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NON-LOCALLY-FINITE PARAHYPERBOLIC NETWORKS

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Abstract. In the classification theory of Riemann surfaces the research for positive or bounded solutions of $\Delta u = Pu$, where $P \geq 0$ is a C^1-function, has played an important role in establishing the similarities between the solutions of this differential equation and the classical harmonic functions. In the discrete potential theory, the Schrödinger operators are to some extent like the equation $\Delta u = Pu$. In this note, we develop on an infinite graph, a theory of functions to reflect the properties of the above solutions, without the use of derivatives. This can be used to study discrete Schrödinger and Helmholtz equations in non-locally-finite networks.

1. Introduction

In the context of the classification theory of Riemann surfaces, many interesting researches (Ozawa [6]) have been carried around the partial differential equation of elliptic type $\Delta u = Pu$ on a Riemann surface R where $P \geq 0$ is a C^1-function. Myrberg [4],[5] showed that there always exists the Green function of $\Delta u = Pu$ on R (when P is not the zero function); Ozawa [7], [8] studied different classes of positive solutions of $\Delta u = Pu$ on R; Royden [9] studied the bounded solutions of $\Delta u = Pu$ on R and compared bounded solutions of $\Delta u = Pu$ and $\Delta u = Qu$ when $Q \geq P$.

Even though many of their results were derived by using partial derivatives on R, yet it is possible to prove them by methods in the Brelot axiomatic potential theory [3] without reference to derivatives of functions. In this note, we illustrate the procedure in the context of discrete potential theory which includes also a discrete version of the Helmholtz equation.

2. Preliminaries

Let T be an infinite tree with $\{p(x,y)\}$ as a set of nearest transition probabilities. For a real-valued function u on T, define the Laplacian at a vertex x as $\Delta u(x) =$

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$\sum_y p(x,y)[u(y)-u(x)]$. Along with the Laplacian operator, on occasions we consider the Schrödinger operators $\Delta u(x) = q(x)u(x)$ also where $q(x) \geq 0$ on T and q is not the zero function. These operators are similar to the equation $\Delta u = Pu$ on a Riemann surface.

Now writing $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$, we note that a solution u of $\Delta_q u = 0$ in T is of the form $u(x) = \sum_y \frac{p(x,y)}{1+q(x)} u(y)$. Generalising this situation, we can fix a set $\{p'(x,y)\}$ of weights on the edges of T such that $0 \leq p'(x,y) \leq p(x,y)$ for any pair x, y in T; $p'(x,y) > 0$ if and only if x and y are joined by an edge; and $p(x,y) > p'(x,y)$ for at least one pair of vertices x, y; and then consider the properties of functions u on T which satisfy the condition $u(x) = \sum_y p'(x,y)u(y)$, similar to the solutions of the Schrödinger operators.

These preliminary remarks are intended to indicate the direction in which the following study of functions on an infinite graph will be carried out.

3. Potentials on non-locally-finite networks

Let X be an infinite graph with a countably infinite number of vertices and a countably infinite number of edges. If x and y are two vertices joined by an edge $[x,y]$, we write $x \sim y$ and say that x and y are neighbours. We assume that X is connected, that is, given any pair of vertices x, y, there exists a finite path $\{x = x_0, x_1, \ldots, x_n = y\}$ connecting them. With any pair of vertices x and y is associated a non-negative number $c(x,y)$ such that $c(x,y) > 0$ if and only if $x \sim y$ and $c(x) = \sum_{y \sim x} c(x,y) \leq 1$ for any $x \in X$, with the restriction that $c(x_0) < 1$ for at least one $x_0 \in X$.

Let E be a subset of X. A vertex $x \in E$ is said to be an interior vertex of E if all the neighbours of x are also in E. We denote by $\mathring{E}$ the set of all interior vertices of E and $\partial E = E \setminus \mathring{E}$. Note that we do not place the condition that the number of neighbours of any vertex is finite, but we consider in this paper only real-valued functions u such that if u is defined on E, then u satisfies the condition $\sum_y c(x,y)|u(y)| < \infty$ for any $x \in \mathring{E}$. Write $Au(x) = \sum_y c(x,y)u(y)$. A function u defined on E is said to be $\{X, c\}-$superharmonic or simply $c-$superharmonic (respectively, $c-$subharmonic or $c-$harmonic) on E if $u(x) \geq Au(x)$ (respectively $u(x) \leq Au(x)$ or $u(x) = Au(x)$) for every $x \in \mathring{E}$. If $p \geq 0$ is a $c-$superharmonic function such that any $c-$subharmonic function majorized by p is non-positive, then p is called a $c-$potential.

Some properties of $c-$superharmonic functions:

We have not assumed the local finiteness of X (X is locally finite if each vertex of X has only a finite number of vertices). Hence all the properties of $\Delta-$superharmonic functions given in [1], using the Laplacian operator Δ defined in Section 2, may not be valid in the present situation. For example, the limit of a sequence of $c-$superharmonic functions may not be $c-$superharmonic. The properties of $c-$superharmonic functions needed in this paper are listed below. They can be proved with slight modifications from the case of a locally finite network.

(3.1) If u_n is an increasing sequence of non-negative $c-$superharmonic (respectively, $c-$harmonic) functions on a subset E of X and if $u = \sup u_n$ is finite at each vertex, then u is $c-$superharmonic (respectively, $c-$harmonic) on E. For, $u_n(x) \geq \sum_y c(x,y)u_n(y)$, if $x \in \mathring{E}$. Write the neighbours of x as $y_1, y_2, \ldots$ and let $a_{mn}(x) = \sum_{i=1}^{m} c(x,y_i)u_n(y_i)$. Then $a_{mn}(x)$ is increasing in m and $u_n(x) \geq \lim_m a_{mn}(x)$. Note that for each m fixed, $a_{mn}(x)$ is an increasing sequence in n. Consequently,

$$\begin{aligned} u(x) = \lim_n u_n(x) &\geq \lim_n \lim_m a_{mn}(x) \\ &= \lim_m \lim_n a_{mn}(x) \\ &= \lim_m \sum_{i=1}^{m} c(x,y_i)u(y_i) \\ &= \sum_y c(x,y)u(y). \end{aligned}$$

The analogous property for the limit of non-negative $c-$harmonic functions is proved similarly.

(3.2) Let $\{u_i\}$ be a family of $c-$superharmonic functions on a subset E of X. Suppose $u_i \geq v$ on E for each i, where v is a function on E such that $\sum_y c(x,y)|v(y)| < \infty$ for each $x \in \mathring{E}$. Let $u(x) = \inf_i u_i(x)$. Then $u(x)$ is $c-$superharmonic on E.

