u-bundle. Let $s^u : X \to E^u/\Phi^u$ be the map so that the equality $s^u \circ p_\infty|_{\langle U_\mathcal{C}\rangle_{X\times\{\pi(u)\}}} = q^{E^u}_{\Phi^u} \circ P(\cdot, u)$ holds. Then, we have $s^u = (\pi^u/^{\Phi^u})^{-1}$. Note that if $\mathcal{C} \subset \mathcal{O}_X$, then $U_\mathcal{C} \in \mathcal{S}_{X^2}(\Delta_X)$. Thus, whenever $\mathcal{C}$ is (C-N) or $\mathcal{C} \subset \mathcal{O}_X$, from Lemma 4.12, $p_\infty|_{\langle U_\mathcal{C}\rangle_{X\times\{\pi(u)\}}} : \langle U_\mathcal{C}\rangle_{X\times\{\pi(u)\}} \to X$ is an identification. Then s^u is continuous. Thus, $\pi^u/^{\Phi^u}$ is a homeomorphism. From Lemma 2.3, we have the conclusion. □

The following example implies that Theorem 1.3 is no longer true if one removes the supposition "strong" from the statement (i):

Example 4.14. Let $\mathrm{pr}_1 : T^2 = S^1 \times S^1 \to S^1$ be the product bundle and $P^{\omega_\alpha} \in \mathcal{PD}_{\mathcal{C}\text{-flat}}(\mathrm{pr}_1, \langle U\rangle)_{S^1}$ as in Example 4.8. Fix an arbitrary $(x_0, a) \in T^2$ and put $x_k := x_{k-1}e^{\frac{2\pi}{5}i}$ for $k \in \{1, \ldots, 10\}$, where $x_{10} = x_5 = x_0$. Then, we have $(P^{\omega_\alpha})^{(x_0,a)}(x_{10}, \ldots, x_0) = e^{2\pi i\alpha\sum_{k=1}^{10}\theta(x_k,x_{k-1})} = e^{2\pi i\alpha\cdot 2}$ and

$$\Phi^{(x_0,a)} = \{e^{2\pi i\alpha k} \mid k \in \mathbb{Z}\}.$$

If α is a rational number, then $\Phi^{(x_0,a)}$ is a finite set while if α is an irrational number, it is dense in S^1 and is not discrete with respect to the relative topology.

4.3. Local holonomy groups of parallel displacements. In Subsection 4.2, we have completed the proof of the main Theorem 1.3. Here Theorem 1.3 is concerned with the strong holonomy groups of flat parallel displacements. In this subsection, we would like to assert Proposition 4.16 which is concerned with the local holonomy groups of flat parallel displacements.

We introduced in [5] the notion of local holonomy group of a parallel displacement and studied its fundamental properties. We will review the definition of local holonomy groups. For a symmetric subspace $U \subset X^2$ with $\Delta_X \subset U$ and $x \in X$, put

$$\langle U\rangle^1_x := \{(x, x_{n-1}, \ldots, x_1, x) \in \langle U\rangle_x \mid (x_k, x) \in U \text{ for all } k \in \{1, \ldots, n-1\}\}.$$

Then $\langle U\rangle^1_x$ is a submonoid of $\langle U\rangle_x$.

Definition 4.15 (cf. [5])**.** Let $\pi : E \to X$ be a principal G-bundle and $P \in \mathcal{PD}(\pi, \mathfrak{S})_G$. For $u \in E$, we call a subgroup

$$\Phi^{u,1} := \Phi^{u,1}(P) := P^u(\langle \mathfrak{S}_{(1)}\rangle^1_{\pi(u)})$$

of Φ^u the *local holonomy group of P with reference point u.*

In the smooth category, local holonomy groups are trivial if there exists a flat connection. As an analogue of this fact, we see that the local holonomy groups of parallel displacements are trivial if there exist a certain covering $\mathcal{C}$ of X and a $\mathcal{C}$-flat parallel displacement.

Proposition 4.16. *Let $\mathcal{C}$ be a covering of X, $P \in \mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi, \langle U_{\mathcal{C}}\rangle)_G$, and $u \in E$. Suppose that $\mathcal{C}$ satisfies the following condition:*

$$C \cap C' \cap C'' \neq \emptyset$$

for any $C, C', C'' \in \mathcal{C}$ with $C \cap C' \neq \emptyset$, $C' \cap C'' \neq \emptyset$ and $C'' \cap C \neq \emptyset$. Then, the local holonomy group $\Phi^{u,1}$ is trivial.

Proof. Let $\mathbf{x} = (\pi(u), x_{n-1}, \ldots, x_1, \pi(u))$ be an element of $\langle U_{\mathcal{C}}\rangle^1_{\pi(u)}$. Since $(x_1, \pi(u)), (x_2, x_1), (x_2, \pi(u)) \in U_{\mathcal{C}}$, there exists $(C, C', C'') \in \mathcal{C}^3$ such that $x_1, \pi(u) \in C$, $x_2, x_1 \in C'$, and $x_2, \pi(u) \in C''$. Thus, $x_1 \in C \cap C'$, $x_2 \in C' \cap C''$, and $\pi(u) \in C'' \cap C$. Then, from the assumption, $C \cap C' \cap C'' \neq \emptyset$. Let $y \in C \cap C' \cap C''$. Then, from Proposition 4.7, we have

$$\begin{aligned} P_{\mathbf{x}} &= P_{(\pi(u), x_{n-1}, \ldots, x_2, x_1, y, \pi(u))} \\ &= P_{(\pi(u), x_{n-1}, \ldots, x_2, y, \pi(u))} = P_{(\pi(u), x_{n-1}, \ldots, x_2, \pi(u))}. \end{aligned}$$

Repeating this operation, we get $P_{\mathbf{x}} = P_{\pi(u)}$. Therefore, $P^u(\mathbf{x}) = 1_{\Phi^{u,1}}$. □

5. A classification theorem

The main purpose in this section is to assert a classification theorem in a category of principal bundles with flat parallel displacements (see Theorem 5.2), and to give a sufficient condition for the existence of an initial object in the category (see Theorem 5.3).

Throughout in this section, we assume that X is a topological space, $v_0 \in X$, and $\mathcal{C}$ is a covering of X. At first, we introduce the following category $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})$. Objects in $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})$ are such quadruples (π, G, P, u), where $\pi : E \to X$ is a principal G-bundle, $P \in \mathcal{PD}_{\mathcal{C}\text{-flat}}(\pi, \mathfrak{S})_G$ is a $\mathcal{C}$-flat parallel displacement, and $u \in E_{v_0}$. Morphisms in $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})$ are such homomorphisms $(h, id_X, \rho) : (\pi, G, P, u) \to (\pi', G', P', u')$ preserving P and P', that is,

$$h(P(\mathbf{x}, v)) = P'(\mathbf{x}, h(v))$$

for $(\mathbf{x}, v) \in (p_0|_{\mathfrak{S}})^*E$, and satisfying $h(u) = u'$. We denote by (h, ρ) the morphism (h, id_X, ρ), and by $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})_0$ (resp. $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})_1$) the collection of objects (resp. morphisms) in $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})$.

