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THE DIRICHLET PROBLEM FOR ∞-HARMONIC FUNCTIONS ON A NETWORK

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Abstract. We define the notion of ∞-harmonic functions on a network as a discrete version of that on a euclidean domain, and show some properties of such functions. We discuss the Dirichlet problems for discrete ∞-harmonic functions. We also show that limits of discrete p-harmonic functions as $p \to \infty$ are in fact discrete ∞-harmonic.

1. Introduction

An ∞-harmonic function in a euclidean domain $D \subset \mathbb{R}^d$ ($d \geq 2$) is defined to be a viscosity solution of the equation

$$\Delta_\infty u := \frac{1}{2}\nabla u \cdot \nabla |\nabla u|^2 = 0 \tag{1}$$

in D (see [1, 2, 3]). For $1 < p < \infty$, a p-harmonic function in D is a continuous weak solution to the p-Laplace equation

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{2}$$

in D. If u_n is p_n-harmonic in D with $p_n \to \infty$ and $u_n \to u$, then u is ∞-harmonic in D (see [1]). This fact shows that (1) is the limiting equation of (2) as $p \to \infty$, and explains the terminology ∞-harmonic.

The purpose of this paper is to define the notion of ∞-harmonic functions on a network as a discrete version of that on a euclidean domain and obtain some properties related to such functions. A discrete analogue of the p-Laplacian Δ_p can be readily defined on a network (see, e.g., [6, 7, 5]). However, there seems to be no appropriate discrete version of the ∞-Laplacian. One may define ∞-harmonic functions on a network as limits of p-harmonic functions as $p \to \infty$; but this definition is somewhat indirect and not so appropriate to handle with to obtain local properties.

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An ∞-harmonic function in a euclidean domain D also arises in the Lipschitz extension problem; $u \in W^{1,\infty}(D)$ is called an absolutely minimal Lipschitz extension in D if

$$\|\nabla u\|_{L^\infty(V)} \le \|\nabla v\|_{L^\infty(V)}$$

for any domain $V \subset D$ and any v with $u - v \in W_0^{1,\infty}(V)$. It is known that an absolutely minimal Lipschitz extension is ∞-harmonic (see [2]).

This suggests our definition of a discrete ∞-harmonicity on a network. We define the ∞-harmonicity of a function on vertices by means of its ∞-mean value around a vertex as in [4]. By using discrete derivative of a function, we obtain a useful criterion as Theorem 3.1 for ∞-harmonicity. Most of properties of classical discrete harmonic functions hold. We discuss the Dirichlet problem for ∞-harmonic functions on a network. We shall introduce in Section 4 an ideal boundary of a network. Roughly speaking, this ideal boundary is the set of infinite paths. Given a function on the ideal boundary, we shall show in Theorem 5.7 the existence of ∞-harmonic functions satisfying the boundary condition. As in the classical theory, a set of ∞-superharmonic functions and a set of ∞-subharmonic functions give the upper solution and the lower solution of our Dirichlet problem. It is shown that these solutions take the given boundary value if the boundary value is a bounded Lipschitz function. We show in Theorem 5.9 that the solutions to the Dirichlet problem give optimal solutions to an ∞-variational problem. We show a boundary maximum principle for the sum of two ∞-subharmonic functions in Lemma 5.11. With the aid of this result, we show in Theorem 5.12 that the solution to the Dirichlet problem is unique. Finally we show in Section 6 that the limit of p-harmonic functions as $p \to \infty$ is ∞-harmonic.

2. Preliminaries

Let (V, E) be a locally finite and connected infinite graph without self-loops, where V is the set of vertices and E is the set of edges. This means that V is a countable set and that an element of E is an ordered pair (x, y) of vertices $x, y \in V$. We assume that $(y, x) \in E$ if $(x, y) \in E$. Let

$$\partial x = \{y \in V; (x, y) \in E\}, \qquad Nx = \partial x \cup \{x\},$$
$$\overline{D} = D \cup \{x \in V; (x, y) \in E \text{ for some } y \in D\}.$$

From our assumptions

(1) $(x, x) \notin E$ for $x \in V$;
(2) ∂x is a finite set for each $x \in V$;
(3) for each $x, y \in V$, there is a sequence $\{x_i\}_{i=0}^{l}$ of distinct vertices such that $x = x_0$, $y = x_l$ and $(x_{j-1}, x_j) \in E$ for $j = 1, 2, \ldots, l$.

A sequence in (3) is called a *path* from x to y.

A *resistance* r is a positive function on E. We assume that $r(y, x) = r(x, y)$ for each edge $(x, y) \in E$. A *network* is a triplet (V, E, r), where (V, E) is a graph and r is a resistance.

Let $L(D)$ be the set of real valued functions in a subset $D \subset V$. For $u \in L(D)$ and $(x, y) \in E$ with $x, y \in D$ we define the *discrete derivative* ∇u at (x, y) as

$$\nabla u(x,y) = \frac{u(y) - u(x)}{r(x,y)}.$$

We define the ∞-*Dirichlet seminorm* $D_\infty[u]$ of $u \in L(V)$ by

$$D_\infty[u] = \sup_{(x,y)\in E} |\nabla u(x,y)|.$$

Let $\mathbf{D}^{(\infty)}$ be the set of functions in V with finite ∞-Dirichlet seminorms.