Since $|u(x)| \leq |u_i(x)| + |v(x)|$ for any $x \in E$, and all u_i $\sum_y c(x,y)u(y)$ is well-defined for each x on $\mathring{E}$.

If $x \in \mathring{E}$, for every i,

$$\begin{aligned} u_i(x) &\geq \sum_y c(x,y)u_i(y) \\ &\geq \sum_y c(x,y)u(y), \end{aligned}$$

and hence $u(x) = \inf u_i(x) \geq \sum_y c(x,y)u(y)$.

A sequence $f_n(x)$ in X is said to be *locally uniformly convergent* to $f(x)$ if for any $\epsilon > 0$ and any x in X, there exists an integer m (depending on ϵ and x) such that if $n \geq m$, then $|f_n(y) - f(y)| < \epsilon$ for all $y \in V(x)$, where $V(x)$ is the neighbourhood set of x consisting of x and all $y \sim x$.

(3.3) **Proposition.** Let $u_n(x)$ be a sequence of harmonic functions on X converging locally uniformly to a finite function $u(x)$. Then u is harmonic on X.

Proof. For any x, if $y \sim x$, then

$$\begin{aligned} |u(y)| &\leq |u_n(y)| + |u_n(y) - u(y)| \\ &\leq |u_n(y)| + \epsilon \text{ when } n \geq m. \end{aligned}$$

Hence $\sum_y c(x,y)|u(y)| \leq \sum_y c(x,y)|u_n(y)| + \epsilon \sum_y c(x,y) < \infty$.

Now, $u_n(x)$ being harmonic on X, for any x, $u_n(x) = \sum_y c(x,y)u_n(y)$. Also,

$$\begin{aligned} \left| \sum_y c(x,y)u_n(y) - \sum_y c(x,y)u(y) \right| &\leq \sum_y c(x,y)\,|u_n(y) - u(y)| \\ &\leq \epsilon\; c(x) \text{ when } n \geq m. \end{aligned}$$

Consequently $u(x) = \lim_n u_n(x) = \lim_n \sum_y c(x,y)u_n(y) = \sum_y c(x,y)u(y)$. Thus, it is proved that u is harmonic on X. □

(3.4) **Domination Principle** Let p be a $c-$potential with $c-$harmonic support E (that is, $p(x)$ is $c-$harmonic at every vertex $x \notin E$). Suppose $s > 0$ is a $c-$superharmonic function on X such that $s \geq p$ on E. Then $s \geq p$ on X.

For, let $q = \inf(s,p)$ and write $u = p - q$. Then $u \geq 0$ on X and $u = 0$ on E. Hence $u(x)$ is $c-$subharmonic at each vertex $x \in E$; and if $x_0 \notin E$, then $p(x)$ is $c-$harmonic at x_0 and $q(x)$ is $c-$superharmonic at x_0 so that $u(x)$ is $c-$subhharmonic at x_0. In other words, $u(x)$ is $c-$subharmonic in X, also $u(x)$ is majorised by the $c-$potential $p(x)$. Hence $u \leq 0$, so that $u = 0$ on X. Now $p = q$ on X implies that $s \geq p$ on X.

(3.5) **Minimum Principle** Let u be a $c-$superharmonic function defined on a finite set E. If $u \geq 0$ on ∂E, then $u \geq 0$ on E.

For, suppose u takes negative values on E. Let $-m = \inf_{x \in E} u(x), m > 0$ Then there exists a vertex $x_0 \in E$ such that $u(x_0) = -m$. Since $u \geq 0$ on ∂E, x_0 should be in $\mathring{E}$. Let z be a vertex in $X \setminus E$. Since X is connected, there is a path $\{x_0, x_1, \ldots, x_n = z\}$ connecting x_0 and z. Let i be the smallest index such that $x_i \in \mathring{E}$ and $x_{i+1} \notin \mathring{E}$. In this case since $x_i \in \mathring{E}$ and $x_{i+1} \sim x_i$ we should have $x_{i+1} \in \partial E$.

Now $-m = u(x_0) \geq \sum_{y \sim x_0} c(x_0,y)u(y) \geq c(x_0)(-m)$ which gives $c(x_0) \geq 1$. But by assumption on $c(x,y), c(x) \leq 1$ for all x. Hence $c(x_0) = 1$, and $\sum_{y \sim x_0} c(x_0,y)[u(y) + m] = 0$. Since each term in this sum is non-negative we conclude $u(y) = -m$ if $y \sim x_0$. In particular $u(x_1) = -m$. Proceeding similarly we obtain $u(x_0) = u(x_1) = \cdots = u(x_i) = u(x_{i+1}) = -m$; but $x_{i+1} \in \partial E$ so that $u(x_{i+1}) \geq 0$. This contradiction shows that u does not take negative values on E, that is $u \geq 0$ on E.

Uniqueness: From the above Minimum Principle, it follows that if h is a $c-$harmonic function on a finite set E such that $h = 0$ on ∂E, then $h = 0$ on E.

Riesz Decomposition Let $s > 0$ be a $c-$superharmonic function on X. Take the family $\Im$ of all $c-$subharmonic functions u such that $u \leq s$. Then $\Im$ is an upper directed family of $c-$subharmonic functions. Hence as in (3.2), $h(x) = \sup_{u \in \Im} u(x)$ is $c-$subharmonic in X. In fact, $h(x)$ is $c-$harmonic in X. For, consider for an arbitrary vertex z in X.

$$v(x) = \begin{cases} h(x), & \text{if } x \neq z; \\ \sum_{y \sim z} c(z,y)h(y), & \text{if } x = z. \end{cases}$$

Then $v(z) = \sum_{y \sim z} c(z,y)h(y) \geq h(z)$ so that $v \geq h$ on X. It is easy to see then $v(x)$ is $c-$subharmonic in X and also $v \leq s$ on X, so that $v \in \Im$. Hence by the definition of h, $v \leq h$ on X. We conclude therefore $v = h$ on X. In particular $h(x)$ is $c-$harmonic at $x = z$. Since z is arbitrary in X, $h(x) \leq s(x)$ for all x in X.