Remark 5.1. Note that $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})$ is an isomorphism-closed full subcategory of the category $\mathbf{C}(X, v_0, \mathfrak{S})$ of principal bundles with parallel displacements introduced in [5, Section 8].

Let $\mathbf{G}$ be the category of topological groups. We denote by $\mathbf{G}_0$ (resp. $\mathbf{G}_1$) the collection of objects (resp. morphisms). Let $\widehat{G} \in \mathbf{G}_0$. An equivalence relation on $\{\rho \in \mathbf{G}_1 \mid \mathrm{dom}\rho = \widehat{G}\}$ is defined as follows. Two morphisms $\rho, \rho' \in \mathbf{G}_1$ with $\mathrm{dom}\rho = \mathrm{dom}\rho' = \widehat{G}$ are *equivalent* if there exists a topological group isomorphism $\tau : \mathrm{cod}\rho \to \mathrm{cod}\rho'$ such that $\rho' = \tau \circ \rho$. We denote by $[\rho]$ the equivalence class of ρ.

Once it is shown that an initial object exists, we obtain the following theorem:

Theorem 5.2 (Classification Theorem (flat version))**.** *Suppose that there exists an initial object $(\widehat{\pi}, \widehat{G}, \widehat{P}, \widehat{u})$ in $\mathbf{C}_{\mathcal{C}\text{-}flat}(X, v_0, \mathfrak{S})$ with $\widehat{\pi} : \widehat{E} \to X$. Two maps*

$$\{\rho \in \mathbf{G}_1 \mid \mathrm{dom}\rho = \widehat{G}\} \xrightarrow{\Lambda'} \mathbf{C}_{\mathcal{C}\text{-}flat}(X, v_0, \mathfrak{S}) \xrightarrow{\Theta'} \{\rho \in \mathbf{G}_1 \mid \mathrm{dom}\rho = \widehat{G}\}$$

are defined as follows. For $\rho \in \mathbf{G}_1$ with $\mathrm{dom}\rho = \widehat{G}$, put

$$\Lambda'(\rho) := (\widehat{\pi}^\rho, \mathrm{cod}\rho, \widehat{P}^\rho, [\widehat{u}, 1_{\mathrm{cod}\rho}]).$$

For an object (π, G, P, u) in $\mathbf{C}_{\mathcal{C}\text{-}flat}(X, v_0, \mathfrak{S})$ with the unique morphism $(h, \rho) : (\widehat{\pi}, \widehat{G}, \widehat{P}, \widehat{u}) \to (\pi, G, P, u)$ in $\mathbf{C}_{\mathcal{C}\text{-}flat}(X, v_0, \mathfrak{S})$, put $\Theta'(\pi, G, P, u) := \rho$. Then $\Theta' \circ \Lambda' = id$, and both $\Lambda'(\Theta'(\pi, G, P, u))$ and (π, G, P, u) are isomorphic for each $(\pi, G, P, u) \in \mathbf{C}_{\mathcal{C}\text{-}flat}(X, v_0, \mathfrak{S})_0$. Moreover, for $\rho, \rho' \in \mathbf{G}_1$ with $\mathrm{dom}\rho = \mathrm{dom}\rho' = \widehat{G}$, ρ and ρ' are equivalent if and only if $\Lambda'(\rho)$ and $\Lambda'(\rho')$ are isomorphic. Thus, the induced map

$$\{[\rho] \mid \rho \in \mathbf{G}_1, \mathrm{dom}\rho = \widehat{G}\} \xrightarrow{\overline{\Lambda'}} \{[(\pi, G, P, u)] \mid (\pi, G, P, u) \in \mathbf{C}_{\mathcal{C}\text{-}flat}(X, v_0, \mathfrak{S})_0\}$$

is bijective, where $[(\pi, G, P, u)]$ is the isomorphic class of (π, G, P, u).

Proof. The proof of this theorem is similar to that of Theorem 8.1 (Classification theorem) in [5]. For $(\rho : \widehat{G} \to G) \in \mathbf{G}_1$, let $(\theta^\rho, \rho) : (\widehat{\pi}, \widehat{G}, \widehat{P}, \widehat{u}) \to (\widehat{\pi}^\rho, G, \widehat{P}^\rho, [\widehat{u}, 1_G])$ be the morphism in $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})$ given by $\theta^\rho(v) := [v, 1_G]$ for $v \in \widehat{E}$ (see Section 2). Since $(\widehat{\pi}, \widehat{G}, \widehat{P}, \widehat{u})$ is an initial object, (θ^ρ, ρ) is the unique morphism from $(\widehat{\pi}, \widehat{G}, \widehat{P}, \widehat{u})$ to $(\widehat{\pi}^\rho, G, \widehat{P}^\rho, [\widehat{u}, 1_G])$. Thus $\Theta'(\Lambda'(\rho)) = \Theta'(\widehat{\pi}^\rho, G, \widehat{P}^\rho, [\widehat{u}, 1_G]) = \rho$.

Let $(\pi, G, P, u) \in \mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})_0$ and $(h, \rho) : (\widehat{\pi}, \widehat{G}, \widehat{P}, \widehat{u}) \to (\pi, G, P, u)$ be the unique morphism in $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \mathfrak{S})$. Let $(h^\rho, id_G) : (\widehat{\pi}^\rho, G) \to (\pi, G)$ be the (X, G)-morphism given by $h^\rho([v, a]) := h(v)a$ for $[v, a] \in \tilde{E}^\rho$ (see Section 2). From Lemma 2.4, $(h^\rho, id_G) : (\widehat{\pi}^\rho, G, \widehat{P}^\rho, [\widehat{u}, 1_G]) \to (\pi, G, P, u)$ is an isomorphism. Thus $\Lambda(\Theta(\pi, G, P, u))$ and (π, G, P, u) are isomorphic.

Suppose that $[\rho : \widehat{G} \to G] = [\rho' : \widehat{G} \to G']$ and let $\tau : G \to G'$ be a topological group isomorphism such that $\rho' = \tau \circ \rho$. A map $k : \widehat{E}^\rho \to \widehat{E}^{\rho'}$ is defined by $k([v, a]) := [v, \tau(a)]$ for $[v, a] \in \widehat{E}^\rho$. We can see that $(k, \tau) : (\widehat{\pi}^\rho, G, \widehat{P}^\rho, [\widehat{u}, 1_G]) \to (\widehat{\pi}^{\rho'}, G, \widehat{P}^{\rho'}, [\widehat{u}, 1_{G'}])$ is an isomorphism.