Clearly, if $h_1(x) \geq 0$ is a $c-$harmonic function such that $h_1(x) \leq s(x)$, then $h_1(x) \leq h(x)$ so we call $h(x)$ as the greatest $c-$harmonic minorant of $s(x)$ and $p(x) = s(x) - h(x)$ as a $c-$potential. Further the decomposition of $s(x) = p(x) + h(x)$ as sum of a $c-$potential and a non-negative $c-$harmonic function is unique. We shall agree to say that a $c-$superharmonic function $p(x) \geq 0$ is a $c-$potential if and only if for any non-negative $c-$subharmonic function $u(x) \leq p(x)$, we have $u(x) = 0$ for any x in X. Consequently, if $p(x)$ is a $c-$potential and $u(x)$ is any $c-$subharmonic function such that $u(x) \leq p(x)$ then $u(x) \leq 0$. For, $\sup(u, 0) = u^+ \leq p$ and since u^+ is $c-$subharmonic, $u^+ = 0$ so that $u \leq 0$.

(3.6) Let E be a subset of X. Suppose u is $c-$superharmonic and v is $c-$subharmonic on E such that $u \geq v$. Let $\Im$ be the family of all $c-$subharmonic functions v_i on E such that $v_i \leq u$. Then $h = \sup_{v_i \in \Im} v_i$ is a $c-$harmonic function on E (named the greatest $c-$harmonic minorant of u) such that $u \geq h \geq v$.

Using the above result, we can obtain the following Dirichlet solution as in [1, Theorem 3.1.7].

(Dirichlet solution) Let F be an arbitrary subset of X, and $E \subset \mathring{F}$. Let f be a real-valued function on $F \setminus E$. Suppose there exist two functions u and v on F such that at each vertex in E, u is $c-$superharmonic and v is $c-$subharmonic; $u \geq f \geq v$ on $F \setminus E$; and $u \geq v$ on F. Then there exists a function h on F such that $h = f$ on $F \setminus E$ and h is $c-$harmonic at each vertex in E.

In particular, if E is a finite set in X and if f is a real-valued function on ∂E, then there exists a unique $c-$harmonic function h on E such that $h = f$

on ∂E. For, since E is a finite set, for some $M > 0, -M \leq f(x) \leq M$ on ∂E. Note that $c(x) \leq 1$ implies that M is $c-$superarmonic. Hence the Dirichlet solution h with boundary value f exists on E. The uniqueness of h follows from the Minimum Principle.

(3.7) **Proposition.** Let E be a finite set in X. Then any $c-$harmonic function on E is a linear combination of n $c-$harmonic functions where n is the cardinality of the set ∂E.

Proof. For $y \in \partial E$, let $P_y(x)$ denote the Dirichlet solution in E with boundary values $\delta_y(x)$ on ∂E. Let now h be a $c-$harmonic function on E. Define $u(x) = \sum_{y \in \partial E} h(y) P_y(x)$. Then $u(x)$ is $c-$harmonic on E such that $u(y) = h(y)$ for every $y \in \partial E$. Hence $u = h$ on E by the Minimum Principle. Consequently, h is a linear combination of $\{P_y(x)\}_{y \in \partial E}$. □

(3.8) Let $f \geq 0$ be a real-valued function on X. Let $\Im$ be the family of all superharmonic functions u on X such that $f \leq u$ on X. If the family $\Im$ is non-empty, then $Rf = \inf_{u \in \Im} u$ is a $c-$superharmonic function on X such that Rf is $c-$harmonic at every vertex x where f is $c-$subharmonic (in particular, if $f(z) = 0$, then Rf is $c-$harmonic at the vertex z). If $\Im$ contains a $c-$potential on X, then Rf is a $c-$potential on X.

A consequence: Since 1 is $c-$superharmonic but not $c-$harmonic on X (by our assumptions that $c(x) = \sum_{y \sim x} c(x, y) \leq 1$ for each $x \in X$ and that there exists some x_0 in X such that $c(x_0) < 1$), then there are always $c-$potentials on X. Take $g(x) = \delta_e(x)$ where e is a vertex in X and δ_e is the Dirac function. Then $Rg(x)$ is $c-$superharmonic (not $c-$harmonic) at the vertex e and $c-$harmonic at every vertex $x \neq e$. That is, $A[Rg(x)] = Rg(x)$ if $x \neq e$ and $A[Rg(x)] < Rg(x)$ if $x = e$. Choose the constant α such that $\alpha(I - A)Rg(x) = \delta_e(x)$. Then $\alpha Rg(x)$ denoted now by $G_e(x)$ is the $c-$Green function on X with point $c-$harmonic singularity at e, that is $G_e(x)$ is a $c-$potential on X, $c-$harmonic at every vertex $x \neq e$ and $(I - A)G_e(x) = \delta_e(x)$ for all $x \in X$.

Proposition 3.1. *Let $s > 0$ be a $c-$superharmonic function in X. Then $s(x) = \sum_{y \in X} [(1 - A)s(y)]\, G_y(x) + h(x)$ where $h(x)$ is a non-negative $c-$harmonic function in X.*

Proof. We have already remarked that s is the unique sum of a $c-$potential p and a non-negative $c-$harmonic function h in X. Hence it remains to show that $p(x)$ has the series expansion given above.

Let $\{E_n\}$ be an increasing sequence of finite sets such that $X = \bigcup_n E_n$. Let $p_n(x) = \sum_{y \in E_n} (1 - A)p(y)G_y(x)$. Then p_n is a $c-$potential on X. Let $s_n(x) = p(x) - p_n(x)$ on X. Since $(1 - A)G_y(x) = \delta_y(x)$, we see that for $y \in E_n$, $(1 - A)p_n(y) = (1 - A)p(y)$ so that $(1 - A)s_n(y) = 0$ if $y \in E_n$; if $x \notin E_n$, $(1 - A)p_n(x) = 0$ while

$(1-A)p(x) \leq 0$ so that $(1-A)s_n(x) \leq 0$. Consequently, $(1-A)s_n(x) \leq 0$ for all $x \in X$. That is $s_n(x)$ is $c-$superharmonic and $p_n + s_n = p \geq 0$ so that $-s_n \leq p_n$ on X. This implies that $-s_n \leq 0$ since $-s_n$ is $c-$subharmonic and p_n is a $c-$potential. Consequently, $p = p_n + s_n \geq p_n$ for any n. Allow $n \to \infty$ to obtain the inequality $p(x) \geq \sum_{y \in X} (1-A)p(y)G_y(x)$.