Conversely, let $(k, \tau) : (\widehat{\pi}^\rho, G, \widehat{P}^\rho, [\widehat{u}, 1_G]) \to (\widehat{\pi}^{\rho'}, G, \widehat{P}^{\rho'}, [\widehat{u}, 1_{G'}])$ be an isomorphism. Then $(k \circ \theta^\rho, \tau \circ \rho) : (\widehat{\pi}, \widehat{G}, \widehat{P}, \widehat{u}) \to (\widehat{\pi}^{\rho'}, G, \widehat{P}^{\rho'}, [\widehat{u}, 1_{G'}])$ is a morphism. Since $(\widehat{\pi}, \widehat{G}, \widehat{P}, \widehat{u})$ is an initial object, the uniqueness of morphism implies $(k \circ \theta^\rho, \tau \circ \rho) = (\theta^{\rho'}, \rho')$. Thus $[\rho] = [\rho']$. ☐

In the smooth category, flat principal G-bundles over a fixed base space are classified by group homomorphisms from the fundamental group of the base space to G (e.g. [8, 10]). We can think of Theorem 5.2 as one of the topological counterparts of the classification theorem for flat bundles in the smooth category.

Next, we construct an initial object in $\mathbf{C}_{\mathcal{C}\text{-flat}}(X, v_0, \langle U_{\mathcal{C}} \rangle)$. We assume the following conditions:

(D1) $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ is a countable (C-N) compact covering, and $\langle U_\mathcal{C}\rangle$ is a closed set in $X^\sqcup$, where $U_\mathcal{C} = \bigcup_{n\in\mathbb{N}} C_n \times C_n$,

(D2) X is a Hausdorff space which has the weak topology with respect to $\mathcal{C}$, and $\langle U_\mathcal{C}\rangle$-connected.

An equivalence relation in $\langle U_\mathcal{C}\rangle$ is generated by the relations

$$(x_n, \ldots, x_i, \ldots, x_0) \sim' (x_n, \ldots, \hat{x}_i, \ldots, x_0)$$

whenever either $x_i = x_{i-1}$, $x_{i+1} = x_{i-1}$, or there exists $n \in \mathbb{N}$ such that $x_{i+1}, x_i, x_{i-1} \in C_n$, where the symbol $\hat{x}$ denotes deletion. The natural projection is denoted by q'. Put $[x_n, \ldots, x_0]' := q'(x_n, \ldots, x_0)$ and

$$\begin{aligned}
&\widehat{\langle U_\mathcal{C}\rangle}_\mathcal{C} := \langle U_\mathcal{C}\rangle / \sim',\\
&\widehat{E}_\mathcal{C} := \{[x_n, \ldots, x_1, x_0]' \in \widehat{\langle U_\mathcal{C}\rangle}_\mathcal{C} \mid x_0 = v_0\},\\
&\widehat{G}_\mathcal{C} := \{[x_n, \ldots, x_1, v_0]' \in \widehat{E}_\mathcal{C} \mid x_n = v_0\},
\end{aligned}$$

where a topology of $\widehat{\langle U_\mathcal{C}\rangle}_\mathcal{C}$ is the quotient topology and consider $\widehat{E}_\mathcal{C}$ and $\widehat{G}_\mathcal{C}$ as subspaces. We can see that two maps $q'' = q'|_{\langle U_\mathcal{C}\rangle_{X\times\{v_0\}}} : \langle U_\mathcal{C}\rangle_{X\times\{v_0\}} \to \widehat{E}_\mathcal{C}$ and $q''' = q'|_{\langle U_\mathcal{C}\rangle_{v_0}} : \langle U_\mathcal{C}\rangle_{v_0} \to \widehat{G}_\mathcal{C}$ are identifications. A map $\widehat{\pi}_\mathcal{C} : \widehat{E}_\mathcal{C} \to X$ is defined by $\widehat{\pi}_\mathcal{C}([\mathbf{x}]') := p_\infty(\mathbf{x})$. Since $\widehat{\pi}_\mathcal{C} \circ q'' = p_\infty|_{\langle U_\mathcal{C}\rangle_{X\times\{v_0\}}}$ and q'' is an identification, $\widehat{\pi}_\mathcal{C}$ is a continuous map, that is, $\widehat{\pi}_\mathcal{C}$ is a bundle. Since X is $\langle U_\mathcal{C}\rangle$-connected, $\widehat{\pi}_\mathcal{C}$ is surjective. A binary operation $\nu : \widehat{G}_\mathcal{C} \times \widehat{G}_\mathcal{C} \to \widehat{G}_\mathcal{C}$ and a unary operation $\cdot^{-1} : \widehat{G}_\mathcal{C} \to \widehat{G}_\mathcal{C}$ are defined so that $\nu \circ (q''' \times q''') = q''' \circ \bullet$ and $\cdot^{-1} \circ q''' = q''' \circ \cdot^{-}$ hold respectively. A right action $\mu : \widehat{E}_\mathcal{C} \times \widehat{G}_\mathcal{C} \to \widehat{E}_\mathcal{C}$ is defined so that $\mu \circ (q'' \times q''') = q'' \circ \bullet$ holds. For the sake of simplicity, we denote the restrictions

$${}_{\widehat{E}_\mathcal{C} \times_X \widehat{E}_\mathcal{C}}|(q'' \times q'')|_{\langle U_\mathcal{C}\rangle_{X\times\{v_0\}} \times_X \langle U_\mathcal{C}\rangle_{X\times\{v_0\}}}$$

and

$${}_{\langle U_\mathcal{C}\rangle \times_X \widehat{E}_\mathcal{C}}|(id_{\langle U_\mathcal{C}\rangle} \times q'')|_{\langle U_\mathcal{C}\rangle \times_X \langle U_\mathcal{C}\rangle_{X\times\{v_0\}}}$$

by $(q'' \times q'')\lceil_{\widehat{E}_\mathcal{C} \times_X \widehat{E}_\mathcal{C}}$ and $(id_{\langle U_\mathcal{C}\rangle} \times q'')\lceil_{\langle U_\mathcal{C}\rangle \times_X \widehat{E}_\mathcal{C}}$ respectively. A map $\hat{T}_\mathcal{C} : \widehat{E}_\mathcal{C} \times_X \widehat{E}_\mathcal{C} \to \widehat{G}_\mathcal{C}$ is defined so that

$$\hat{T}_\mathcal{C} \circ (q'' \times q'')\lceil_{\widehat{E}_\mathcal{C} \times_X \widehat{E}_\mathcal{C}} = q''' \circ \bullet \circ (\cdot^{-} \times id_{\langle U_\mathcal{C}\rangle_{X\times\{v_0\}}})$$

holds. A map $\widehat{P}_\mathcal{C} : \langle U_\mathcal{C}\rangle \times_X \widehat{E}_\mathcal{C} \to \widehat{E}_\mathcal{C}$ is defined so that

$$\widehat{P}_\mathcal{C} \circ (id_{\langle U_\mathcal{C}\rangle} \times q'')\lceil_{\langle U_\mathcal{C}\rangle \times_X \widehat{E}_\mathcal{C}} = q'' \circ \bullet$$

holds.