Denoting the right side infinite sum as $q(x)$, we remark that $q(x)$ is $c-$superharmonic on X since it is the limit of an increasing sequence of $c-$potentials. Since $p(x)$ is a $c-$potential and $q(x) \leq p(x)$, we conclude that $q(x)$ also is a $c-$potential. Let $v(x) = p(x) - q(x)$. Then

$$(1-A)v(x) = (1-A)p(x) - (1-A)q(x) = 0,$$

so that $v(x)$ is $c-$harmonic on X. Thus $p(x) = q(x) + v(x)$. Introduce now the uniqueness of decomposition of a non-negative $c-$superhhamronoic function as the sum of a $c-$potential and a $c-$harmonic function to conclude that $v = 0$. Hence $p(x) = \sum_{y \in X} [(1-A)p(y)]\, G_y(x)$. Finally, we have the expression $s(x) = \sum_{y \in X} [(1-A)s(y)]\, G_y(x)$ for all $x \in X$, since $(1-A)s(y) = (1-A)p(y)$ for any y in X. □

4. Network classification

Since 1 is $c-$superharmonic on X, by the representation given above, 1 is the sum of a $c-$potential p and a non-negative $c-$harmonic function h on X. It is possible that h is 0 on X. There are many differences in the study of $c-$potentials on X, depending on whether $h = 0$ or $h > 0$, that is whether the $c-$superharmonic function 1 is a $c-$potential on X or not. Accordingly we introduce a definition as follows:

Definition 4.1. X is said to be parahyperbolic if and only if the constant 1 is a $c-$potential on X. Otherwise X is said to be bounded hyperbolic.

Note: The term parahyperbolic is chosen to indicate that though X has positive $c-$potentials, X manifests many characteristics of a parabolic Riemannian manifold. The term bounded hyperbolic indicates that there are non-zero bounded $c-$harmonic functions in X.

For a real-valued function $f \geq 0$ on X, we say that $\varliminf_{x \to \infty} f(x) = 0$ if for any given $\epsilon > 0$ and any finite set A, $f(x) < \epsilon$ for some $x \notin A$. $\varliminf_{x \to \infty} f(x) = \alpha > 0$ would mean that $f(x) \geq \frac{\alpha}{2}$ for all x outside a finite set.

Proposition 4.2. *X is bounded hyperbolic if and only if for any $c-$potential $p > 0$ on X, $\varliminf_{x \to \infty} p(x) = 0$.*

Proof. Suppose $\varliminf_{x \to \infty} p(x) = \alpha > 0$ for some $c-$potential p on X. Then $p(x) \geq \frac{\alpha}{2}$ outside a finite set A. This implies that $p(x) \geq \beta$ on X for some $\beta > 0$. Since

p is a $c-$potential and β is $c-$superharmonic on X, then β (and hence 1) is a $c-$potential on X. That is, X is parahyperbolic. Hence if X is bounded hyperbolic, then $\underline{\lim}_{x\to\infty} p(x) = 0$ for any $c-$potential > 0 in X.

On the other hand, since 1 is a $c-$potential in a parahyperbolic network, the condition that $\underline{\lim}_{x\to\infty} p(x) = 0$ for any $c-$potential $p > 0$ on X would mean that X is bounded hyperbolic.

□

Remark 4.3. For some vertex e in X, suppose the $c-$Green function $G_e(x)$ with pole e satisfies the condition $\underline{\lim}_{x\to\infty} G_e(x) > 0$. Then X is parahyperbolic and for any $c-$superharmonic function $s > 0$ on X, $\underline{\lim}_{x\to\infty} s(x) > 0$. For by the Domination Principle, $s(x) \geq \frac{s(e)}{G_e(e)} G_e(x)$ for any $x \in X$. Again, by the Domination Principle, if $\underline{\lim}_{x\to\infty} G_e(x) = 0$, then $\underline{\lim}_{x\to\infty} p(x) = 0$ for any $c-$potential p with finite $c-$harmonic support in X. For, if E is the finite $c-$harmonic support of p, then $p(x) \leq \alpha G_e(x)$ on E for some $\alpha > 0$, consequently $p(x) \leq \alpha G_e(x)$ on X. In particular, if for some e in X, $\underline{\lim}_{x\to\infty} G_e(x) = 0$ (respectively, $\underline{\lim}_{x\to\infty} G_e(x) > 0$) then for any vertex a, $\underline{\lim}_{x\to\infty} G_a(x) = 0$ (respectively, $\underline{\lim}_{x\to\infty} G_a(x) > 0$.)

The following Proposition 4.4, Lemma 4.7 and Theorem 4.8 are proved in the non-locally finite case as was done in the locally finite case [1, Section 4.3].

Proposition 4.4. *Let X be a parahyperbolic network. Let E be an arbitrary subset of X. Suppose s is a lower bounded $c-$superharmonic function on E such that $s \geq 0$ on ∂E. Then $s \geq 0$ on E.*

Proof. Let $u = \inf(s, 0)$ on E, extended by 0 outside E. Then u is a $c-$superharmonic function on X; if $s \geq -m$ on E for some $m > 0$, then $u \geq -m$ on X. Then $-u$ is $c-$subharmonic and $-u \leq m$ on X. Since X is parahyperbolic, m is a $c-$potential on X and hence $-u \leq 0$ on X. This implies that $s \geq 0$ on E. □

Remark 4.5. The above Minimum Principle is valid in a bounded hyperbolic network if E is a finite subset of X, but not necessarily if E is an infinite set. For an example to show that the Minimum Principle is not valid when E is an infinite set in a bounded hyperbolic network, consider a fixed vertex e with a finite number of neighbours and let $V(e)$ be the set of e and all its neighbours. Take $E = X \setminus \{e\}$. Then $V(e) \setminus \{e\}$ is the boundary ∂E of E. Now, since X is bounded hyperbolic, there exists a $c-$harmonic function h, $0 < h < 1$, on X. Let $u = h - R_h^{V(e)}$, where $R_h^{V(e)}(x) = \inf_{s\in\Im} s(x)$ for $x \in X$, $\Im$ being the family of all positive $c-$superharmonic functions on X such that $s(x) \geq h(x)$ on $V(e)$. Note that $R_h^{V(e)} \leq h$ on X, $R_h^{V(e)} = h$ on $V(e)$, $R_h^{V(e)}$ is $c-$harmonic at each vertex in $X \setminus V(e)$ and $R_h^{V(e)}$ is a $c-$potential since $V(e)$ is a finite set. Then u is a bounded $c-$harmonic function on E, such that

$u = 0$ on ∂E. If the Minimum Principle is valid on X, u should be 0 on E. Since $u = 0$ on $V(e)$, then u is identically 0 on X, which would imply that $h = R_h^{V(e)}$ on X. This is not possible, since the left side is a positive $c-$harmonic function on X and the right side is a positive $c-$potential on X.

Remark 4.6. (Uniqueness of the Dirichlet solution in an infinite subset):
From our earlier reference to the Dirichlet solution, it is clear that if E is an arbitrary set in X and if f is a bounded function on ∂E, then there exists a bounded $c-$harmonic function H on E such that $H = f$ on ∂E. It follows from Proposition 4.4 that this function H is uniquely determined if X is parahyperbolic. On the other hand, if X is bounded hyperbolic, then this Dirichlet solution may not be unique if E is an infinite subset. For, as in the example given in Remark 2, take $E = X \setminus \{e\}$. Then $\partial E = V(e) \setminus \{e\}$. If $f = 0$ on ∂E, then the bounded Dirichlet solution on E can be $H = 0$ or $H = h - R_h^{V(e)}$.

Lemma 4.7. *Let $c(x) = \sum_{y\sim x} c(x, y)$ for each x in X. If $G_y(x)$ is the $c-$Green function with vertex $c-$harmonic singularity at y, then there exists a non-negative $c-$harmonic function h on X such that $\sum_{z\in X}[1 - c(z)]G_z(x) + h(x) = 1$ for all x in X.*

Proof. If s is the constant function 1, then for any $z \in X$,

$$(I - A)s(z) = s(z) - \sum_{a\sim z} c(z, a)s(a) = 1 - c(z).$$

Now, writing 1 as the unique sum of a $c-$potential and a non-negative $c-$harmonic function h on X, we have for all x in X,

$$\begin{aligned} 1 = s(x) &= \sum_{z\in X} [(I - A)s(z)]\, G_z(x) + h(x), \text{(Proposition 3.1)} \\ &= \sum_{z\in X} [1 - c(z)]G_z(x) + h(x). \end{aligned}$$

□

Theorem 4.8. *X is parahyperbolic if and only if $\sum_{z\in X}[1 - c(z)]G_z(x) = 1$ for all $x \in X$.*

Proof. By definition, X is parahyperbolic if and only if 1 is a $c-$potential on X. Now from the above lemma, 1 is a $c-$potential if and only if $h = 0$, that is if and only if $\sum_{z\in X}[1 - c(z)]G_z(x) = 1$. □

Theorem 4.9. *Let $\{X, c(x, y)\}$ be an infinite network as above. Let $\{c'(x, y)\}$ be another system of conductance on X such that $0 \le c'(x, y) \le c(x, y)$ for every pair x, y in X and $c'(x, y) > 0$ if and only if $x \sim y$. If $\{X, c\}$ is parahyperbolic, then $\{X, c'\}$ is also parahyperbolic.*

Proof. First note that if $u \geq 0$ is $\{X, c\}-$superharmonic at a vertex x, then u is $\{X, c'\}-$superharmonic at x. For

$$\begin{aligned} u(x) &\geq \sum_{y \sim x} c(x,y)u(y) \\ &\geq \sum_{y \sim x} c'(x,y)u(y). \end{aligned}$$

Now 1 is a $\{X, c\}-$potential by hypothesis. Then 1 is $\{X, c'\}-$superharmonic on X. Let v be any $\{X, c'\}-$subharmonic function such that $0 \leq v \leq 1$ on X. Note that v is $\{X, c\}-$subharmonic also; since 1 is $\{X, c\}-$potential, $v = 0$. This means that 1 is a $\{X, c'\}-$potential. Hence $\{X, c'\}$ is parahyperbolic. □

Corollary 4.10. *If there is a non-zero bounded $\{X, c'\}-$harmonic function on X, then there is at least one bounded positive $\{X, c\}-$harmonic function on X.*

Proof. By hypothesis, there is a non-zero $\{X, c'\}-$harmonic function b on X such that $|b| \leq 1$ on X. Since $|b|$ is $\{X, c'\}-$subharmonic function on X, the constant 1 cannot be a $\{X, c'\}-$potential. That is, X is $\{X, c'\}-$ bounded hyperbolic. Then, by the above theorem, X is $\{X, c\}-$ bounded hyperbolic. Consequently, there exists a $\{X, c\}-$harmonic function on X, $0 < h < 1$. □

5. Networks without positive superharmonic functions

In the introduction, we mentioned about the equation $\Delta u = Pu$ in the context of the classification of Riemann surfaces R. The condition that P is non-negative, non-zero ensures that the positive constants are $P-$superharmonic (not $P-$harmonic) functions on R. Consequently, potential-theoretic methods can be used here. However for the existence of positive $P-$superharmonic functions on R, the condition that $P(x) \geq \dfrac{\Delta\xi(x)}{\xi(x)}$ is sufficient, where $\xi > 0$ is a C^2-function on R. This condition permits $P(x)$ to possibly take non-positive values also.

In the context of an infinite network X, with conductance $t(x, y) \geq 0$ such that $t(x) = \sum_y t(x, y) < \infty$ for all x, this situation is described as follows: Take $\Delta u(x) = P(x)u(x)$ where $\Delta u(x) = \sum_{y \sim x} t(x, y)[u(y) - u(x)] = -t(x)u(x) + Au(x)$, $t(x) = \sum_{y \sim x} t(x, y)$ and $Au(x) = \sum_{y \sim x} t(x, y)u(y)$; and $P(x) \geq \dfrac{\Delta\xi(x)}{\xi(x)}$ with $\xi(x) > 0$ for each $x \in X$. In particular, $q(x) = t(x) + P(x) \geq \sum_{y \sim x} t(x, y)\dfrac{\xi(y)}{\xi(x)} > 0$. (This condition is found in Bendito et al.[2] in a different treatment of Schrödinger operators in a finite graph with symmetric conductance.) Recall that u is said to be $q-$superharmonic at x if $q(x)u(x) = [t(x) + P(x)]u(x) \geq Au(x)$.

However, it is possible that $q(x) = t(x) + P(x) > 0$ for every $x \in X$, yet there is no positive $q(x)-$superharmonic on X. Consider the following
Example: Let $X = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$ be a linear tree with conductance

$t(x_i, x_{i+1}) = t(x_{i+1}, x_i) = \frac{1}{2}$ for all i. Let $P(x) = -\frac{1}{2}$ for all x in X. Then $q(x) = t(x) + P(x) = \frac{1}{2}$ for all x. Suppose $s(x) \geq 0$ is a $q-$superharmonic on X. Then we should have for all i,

$$\frac{1}{2}s(x_i) \geq \frac{1}{2}s(x_{i-1}) + \frac{1}{2}s(x_{i+1}).$$

This is possible only if $s = 0$.