Under the conditions (D1) and (D2), we have the following theorem:

Theorem 5.3. *Suppose the following conditions hold*:

(D1) $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ *is a countable* (*C-N*) *compact covering, and* $\langle U_\mathcal{C}\rangle$ *is a closed set in* $X^\sqcup$, *where* $U_\mathcal{C} = \bigcup_{n\in\mathbb{N}} C_n \times C_n$,

(D2) X *is a Hausdorff space which has the weak topology with respect to* $\mathcal{C}$, *and* $\langle U_\mathcal{C}\rangle$*-connected.*

Then, the quadruple $(\widehat{\pi}_{\mathcal{C}}, \widehat{G}_{\mathcal{C}}, \widehat{P}_{\mathcal{C}}, [v_0]')$ *is an initial object in* $\mathbf{C}_{\mathcal{C}\text{-}\mathit{flat}}(X, v_0, \langle U_{\mathcal{C}}\rangle)$*. In particular,* $\widehat{\pi}_{\mathcal{C}}$ *is locally trivial.*

We prepare the following lemmas for proving the above theorem.

Lemma 5.4 (cf. [5, (ii) in Theorem 4.4])**.** *Let* G *be a topological group,* $\pi : E \to X$ *a principal* G*-bundle, and* $P \in \mathcal{PD}(\pi, \mathfrak{S})_G$*. Suppose that* X *is* $\mathfrak{S}$*-connected and* $\mathfrak{S}_{(1)} \in \mathcal{S}_{X^2}(\Delta_X)$*. Then* π *is a locally trivial* G*-bundle.*

We denote by $\mathbf{C}$ a category of principal G-bundles with parallel displacements introduced in [5], and by $\mathbf{C}_0$ its objects. In [5, Definition 7.1], we call a quadruple $(\pi^u, \Phi^u, P\lceil_{E^u}, u)$ the *strong holonomy reduction of* $(\pi, G, P, u) \in \mathbf{C}_0$ if $(\pi^u, \Phi^u, P\lceil_{E^u}, u) \in \mathbf{C}_0$ when Φ^u (resp. E^u) is the strong holonomy group (resp. the strong holonomy bundle).

Lemma 5.5 (cf. [5, Theorem 7.1])**.** *Let* (π, G, P, u) *(resp.* (π', G', P', u')*)*$\in \mathbf{C}_0$*, where* $\pi : E \to X$ *(resp.* $\pi' : E' \to X$*) is a principal* G *(resp.* G'*)-bundle,* $P \in \mathcal{PD}(\pi, \mathfrak{S})_G$ *(resp.* $P' \in \mathcal{PD}(\pi', \mathfrak{S}')_{G'}$*) is a parallel displacement, and* $u \in E$ *(resp.* $u' \in E'$*). Suppose that* X *is* $\mathfrak{S}$*-connected and* $(\pi^u, \Phi^u, P\lceil_{E^u}, u)$ *is the strong holonomy reduction of* (π, G, P, u)*. Let* $f : X \to X'$ *be a continuous map preserving* $\mathfrak{S}$ *and* $\mathfrak{S}'$*, and satisfying* $f(\pi(u)) = \pi'(u')$*. If* $f^{\sqcup}(\mathrm{Ker}P^u) \subset \mathrm{Ker}P'^{u'}$*, then there exists a unique morphism* $(h^{u'u}, f, \rho^{u'u}) : (\pi^u, \Phi^u, P\lceil_{E^u}, u) \to (\pi', G', P', u')$ *in* $\mathbf{C}$*. In particular,* $h^{u'u}(E^u) \subset E'^{u'}$ *and* $\rho^{u'u}(\Phi^u) \subset \Phi^{u'}$*.*

Lemma 5.6 (cf. [5, Lemma 7.7])**.** *Let* X *and* X' *be Hausdorff spaces. Suppose that* X *(resp.* X'*) has a weak topology with respect to a compact covering* $(X_n)_{n\in\mathbb{N}}$ *(resp.* $(X'_m)_{m\in\mathbb{N}}$*). Then, for any Hausdorff spaces* Y*,* Y' *and identifications* $f : X \to Y$*,* $f' : X' \to Y$*, the product* $f \times f' : X \times X' \to Y \times Y'$ *is an identification.*

Lemma 5.7 (cf. [5, Lemma 7.8])**.** *Let* X *be a Hausdorff space and* $\mathfrak{S} \in \mathcal{AS}(X)$*. Suppose that* X *has the weak topology with respect to a countable compact covering, and* $\mathfrak{S}$ *is a closed set in* $X^{\sqcup}$*. Then for any* $x \in X$*,* $\mathfrak{S}$*,* $\mathfrak{S}_{X\times\{x\}}$*, and* $\mathfrak{S}_x$ *have the weak topology with respect to countable compact coverings respectively.*

Lemma 5.8. $\widehat{E}_{\mathcal{C}}$ *is a Hausdorff space.*

Proof. Let $[\mathbf{x}]', [\mathbf{y}]' \in \widehat{E}_{\mathcal{C}}$ with $\mathbf{x} = (x_n, \dots, v_0)$ and $\mathbf{y} = (y_m, \dots, v_0)$ such that $[\mathbf{x}]' \neq [\mathbf{y}]'$. First, suppose that $x_n \neq y_m$. Since X is a Hausdorff space, there exist open neighborhoods U_{x_n} and U_{y_m} of x_n and y_m respectively such that $U_{x_n} \cap U_{y_m} = \emptyset$. Then, $\widehat{\pi}_{\mathcal{C}}^{-1}(U_{x_n})$ and $\widehat{\pi}_{\mathcal{C}}^{-1}(U_{y_m})$ are open neighborhoods of $[\mathbf{x}]'$ and $[\mathbf{y}]'$ respectively, and $\widehat{\pi}_{\mathcal{C}}^{-1}(U_{x_n}) \cap \widehat{\pi}_{\mathcal{C}}^{-1}(U_{y_m}) = \emptyset$. Next, suppose that $x_n = y_m$. Since $\mathcal{C}$ is (C-N), there exists $V_{x_n} \in \mathcal{O}_X(x_n)$ such that (V_{x_n}, x_n) is $\mathcal{C}$-compatible. Put

$$U_{[\mathbf{x}]'} := q''((V_{x_n} \times \{x_n\}) \bullet \{\mathbf{x}\}).$$

Then, $[\mathbf{x}]' \in U_{[\mathbf{x}]'}$. We will show that $U_{[\mathbf{x}]'}$ is an open set in $\widehat{E}_{\mathcal{C}}$. Let $\mathbf{z} = (z_r, \dots, v_0) \in q''^{-1}(U_{[\mathbf{x}]'})$. Then, we have $z_r \in V_{x_n}$ and $\mathbf{z} \sim' (z_r, x_n) \bullet \mathbf{x}$. From Lemma 4.13 (ii), there exists an open neighborhood W of $\mathbf{z}$ in $\langle U_{\mathcal{C}}\rangle_{X\times\{v_0\}}$ such that $W \subset (\langle U_{\mathcal{C}}\rangle_{X\times\{v_0\}})_{(r)}$ and for any $\mathbf{w} = (w_r, \dots, v_0) \in W$, $\mathbf{w} \sim' (w_r, z_{r-1}, \dots, z_1, v_0)$. Put $W' := (V_{x_n} \times X^r) \cap W$. Then, W' is an open neighborhood of $\mathbf{z}$ in