Definition 5.1. Let $q(x)$ be a real-valued function defined on a network X. Let us write $A_q u(x) = Au(x) - q(x)u(x)$ for x in X. A real-valued function u is said to be A_q-superharmonic on a subset E, if $q(x)u(x) \geq Au(x)$ for each $x \in \overset{\circ}{E}$. A_q-harmonic and A_q-subharmonic functions are defined accordingly.

Proposition 5.2. *If $q(x_0) \leq 0$ for some x_0 in X, then 0 is the only non-negative A_q-superharmonic function on X.*

Proof. Suppose $s \geq 0$ is A_q-superharmonic on X. Then, $As(x_0) \leq q(x_0)s(x_0) \leq 0$. This implies that $s(y) = 0$ for every $y \sim x_0$. We know that if a non-negative A_q-superharmonic function s on X takes the value 0 at a vertex in X, then s is the zero function. For if $s(a) = 0$ then $0 = q(a)s(a) \geq \sum_{y \sim a} t(a, y)s(y) \geq 0$ which implies that $s(y) = 0$ if $y \sim a$. This leads to the conclusion $s = 0$, since X is connected. □

Corollary 5.3. *Let $q(x_0) \leq 0$ for some x_0 in X. If u is A_q-superharmonic and v is A_q-subharmonic on X such that $u \geq v$, then $u = v$ is A_q-harmonic on X.*

Proof. For $s = u - v \geq 0$ is A_q-superharmonic on X, so that $s = 0$. Hence $u = v$ is A_q-harmonic on X. □

In this section, we assume that 0 is the only non-negative $q-$superharmonic function on X and develop a potential theory on X.

Proposition 5.4. *Let f be a real-valued function on X such that $\sum_y t(x, y)|f(y)| < \infty$ for each $x \in X$. Suppose the family $\Im$ of A_q-superharmonic functions s majorizing f is non-empty. Then $Rf(x) = \inf_{s \in \Im} s(x)$ is A_q-superharmonic on X and A_q-harmonic at each vertex a where f is A_q-subharmonic and $q(a) < 0$.*

Proof. First note that $\Im$ is lower directed. For, if $s_1, s_2 \in \Im$ then set $s = \inf(s_1, s_2)$. At a vertex z, suppose $s(z) = s_1(z)$. Then

$$\begin{aligned} q(z)s(z) = q(z)s_1(z) \quad &\geq \sum_{y \sim z} t(z, y)s_1(y) \\ &\geq \sum_{y \sim z} t(z, y)s(y). \end{aligned}$$

Hence $s \in \Im$ so that $\Im$ is lower directed. Consequently, since X has only a countable number of vertices, there exists a decreasing sequence s_n in $\Im$ such that $Rf(x) =$

$\inf_n s_n(x) = \lim_n s_n(x)$, for $x \in X$. Then for any vertex z in X,

$$\begin{aligned} q(z)s_n(z) &\geq \sum_y t(z,y)s_n(y) \\ &\geq \sum_y t(z,y)Rf(y). \end{aligned}$$

Hence, when $n \to \infty$, we have $q(z)Rf(z) \geq \sum_y t(z,y)Rf(y)$. That is $Rf(z)$ is A_q-superharmonic at the vertex z.

Let now $f(x)$ be A_q-subharmonic at a vertex a. Then,

$$\begin{aligned} q(a)f(a) &\leq \sum_y t(a,y)f(y) \\ &\leq \sum_y t(a,y)Rf(y) \\ &\leq q(a)Rf(a). \end{aligned}$$

Since $q(a) < 0$, we have then $f(a) \geq Rf(a)$ and hence $f(a) = Rf(a)$. This shows that $q(a)Rf(a) = \sum_y t(a,y)Rf(y)$, that is $Rf(x)$ is A_q-harmonic at the vertex a. □

Remark 5.5. It is possible that $q(x) > 0$ for all x, yet there is no positive $q-$superharmonic function on X. In that case, we remove the condition $q(a) < 0$ and prove the above proposition 5.4 by using the Poisson modification (similar to the one used above while obtaining the Riesz decomposition).

In the sequel, to prove certain results, our method requires that there are no closed paths in the network X. Hence we are going to assume that X is an infinite tree T, as before possibly non-locally finite and $\{t(x,y)\}$ as conductance. **We make the assumption that every non-terminal vertex in T has at least two non-terminal vertices as neighbours and that $q(y) \neq 0$ if y is a terminal vertex in T.** Apart from the trees without terminal vertices like homogeneous trees and binary trees, some other examples satisfying the first assumption are: i) a linear tree $[x_0, x_1, x_2, \ldots]$ where $x_i \sim x_{i+1}$ for $i \geq 0$; ii) a tree consisting of non-terminal vertices $\{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$, $x_i \sim x_{i+1}$ together with terminal vertices $\{a_i\}$ and $\{b_i\}$, $-\infty < i < \infty$, where a_i and b_i have x_i as neighbours. With this assumption on T, we can prove the following lemma as indicated in [1, p.112]: (Recall that for a subset E in T, $V(E)$ stands for the set consisting of E and the neighbours of each vertex in E.)

Lemma 5.6. *Let E be an arbitrary connected subset of T and $F = V(E)$. Suppose u is a real-valued function defined on F. Then there exists a function v on T such that $v = u$ on F and $A_q v(x) = 0$ for each $x \in T \setminus \mathring{F}$.*

Proof. The proof is similar to that of [1, Theorem 5.1.2]. Let $z \in \partial F$. Then z has only one neighbour x_0 in F. Note that z is not a terminal vertex. Since by the assumption on T, the non-terminal vertex z should have at least two non-terminal vertices as neighbours, z should have at least one non-terminal vertex as a neighbour outside F. Let $A = \{y_1, y_2, \dots\}$ be the neighbours of z outside F. Let A_1 denote the set of all terminal vertices in A and $A_2 = A \setminus A_1$. Then $A_2 \neq \emptyset$.

Define $v(x) = u(x)$ if $x \in F$; if $y \in A_1$, then take $v(y) = \dfrac{t(y,z)}{q(y)}u(z)$; and if $y \in A_2$ then take $v(y) = \lambda$, a constant where the constant λ is chosen so that $A_q v(z) = 0$. That is

$$q(z)v(z) = q(z)u(z) = t(z, x_0)u(x_0) + u(z)\sum_{y \in A_1} t(z,y)\frac{t(y,z)}{q(y)} + \lambda \sum_{y \in A_2} t(z,y).$$

This procedure can be used with respect to each one of the vertices on ∂F to get an extension of u from F to a function v on the set $V(F)$ such that $A_q v(z) = 0$ at every $z \in \partial F$. Thus, v is a function defined on $V(F)$ such that $v = u$ on F and $A_q v(z) = 0$ for each $z \in \partial F$. Then v is similarly extended to $V[V(F)]$. Since T is connected, eventually v is defined at any vertex x in T such that $A_q v(x) = 0$ at each $x \in T \setminus \mathring{F}$ and $v = u$ on F. □

Theorem 5.7. *For any e in T, there exists a $q(x)-$superharmonic function $g_e(x)$ on T such that $(-A_q)g_e(x) = q(x)g_e(x) - Ag_e(x) = \delta_e(x)$ for $x \in T$.*

Proof. Let the neighbours of e be $\{z_1, \dots, z_i, z'_1, \dots, z'_j\}$ of which $\{z_1, \dots, z_i\}$ are non-terminal vertices and $\{z'_i, \dots, z'_j\}$ are terminal vertices. Note that i or j or both can be infinite. For a terminal vertex z', let $u(z') = \dfrac{t(z',e)}{q(z')}u(e)$ and $u(z_1) = \cdots = u(z_i) = \lambda$ where λ and $u(e)$ are chosen so that $u(e)\left[q(e) - \sum\limits_{k=1}^{j} t(e,z'_k)\dfrac{t(z'_k,e)}{q(z'_k)}\right] > \lambda \sum\limits_{k=1}^{i} t(e,z_k)$.

Thus u is defined on $F = V(e)$, $A_q u(z'_k) = 0$ for $k = 1, \dots, j$ and $(-A_q)u(e) > 0$. Now use the above lemma 5.6 to construct v on T such that $v = u$ on F and $A_q v(x) = 0$ for each $x \in T \setminus \mathring{F}$. This means that v is defined on T such that $(-A_q)v(e) > 0$ and $(-A_q)v(x) = 0$ if $x \neq e$. Define $g_e(x) = \dfrac{v(x)}{(-A_q)v(e)}$ for $x \in T$ and remark that $(-A_q)g_e(x) = \delta_e(x)$ for all x in T. □

Remark 5.8. The only non-negative A_q-harmonic function in T is 0. However there are many non-zero A_q-harmonic functions in T. This can be seen as follows: In the proof of the above Theorem 5.7, choose λ and the non-zero value $u(e)$ so that

$$u(e)\left[q(e) - \sum_{k=1}^{j} t(e,z'_k)\frac{t(z'_k,e)}{q(z'_k)}\right] = \lambda \sum_{k=1}^{i} t(e,z_k),$$

which is possible since $\sum_{k=1}^{i} t(e, z_k) > 0$. Then we construct the function v on T such that $v = u$ on $F = V(e)$ and $A_q v(x) = 0$ if $x \in T \setminus \mathring{F}$. Consequently, v is a A_q-harmonic function on T.

Another remark, as a consequence of Lemma 5.6, concerns the analytic property of harmonic functions on T : we have termed a class of functions on a network X as harmonic functions if these functions have the sheaf property, the property of local solvability of the Dirichlet problem and the Harnack property. Thus we have considered Laplace harmonic functions, Schrödinger harmonic functions, $c-$harmonic functions etc. on X. We shall say that a class of $\Im$ of harmonic functions on X has the analytic property if for any $h \in \Im$, $h = 0$ in a neighbourhood $V(e)$ of a vertex e implies that $h = 0$ on X.

Proposition 5.9. *Let E be a connected subset of T with more than one vertex and $F = V(E)$. Suppose some e in E has at least two non-terminal vertices in $F \setminus E$ as its neighbours. Then the A_q-harmonic functions in T do not possess the analytic property.*

Proof. Let $a, b \in F \setminus E$ be two non-terminal vertices that are neighbours of e in E. Note that $a, b \in \partial F$. Let $e \neq z \in E$. Since E is connected neither a nor b is a neighbour of z. Define u on F such that $u(a) \neq 0$, $u(b) = -\frac{t(e,a)}{t(e,b)} u(a)$ and $u(x) = 0$ for all other x in F. Note that $A_q u(y) = 0$ for all $y \in \mathring{F}$.

Then, by Lemma 5.6, there exists a function v on T such that $v = u$ on F and $A_q v(x) = 0$ for all $x \in T \setminus \mathring{F}$. Consequently, v is A_q-harmonic on T such that $v = 0$ on $\mathring{F} \supset E$. In particular, v is a non-zero A_q-harmonic function on T and $v = 0$ on $V(z)$. Hence the A_q-harmonic functions in this case do not possess the analytic property. □

6. A discrete version of the Helmholtz equation

As an illustration of what was discussed above in a connected infinite network X (which may not be locally finite), let us consider now a discrete Helmholtz equation of the type $\Delta u(x) + k(x)u(x) = 0$ in X, where $k(x) \geq 0$ and $\Delta u(x) = \sum_y t(x, y)[u(y) - u(x)]$. Let us write

$$\begin{aligned} \Delta_k u(x) &= \Delta u(x) + k(x)u(x) \\ &= \sum_y t(x, y)u(y) - [t(x) - k(x)]u(x) \\ &= Au(x) - [t(x) - k(x)]u(x) \end{aligned}$$

and say that a real-valued function u defined on a subset E of X such that $A|u|(x) < \infty$ for each $x \in \mathring{E}$ is a Helmholtz superharmonic ($HH-$superharmonic) function on E if and only if $[t(x) - k(x)]u(x) \geq Au(x)$ for each $x \in \mathring{E}$. That is $\Delta_k u(x) \leq 0$ if $x \in \mathring{E}$. Define similarly $HH-$harmonic functions and $HH-$subharmonic functions on E. A $HH-$superharmonic function $s \geq 0$ on E is said to be a $HH-$potential if for any $HH-$subharmonic v on E such that $v \leq s$ on E, we have $v \leq 0$. If there is a $HH-$potential $p > 0$ on X, we say that X is $HH-$hyperbolic; otherwise X is called a $HH-$parabolic network. In the above nomenclature we leave out the prefix HH if $k = 0$. Unless mentioned otherwise, k stands for a function $k(x) \geq 0$ such that $k(x_0) > 0$ for at least one x_0 in X.