$\langle U_{\mathcal{C}}\rangle_{X\times\{v_0\}}$. Let $\mathbf{w}=(w_r,\dots,v_0)\in W'$. Then, we have $w_r\in V_{x_n}$ and $\mathbf{w}\sim'(w_r,z_{r-1},\dots,z_1,v_0)$. Since $(w_r,z_{r-1})\in U_{\mathcal{C}}$, there exists $k\in\mathbb{N}$ such that $w_r,z_{r-1}\in C_k$. Since $(w_r,x_n)\in V_{x_n}\times\{x_n\}\subset U_{\mathcal{C}}$, there exists $l\in\mathbb{N}$ such that $w_r,x_n\in C_l$. Then, $w_r\in V_{x_n}\cap C_k\cap C_l$ and (C2) imply $x_n\in C_k\cap C_l$. Thus, we get $w_r,x_n,z_{r-1}\in C_k$ and $(w_r,z_{r-1},\dots,z_1,v_0)\sim'(w_r,x_n,z_{r-1},\dots,z_1,v_0)$. By the same argument, there exists $k'\in\mathbb{N}$ such that $x_n,z_r,z_{r-1}\in C_{k'}$ and $(w_r,x_n,z_{r-1},\dots,z_1,v_0)\sim'(w_r,x_n,z_r,z_{r-1},\dots,z_1,v_0)$. Then, we have equivalence relations

$$\begin{aligned}\mathbf{w}&\sim'(w_r,x_n,z_r,z_{r-1},\dots,z_1,v_0)=(w_r,x_n,z_r)\bullet\mathbf{z}\\&\sim'(w_r,x_n,z_r)\bullet(z_r,x_n)\bullet\mathbf{x}\sim'(w_r,x_n)\bullet\mathbf{x}.\end{aligned}$$

Therefore, $\mathbf{w}\in q''^{-1}(U_{[\mathbf{x}]'})$ and $q''^{-1}(U_{[\mathbf{x}]'})$ is an open set in $\langle U_{\mathcal{C}}\rangle_{X\times\{v_0\}}$. Similarly, put $U_{[\mathbf{y}]'}:=q''(V_{y_m}\times\{\mathbf{y}\})$. Then $U_{[\mathbf{y}]'}$ is an open neighborhood of $[\mathbf{y}]'$. Note that now $y_m=x_n$. Suppose that $U_{[\mathbf{x}]'}\cap U_{[\mathbf{y}]'}\neq\emptyset$ and let $[\mathbf{z}]'\in U_{[\mathbf{x}]'}\cap U_{[\mathbf{y}]'}$ with $\mathbf{z}=(z_r,\dots,v_0)\in\langle U_{\mathcal{C}}\rangle_{X\times\{v_0\}}$. Then, we have $(z_r,x_n,\dots,v_0)\sim'\mathbf{z}\sim'(z_r,y_m,\dots,v_0)$. Thus, we get equivalence relations

$$\begin{aligned}\mathbf{x}&\sim'(x_n,z_r,x_n,\dots,v_0)=(x_n,z_r)\bullet(z_r,x_n,\dots,v_0)\\&\sim'(y_m,z_r)\bullet(z_r,y_m,\dots,v_0)=\mathbf{y}.\end{aligned}$$

This contradicts $[\mathbf{x}]'\neq[\mathbf{y}]'$. Therefore, we have $U_{[\mathbf{x}]'}\cap U_{[\mathbf{y}]'}=\emptyset$. This completes the proof of Lemma 5.8. □

Proof of Theorem 5.3. First, we will show that $(\widehat{\pi}_{\mathcal{C}},\widehat{G}_{\mathcal{C}},\widehat{P}_{\mathcal{C}},[v_0]')$ is an object in $\mathbf{C}_{\mathcal{C}\text{-flat}}(X,v_0,\langle U_{\mathcal{C}}\rangle)$. Note that X is a Hausdorff space and has the weak topology with respect to the compact covering $\mathcal{C}=(C_n)_{n\in\mathbb{N}}$ and $\langle U_{\mathcal{C}}\rangle$ is a closed set in $X^{\sqcup}$. From Lemma 5.7, $\langle U_{\mathcal{C}}\rangle_{X\times\{v_0\}}$ and $\langle U_{\mathcal{C}}\rangle_{v_0}$ have the weak topology with respect to countable compact coverings respectively. From Lemma 5.8, $\widehat{E}_{\mathcal{C}}$ is a Hausdorff space, and consequently, so is $\widehat{G}_{\mathcal{C}}$. Thus, from Lemma 5.6, $id_{\langle U_{\mathcal{C}}\rangle}\times q''$, q''', $q''\times q'''$, and $q'''\times q'''$ are all identifications. Using Lemma 2.1, we can see that $(q''\times q'')\lceil_{\widehat{E}_{\mathcal{C}}\times_X\widehat{E}_{\mathcal{C}}}$ and $(id_{\langle U_{\mathcal{C}}\rangle}\times q'')\lceil_{\langle U_{\mathcal{C}}\rangle\times_X\widehat{E}_{\mathcal{C}}}$ are identifications. Therefore, ν, $\cdot^{-1}$, μ, $\widehat{T}_{\mathcal{C}}$, and $\widehat{P}_{\mathcal{C}}$ are all continuous. We can see that $\widehat{\pi}_{\mathcal{C}}/^{\widehat{G}_{\mathcal{C}}}$ is a homeomorphism. Thus, $\widehat{\pi}_{\mathcal{C}}$ is a principal $\widehat{G}_{\mathcal{C}}$-bundle. By the definition, $\widehat{P}_{\mathcal{C}}$ is a $\mathcal{C}$-flat $\widehat{G}_{\mathcal{C}}$-compatible parallel displacement. Moreover, from Lemma 5.4, $\widehat{\pi}_{\mathcal{C}}$ is locally trivial.

Next, we will show that the quadruple $(\widehat{\pi}_{\mathcal{C}},\widehat{G}_{\mathcal{C}},\widehat{P}_{\mathcal{C}},[v_0]')$ is an initial object in $\mathbf{C}_{\mathcal{C}\text{-flat}}(X,v_0,\langle U_{\mathcal{C}}\rangle)$. Note that $(\widehat{E}_{\mathcal{C}})^{[v_0]'}=\widehat{E}_{\mathcal{C}}$ and $\Phi^{[v_0]'}(\widehat{P}_{\mathcal{C}})=\widehat{G}_{\mathcal{C}}$, and the topology of $\widehat{E}_{\mathcal{C}}$ (resp. $\widehat{G}_{\mathcal{C}}$) is the identification topology induced from $\widehat{P}_{\mathcal{C}}(\cdot,[v_0]')=q''$ (resp. $(\widehat{P}_{\mathcal{C}})^{[v_0]'}=q'''$). Thus, $(\widehat{\pi}_{\mathcal{C}},\widehat{G}_{\mathcal{C}},\widehat{P}_{\mathcal{C}},[v_0]')$ is the strong holonomy reduction of itself, that is,

$$((\widehat{\pi}_{\mathcal{C}})^{[v_0]'},\Phi^{[v_0]'},(\widehat{P}_{\mathcal{C}})\lceil_{(\widehat{E}_{\mathcal{C}})^{[v_0]'}},[v_0]')=(\widehat{\pi}_{\mathcal{C}},\widehat{G}_{\mathcal{C}},\widehat{P}_{\mathcal{C}},[v_0]')\in\mathbf{C}_{\mathcal{C}\text{-flat}}(X,v_0,\langle U_{\mathcal{C}}\rangle)_0.$$

Let $(\pi,G,P,u)\in\mathbf{C}_{\mathcal{C}\text{-flat}}(X,v_0,\langle U_{\mathcal{C}}\rangle)_0$. Once we have $\mathrm{Ker}(\widehat{P}_{\mathcal{C}})^{[v_0]'}\subset\mathrm{Ker}P^u$, from Lemma 5.5, there exists a unique morphism $(h^{u[v_0]'},\rho^{u[v_0]'}):(\widehat{\pi}_{\mathcal{C}},\widehat{G}_{\mathcal{C}},\widehat{P}_{\mathcal{C}},[v_0]')\to(\pi,G,P,u)$, and we obtain the conclusion. Thus, we will show $\mathrm{Ker}(\widehat{P}_{\mathcal{C}})^{[v_0]'}\subset$

KerP^u. Let $(v_0, x_{n-1}, \dots, x_1, v_0) \in \mathrm{Ker}(\widehat{P}_{\mathcal{C}})^{[v_0]'}$. Then, $[v_0, x_{n-1}, \dots, x_1, v_0]' = [v_0]'$, that is, $(v_0, x_{n-1}, \dots, x_1, v_0) \sim' v_0$. By the definition of $\sim'$, there exists $m \in \mathbb{N} \cup \{0\}$ and a sequence $(v_0, x_{n-1}, \dots, x_1, v_0) = \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m = v_0$ such that for any $j \in \{1, \dots, m\}$ at least one of the following conditions hold:

(1) there exist $k \in \mathbb{N}$ and $i \in \{1, \dots, k\}$ such that $\mathbf{x}_{j-1} = (y_k, \dots, y_i, \dots, y_0)$, $\mathbf{x}_j = (y_k, \dots, \hat{y}_i, \dots, y_0)$ and $y_i = y_{i-1},\ y_{i+1} = y_{i-1}$, or there exists $C \in \mathcal{C}$ such that $y_{i+1}, y_i, y_{i-1} \in C$.

(2) there exist $k \in \mathbb{N}$ and $i \in \{1, \dots, k\}$ such that $\mathbf{x}_{j-1} = (y_k, \dots, \hat{y}_i, \dots, y_0)$, $\mathbf{x}_j = (y_k, \dots, y_i, \dots, y_0)$ and $y_i = y_{i-1},\ y_{i+1} = y_{i-1}$, or there exists $C \in \mathcal{C}$ such that $y_{i+1}, y_i, y_{i-1} \in C$.

In case (1), whether $y_i = y_{i-1},\ y_{i+1} = y_{i-1}$, or there exists $C \in \mathcal{C}$ such that $y_{i+1}, y_i, y_{i-1} \in C$, we have

$$\begin{aligned} P_{\mathbf{x}_{j-1}} &= P_{(y_k,\dots,y_0)} = P_{(y_k,y_{k-1})} \circ \cdots \circ P_{(y_{i+1},y_i)} \circ P_{(y_i,y_{i-1})} \circ \cdots \circ P_{(y_1,y_0)} \\ &= P_{(y_k,y_{k-1})} \circ \cdots \circ P_{(y_{i+1},y_{i-1})} \circ \cdots \circ P_{(y_1,y_0)} = P_{(y_k,\dots,\hat{y}_i,\dots,y_1,y_0)} = P_{\mathbf{x}_j}. \end{aligned}$$

In case (2), similar to the case (1), we get $P_{\mathbf{x}_{j-1}} = P_{\mathbf{x}_j}$. Then, we have

$$P_{\mathbf{x}_0} = P_{\mathbf{x}_1} = \cdots = P_{v_0}.$$

Therefore, $(v_0, x_{n-1}, \dots, x_1, v_0) \in \mathrm{Ker}P^u$. This completes the proof of Theorem 5.3. □

From this theorem and Proposition 3.5, we have the following corollary:

Corollary 5.9. *Let X be a polyhedron of countable connected simplicial complex Kin the weak topology. Any locally trivial G-bundle π over X is, if the topology of G is discrete, associated with $\widehat{\pi}_{\mathcal{C}_K}$, where $\mathcal{C}_K := \{|\tau| \mid \tau \in K\}$.*

Proof. From Proposition 3.5, there exists $\omega_K \in SF_{\mathcal{C}_K\text{-flat}}(\pi, U_K)$. Then, for any $u \in E$, $(\pi, G, P^{\omega_K}, u) \in \mathbf{C}_{\mathcal{C}_K\text{-flat}}(X, v_0, \langle U_K \rangle)_0$. Thus, there exists a unique morphism $(h^{u[v_0]'}, \rho^{u[v_0]'}) : (\widehat{\pi}_{\mathcal{C}_K}, \widehat{G}_{\mathcal{C}_K}, \widehat{P}_{\mathcal{C}_K}, [v_0]') \to (\pi, G, P^{\omega_K}, u)$, and π is associated with $\widehat{\pi}_{\mathcal{C}_K}$ by $\rho^{u[v_0]'}$. □

We will show that if the base space is a polyhedron of a countable connected simplicial complex K in the weak topology, then $\widehat{\pi}_K := \widehat{\pi}_{\mathcal{C}_K}$ is a universal covering space.

Put $\widehat{\langle U_K \rangle}_K := \widehat{\langle U_{\mathcal{C}_K} \rangle}_{\mathcal{C}_K}$, $\widehat{E}_K := \widehat{E}_{\mathcal{C}_K}$, and $\widehat{G}_K := \widehat{G}_{\mathcal{C}_K}$. A map $\overline{q}' : \tilde{\mathfrak{S}}_K \to \widehat{\langle U_K \rangle}_K$ is given by $\overline{q}'([x_n, \dots, x_0]) := [x_n, \dots, x_0]'$ for $[x_n, \dots, x_0] \in \tilde{\mathfrak{S}}_K$. Put $h_K := {}_{\widehat{E}_K}|\overline{q}'|_{\tilde{E}_K}$ and $\rho_K := {}_{\widehat{G}_K}|h|_{\tilde{G}_K}$.

Proposition 5.10. *Let X be a polyhedron of a countable connected simplicial complex K in the weak topology. Then, $\widehat{\pi}_K$ is a universal covering space associated with Milnor's universal bundle $\tilde{\pi}_K$ (see Example 3.4).*

Proof. Since q is an identification and q' is continuous, $\overline{q}'$ is continuous. Since q is continuous and q' is an identification, $\overline{q}'$ is an identification. Since $\overline{q}'$ is an identification and $\tilde{E}_K$ is a closed set in $\tilde{\mathfrak{S}}_K$ such that $(\overline{q}')^{-1}(\overline{q}'(\tilde{E}_K)) = \tilde{E}_K$, from Lemma 2.1, h_K is also an identification, and by a similar argument, so is ρ_K.