In the Euclidean space $\mathbb{R}$ or $\mathbb{R}^2$, let $u(x)$ be a non-negative C^2-function and $k(x)$ be continuous such that $\Delta_k u(x) = \Delta u(x) + k(x)u(x) \leq 0$. Then $u(x)$ is a classical $\Delta-$superharmonic function, so that $u(x)$ is a constant which should necessarily be 0. A discrete version of this result is the following:

Proposition 6.1. *Let $u \geq 0$ be a $HH-$superharmonic function on a parabolic network X. Then $u = 0$.*

Proof. By assumption $\Delta_k u(x) = \Delta u(x) + k(x)u(x) \leq 0$ on X. This implies that $\Delta u(x) \leq 0$ on X. That is $u(x)$ is a superharmonic function on X. Since $u \geq 0$ and X is parabolic, we should have $u = c$, a constant. Necessarily $u = 0$. □

Proposition 6.2. *Suppose $t(x_0) \leq k(x_0)$ for some x_0 in X. Then 0 is the only non-negative $HH-$superharmonic function on X.*

Proof. Let $s \geq 0$ be a $HH-$superharmonic function on X. Then, $0 \geq [t(x_0) - k(x_0)]s(x_0) \geq \sum_y t(x_0, y)s(y)$. This means that $s(y) = 0$ if $y \sim x_0$. Then, for such a vertex $y \sim x_0$,

$$0 = [t(y) - k(y)]s(y) \geq \sum_z t(y, z)s(z).$$

Hence $s(z) = 0$ if $z \sim y$. This leads to the conclusion $s = 0$, since X is connected. □

Proposition 6.3. *There exists a positive $HH-$superharmonic function on X if and only if there exists some function $\xi > 0$ on X such that $k(x) \leq -\dfrac{\Delta\xi(x)}{\xi(x)}$ for x in X.*

Proof. Suppose $p > 0$ is a $HH-$superharmonic function on X. Then $[t(x) - k(x)]p(x) \geq Ap(x)$ for each x in X. That is $-k(x) \geq \dfrac{\Delta p(x)}{p(x)}$. Conversely, suppose $k(x) \leq -\dfrac{\Delta\xi(x)}{\xi(x)}$ for some function $\xi > 0$. Then $\xi(x)$ is a positive $HH-$superharmonic function on X. □

Corollary 6.4. *If there exists a positive $HH-$superharmonic function on X, then X is hyperbolic.*

Proof. For there exists a function $\xi > 0$ such that $-\dfrac{\Delta\xi(x)}{\xi(x)} \geq k(x)$. Since $k(x_0) > 0$ for some x_0, $\Delta\xi(x) \leq 0$ and $\Delta\xi(x_0) < 0$. That is, $\xi(x)$ is superharmonic but not harmonic on X. Hence X is hyperbolic. □

Example of a hyperbolic network that is not $HH-$hyperbolic: Let X be a homogeneous tree, each vertex having $q+1$ neighbours, $q \geq 2$, and the conductance being $t(x,y) = \dfrac{1}{q+1}$ between any two neighbouring vertices x and y. Let us consider on X the $HH-$harmonic functions defined by Δ_k where $k(x) = 1$ for all $x \in X$.

Fix a vertex e and let $|x|$ denote the distance between e and x. Define $g(e) = \dfrac{q}{q-1}$ and $g(x) = \dfrac{1}{q^{m-1}(q-1)}$ if $|x| = m \geq 1$. Then $\Delta g(x) = -\delta_e(x)$ so that X is hyperbolic, where $\Delta u(x) = \sum\limits_{y\sim x} \frac{1}{q+1}\left[u(y) - u(x)\right]$.

Take now $k = 1$ and write $\Delta_k u(x) = \Delta u(x) + ku(x)$ so that $\Delta_k u(x) = \frac{1}{q+1}\sum\limits_{y\sim x} u(y)$. Suppose now that there exists a $HH-$superharmonic function $s > 0$ on X, so that $\Delta_k s(x) \leq 0$ for each x in X. But $\Delta_k s(x) \leq 0$ for each x means that $s = 0$, a contradiction. Hence X is not $HH-$hyperbolic.

Remark 6.5. Analogous to the example above, we can consider in the line segment $X = (-\pi, \pi)$ the operators $\Delta y = y''$ and $\Delta_1 y = y'' + y$. Then X is $\Delta-$hyperbolic but 0 is the only non-negative Δ_1-superharmonic function on X.

The following remarks are easy to verify:

i) Any non-negative $HH-$superharmonic function on E is superharmonic on E.
ii) Any non-negative subharmonic function on E is $HH-$subharmonic on E.
iii) Any $HH-$potential on E is a potential on E. In particular, if X is $HH-$hyperbolic then X is hyperbolic also.
iv) In a $HH-$hyperbolic network X, for any vertex e in X, there exists a unique $HH-$potential $G_e(x)$ called the $HH-$Green potential on X, such that $(-\Delta_k)G_e(x) = \delta_e(x)$.
Construction: Let $\Im$ be the family of all positive $HH-$superharmonic functions s on X such that $s(x) \geq \delta_e(x)$. Let $u(x) = \inf\limits_{s\in\Im} s(x)$. Then $u(x)$ is $HH-$superharmonic on X, since $\Im$ is a lower directed family of positive $HH-$superharmonic functions. Actually, u is a $HH-$potential on X, since there are $HH-$potentials in the family $\Im$. Note that $\Delta_k u(x) = 0$ if $x \neq e$ since $\delta_e(x) = 0$ if $x \neq e$. Take $G_e(x) = \dfrac{u(x)}{(-\Delta_k)u(e)}$.

Proposition 6.6. *Let X be a $HH-$hyperbolic network, $e \in X$. Then the Green potential $g_e(x)$ with harmonic support at e exists on X and $g_e(x) \leq G_e(x)$ where $G_e(x)$ is the $HH-$Green potential on X with $HH-$harmonic support at e.*

Proof. Since X is $HH-$hyperbolic, as remarked in iii) above, X is hyperbolic also. Hence the Green potential $g_e(x)$ exists. Now

$$\begin{aligned}\Delta(g_e(x) - G_e(x)) &= -\delta_e(x) - \Delta G_e(x)\\ &= \Delta_k G_e(x) - \Delta G_e(x)\\ &= k(x)G_e(x)\\ &\geq 0.\end{aligned}$$

Hence, $g_e(x) - G_e(x) = v(x)$ is subharmonic on X. Since $v(x) \leq g_e(x)$, we conclude that $v(x) \leq 0$ so that $g_e(x) \leq G_e(x)$. □

Remark 6.7. Actually, by using the Domination Principle, we can show that $g_e(x) \leq \frac{g_e(e)}{G_e(e)} G_e(x) \leq G_e(x)$.

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