We can see that a unary operation $\cdot^{-}$ on $\widehat{\langle U_K\rangle_K}$ is the one induced by one on $\tilde{\mathfrak{S}}_K$. Similarly, a partial binary operation on $\widehat{\langle U_K\rangle_K}$ is the one induced by one on $\tilde{\mathfrak{S}}_K$. Namely, these two operations are compatible with $\overline{q}'$. Thus, ρ_K is a group homomorphism and $(h_K, \rho_K) : (\tilde{\pi}_K, \tilde{G}_K) \to (\widehat{\pi}_K, \widehat{G}_K)$ is a homomorphism. Therefore, $\widehat{\pi}_K$ is associated with $\tilde{\pi}_K$.

Next, we will show that $\widehat{\pi}_K$ is a universal covering space. By an elementary argument of homotopy theory, we can see that $\widehat{G}_K$ is isomorphic to $\pi_1(X, v_0)$ as a topological group (see Lemma 5.13). Since $\tilde{E}_K$ is connected and $h_K : \tilde{E}_K \to \widehat{E}_K$ is a surjective continuous map, so is $\widehat{E}_K$. Since X is locally pathwise connected and $\widehat{\pi}_K$ is a homeomorphism locally, $\widehat{E}_K$ is locally pathwise connected. Thus, $\widehat{\pi}_K$ is a covering space. Note that for any covering space $\pi : E \to X$, if it is a principal $\pi_1(X, x)$-bundle, the total space E is simply connected (see Lemma 5.14). Therefore, $\widehat{E}_K$ is simply connected and we have the conclusion. □

In the smooth category, the restricted holonomy group is the subgroup of the holonomy group consisting of parallel displacements arising from all loops which are homotopic to zero. As an analogue of this definition, we can think of $\mathrm{Ker}\rho_K$ as the restricted holonomy group of $\tilde{P}_K$.

Proposition 5.11. *The kernel* $\mathrm{Ker}\rho_K$ *is an open set in* $\tilde{G}_K = \Phi^{[v_0]}$*, and the induced map* $\overline{\rho}_K : \tilde{G}_K/\mathrm{Ker}\rho_K \to \widehat{G}_K \cong \pi_1(X, v_0)$ *is a homeomorphism.*

Proof. The equation $q^{-1}(\mathrm{Ker}\rho_K) = q'^{-1}(\{[v_0]'\})$ implies that $\mathrm{Ker}\rho_K$ is an open set in $\tilde{G}_K$. Since $\widehat{G}_K$ is a discrete group, ρ_K is an open map, and consequently, induced map $\overline{\rho}_K$ is a homeomorphism. □

Remark 5.12. Since $(\widehat{\pi}_{\mathcal{C}_K}, \widehat{G}_{\mathcal{C}_K}, \widehat{P}_{\mathcal{C}_K}, [v_0]')$ is also an object in the category $\mathbf{C}(X, v_0, \langle U_K\rangle)$ of principal bundles with parallel displacements introduced in [5, Section 8], and $(\tilde{\pi}_K, \tilde{G}_K, \tilde{P}_K, [v_0])$ is an initial object in $\mathbf{C}(X, v_0, \langle U_K\rangle)$, there exists a unique morphism from $(\tilde{\pi}_K, \tilde{G}_K, \tilde{P}_K, [v_0])$ to $(\widehat{\pi}_{\mathcal{C}_K}, \widehat{G}_{\mathcal{C}_K}, \widehat{P}_{\mathcal{C}_K}, [v_0]')$. For $(\mathbf{x}, [\mathbf{y}]) \in (p_0|_{U_K})^*\tilde{E}_K$, we have

$$\begin{aligned} h_K(\tilde{P}_K(\mathbf{x}, [\mathbf{y}])) &= h_K([\mathbf{x}], [\mathbf{y}]) = h_K([\mathbf{x} \bullet \mathbf{y}]) \\ &= [\mathbf{x} \bullet \mathbf{y}]' = [\mathbf{x}]'[\mathbf{y}]' = \widehat{P}_K(\mathbf{x}, h_K([\mathbf{y}])). \end{aligned}$$

Then, h_K preserves $\tilde{P}_K$ and $\widehat{P}_K$. Thus, $(h_K, \rho_K) : (\tilde{\pi}_K, \tilde{G}_K, \tilde{P}_K, [v_0]) \to (\widehat{\pi}_{\mathcal{C}_K}, \widehat{G}_{\mathcal{C}_K}, \widehat{P}_{\mathcal{C}_K}, [v_0]')$ is the unique morphism. Moreover, for any $(\pi, G, P, u) \in \mathbf{C}_{\mathcal{C}_K\text{-flat}}(X, v_0, \langle U_K\rangle)_0$, the unique morphism $(h^{u[v_0]}, \rho^{u[v_0]}) : (\tilde{\pi}_K, \tilde{G}_K, \tilde{P}_K, [v_0]) \to (\pi, G, P, u)$ is written as a composition of two morphisms as $(h^{u[v_0]}, \rho^{u[v_0]}) = (h^{u[v_0]'}, \rho^{u[v_0]'}) \circ (h_K, \rho_K)$.

Appendix

We provide two Lemmas 5.13 and 5.14. Proofs are rather elementary, yet we supply them for the sake of completeness.

Lemma 5.13. $\widehat{G}_K$ *is isomorphic to* $\pi(X, v_0)$

Proof. Put

$$\Pi(X) := \bigcup_{(x,y)\in X\times X} M(x,y),$$

where $M(x,y)$ is the set of all homotopic classes of curves joining y to x. Then, $\Pi(X)$ is a fundamental groupoid. A map $\langle U_K\rangle \to \Pi(X)$ is defined as follows. Let $(x_n,\dots,x_0)\in\langle U_K\rangle$ and $c_{(x_n,\dots,x_0)} : I_{(n)} \to X$ be a map given by $c_{(x_n,\dots,x_0)}(\frac{i}{n}) = x_i$ for $i\in\{0,1,\dots,n\}$, where $I_{(n)}$ is a division of $I=[0,1]$ into n equal intervals. Then, assign to $(x_n,\dots,x_0)$ a homotopy class $\left[\overline{c}_{(x_n,\dots,x_0)}\right]$ of induced map $\overline{c}_{(x_n,\dots,x_0)} : I \to X$ of $c_{(x_n,\dots,x_0)}$. We can see that if $(x_n,x_{n-1},\dots,x_1,x_0) \sim' (x_n,y_{m-1},\dots,y_1,x_0)$, $\overline{c}_{(x_n,x_{n-1},\dots,x_1,x_0)}$ and $\overline{c}_{(x_n,y_{m-1},\dots,y_1,x_0)}$ are homotopic. Therefore, a map $\eta : \widehat{\langle U_K\rangle}_K \to \Pi(X)$ is well defined. For $[\mathbf{x}]',[\mathbf{y}]' \in \widehat{\langle U_K\rangle}_K$, we can see $\eta([\mathbf{x}]'[\mathbf{y}]') = \eta([\mathbf{x}]')\eta([\mathbf{y}]')$ whenever $[\mathbf{x}]'[\mathbf{y}]'$ is defined. Thus, η is a pseudo group homomorphism. We will show that $\eta : \widehat{\langle U_K\rangle}_K \to \Pi(X)$ is bijective.

First, we will show that η is surjective. Let $[\gamma]\in\Pi(X)$ with a continuous map $\gamma : I \to X$. From simplicial approximation theorem, there exists a division $I_{(n)}$ of I and a simplicial map $\varphi : I_{(n)} \to K$ such that for each $i\in\{0,\dots,n\}$, $\gamma(V_{I_{(n)}}(\frac{i}{n})) \subset V_K(\varphi(\frac{i}{n}))$. Put

$$\mathbf{x} := (\gamma(0),\varphi(0),\varphi(\tfrac{1}{n}),\dots,\varphi(\tfrac{n-1}{n}),\varphi(1),\gamma(1)) \in \langle U_K\rangle.$$

Then, we have $\eta([\mathbf{x}]') = [\overline{c}_{\mathbf{x}}] = [\gamma]$.

Next, we will show that η is injective. Let $\mathbf{x} = (x_0,x_{n-1},\dots,x_1,x_0)\in\langle U_K\rangle$ with $[\overline{c}_{\mathbf{x}}] = [e_{x_0}]$. Then, there exists a homotopy $F : I\times I \to X$ such that $F(t,0) = \overline{c}_{\mathbf{x}}(t)$ for any $t\in I$ and $F((\{0,1\}\times I)\cup(I\times\{1\})) = \{x_0\}$. From simplicial approximation theorem, there exists a division $(I\times I)_{(nk,m)}$ of $I\times I$ and a simplicial map $G : (I\times I)_{(nk,m)} \to K$ such that for each $(i,j)\in\{0,\dots,nk\}\times\{0,\dots,m\}$, $F(V_{(I\times I)_{(nk,m)}}(\frac{i}{nk},\frac{j}{m})) \subset V_K(G(\frac{i}{nk},\frac{j}{m}))$. Put

$$\mathbf{x}^{(-1)} := (x_0,\overline{c}_{\mathbf{x}}(1),\overline{c}_{\mathbf{x}}(\tfrac{nk-1}{nk}),\dots,\overline{c}_{\mathbf{x}}(\tfrac{2}{nk}),\overline{c}_{\mathbf{x}}(\tfrac{1}{nk}),\overline{c}_{\mathbf{x}}(0),x_0).$$

Then, we get $\mathbf{x} \sim' \mathbf{x}^{(-1)}$. Put

$$\mathbf{x}^{(0)} := (x_0,G(1,0),G(\tfrac{nk-1}{nk},0),\dots,G(\tfrac{2}{nk},0),G(\tfrac{1}{nk},0),G(0,0),x_0).$$

Since $\overline{c}_{\mathbf{x}}(\frac{i}{nk}) = F(\frac{i}{nk},0) \in V_K(G(\frac{i}{nk},0))$ for $i\in\{0,\dots,nk\}$ and $\mathbf{x}^{(0)}\in\langle U_K\rangle$, from Lemma 4.13, we have $\mathbf{x}^{(-1)} \sim' \mathbf{x}^{(0)}$. Put

$$\mathbf{x}^{(1)} := (x_0,G(1,0),G(\tfrac{nk-1}{nk},0),\dots,G(\tfrac{2}{nk},0),G(\tfrac{1}{nk},0),G(0,\tfrac{1}{m}),x_0).$$

Since G is a simplicial map, there exists $\tau\in K$ such that $G(0,0),G(0,\frac{1}{m}),x_0\in|\tau|$. Then, we get

$$\mathbf{x}^{(0)} = (x_0,\dots,G(\tfrac{1}{nk},0),G(0,0),x_0) \sim' (x_0,\dots,G(\tfrac{1}{nk},0),G(0,0),G(0,\tfrac{1}{m}),x_0).$$

Similarly, since there exists $\tau'\in K$ such that $G(\frac{1}{nk},0),G(0,0),G(0,\frac{1}{m})\in|\tau'|$, we have

$$(x_0,\dots,G(\tfrac{1}{nk},0),G(0,0),G(0,\tfrac{1}{m}),x_0) \sim' \mathbf{x}^{(1)},$$

that is, $\mathbf{x}^{(0)} \sim' \mathbf{x}^{(1)}$. Put

$$\mathbf{x}^{(2)} := (x_0, G(1,0), G(\tfrac{nk-1}{nk}, 0), \ldots, G(\tfrac{2}{nk}, 0), G(\tfrac{1}{nk}, \tfrac{1}{m}), G(0, \tfrac{1}{m}), x_0).$$

Then, by similar argument, we get $\mathbf{x}^{(1)} \sim' \mathbf{x}^{(2)}$. Repeating the same operation $(nk+1)m$ times, we have

$$\mathbf{x}^{((nk+1)m)} := (x_0, G(1,1), G(\tfrac{nk-1}{nk}, 1), \ldots, G(\tfrac{2}{nk}, 1), G(\tfrac{1}{nk}, 1), G(0,1), x_0)$$

and $\mathbf{x}^{((nk+1)m)} \sim' (\underbrace{x_0, \ldots, x_0}_{nk+3})$. Thus, we get $\mathbf{x} \sim' (x_0)$ and $\eta : \widehat{\langle U_K \rangle}_K \to \Pi(X)$ is bijective. Therefore, it follows that restrictions $\eta|_{\widehat{E}_K} : \widehat{E}_K \to \bigcup_{x \in X} M(x, v_0)$ and $\eta|_{\widehat{G}_K} : \widehat{G}_K \to \pi_1(X, v_0)$ are bijective. Especially, if $\pi_1(X, v_0)$ is a discrete group, $\eta|_{\widehat{G}_K}$ is a topological group isomorphism. $\square$

Lemma 5.14. *Let $\pi : E \to X$ be a covering space. If it is a principal $\pi_1(X, x_0)-$bundle ($x_0 \in X$), then E is simply connected.*

Proof. Since $\pi_1(X, x_0)$ acts $\pi_1(X, x_0)$ freely on the left, for any $[\gamma] \in \pi_1(X, x_0)$, the isotropy subgroup $\pi_1(X, x_0)_{[\gamma]}$ is equal to $\{1\}$. On the other hand, by general argument, isotropy subgroup $\pi_1(X, x_0)_{[\gamma]}$ of this action is equal to $\pi_*(\pi_1(E, \tilde{x}_0))$, where $\tilde{x}_0 \in E_{x_0}$. Since π_* is injection, we have $\pi_1(E, \tilde{x}_0) = \{1\}$. $\square$

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