3. Local ∞-variational problem

For $x \in V$ and a function $u \in L(Nx)$ let

$$M_u(x) = \max_{y\in\partial x} |\nabla u(x,y)|, \qquad \mu^\infty_{x,u}(t) = \max_{y\in\partial x} \frac{|u(y) - t|}{r(x,y)}$$

for $t \in \mathbb{R}$. Note that $M_u(x) = \mu^\infty_{x,u}(u(x))$. Since $\mu^\infty_{x,u}$ is a convex function such that $\lim_{t\to\pm\infty} \mu^\infty_{x,u}(t) = \infty$ and that it is not constant on any open interval, it follows that there exists a unique ∞-*mean value* $\mathrm{H}^\infty_x u$ such that $\mu^\infty_{x,u}(t) \geq \mu^\infty_{x,u}(\mathrm{H}^\infty_x u)$ for any $t \in \mathbb{R}$.

Let $x \in V$ and $u \in L(Nx)$. If u satisfies $u(x) \leq \mathrm{H}^\infty_x u$ ($u(x) \geq \mathrm{H}^\infty_x u$, $u(x) = \mathrm{H}^\infty_x u$, resp.), then u is said to be ∞-*subharmonic* (∞-*superharmonic*, ∞-*harmonic*, resp.) at x. Let $D \subset V$ and $u \in L(\overline{D})$. If u is ∞-subharmonic (∞-superharmonic, ∞-harmonic, resp.) at each $x \in D$, then u is said to be ∞-*subharmonic* (∞-*superharmonic*, ∞-*harmonic*, resp.) in D. Note that u is ∞-superharmonic if and only if $-u$ is ∞-subharmonic.

We repeatedly use the next theorem, which characterizes ∞-superharmonic functions and ∞-subharmonic functions.

Theorem 3.1. *Let $x \in V$ and u a function on Nx.*

1. *u is ∞-superharmonic at x if and only if there is a vertex $y \in \partial x$ such that $\nabla u(x,y) = -M_u(x)$.*
2. *u is ∞-subharmonic at x if and only if there is a vertex $y \in \partial x$ such that $\nabla u(x,y) = M_u(x)$.*

Proof. Let $t_0 = \mathrm{H}^\infty_x u$. Note that there is $y \in \partial x$ such that either $\nabla u(x,y) = M_u(x)$ or $\nabla u(x,y) = -M_u(x)$.

Case 1: $u(x) < t_0$. Using $M_u(x) = \mu^\infty_{x,u}(u(x)) \geq \mu^\infty_{x,u}(t_0)$ we have

$$\nabla u(x,z) = \frac{u(z) - u(x)}{r(x,z)} > \frac{u(z) - t_0}{r(x,z)} \geq -\mu^\infty_{x,u}(t_0) \geq -M_u(x).$$

for $z \in \partial x$. This means that $\nabla u(x,z) \neq -M_u(x)$ for $z \in \partial x$, and that there is $y_1 \in \partial x$ such that $\nabla u(x, y_1) = M_u(x)$.

Case 2: $u(x) > t_0$. It follows from an argument similar to Case 1 that $\nabla u(x,z) \neq M_u(x)$ for each $z \in \partial x$ and that there is $y_2 \in \partial x$ such that $\nabla u(x,y_2) = -M_u(x)$.

Case 3: $u(x) = t_0$. We show that there is $y_1 \in \partial x$ such that $\nabla u(x, y_1) = M_u(x)$. On the contrary we assume that $\nabla u(x, z) < M_u(x)$ for each $z \in \partial x$. Then

$$\frac{u(z) - t_0}{r(x,z)} = \frac{u(z) - u(x)}{r(x,z)} < M_u(x).$$

There is $\varepsilon > 0$ such that $r(x, z)^{-1}(u(z) - t_0 + \varepsilon) < M_u(x)$ for any $z \in \partial x$. Since

$$\begin{aligned} M_u(x) > \frac{u(z) - (t_0 - \varepsilon)}{r(x,z)} &= \frac{u(z) - u(x)}{r(x,z)} + \frac{\varepsilon}{r(x,z)} \\ &\geq -M_u(x) + \frac{\varepsilon}{r(x,z)} > -M_u(x), \end{aligned}$$

it follows that $\mu^\infty_{x,u}(t_0 - \varepsilon) < M_u(x) = \mu^\infty_{x,u}(u(x)) = \mu^\infty_{x,u}(t_0)$, which contradicts the definition of t_0. This means that there is $y_1 \in \partial x$ such that $\nabla u(x, y_1) = M_u(x)$. Similarly there is $y_2 \in \partial x$ such that $\nabla u(x, y_2) = -M_u(x)$.

Now suppose that u is ∞-superharmonic at x. Then either Case 2 or Case 3 holds. There is $y_2 \in \partial x$ such that $\nabla u(x, y_2) = -M_u(x)$. Conversely, we assume that $\nabla u(x, y_2) = -M_u(x)$ for some $y_2 \in \partial x$. Then Case 1 cannot hold, so that $u(x) \geq t_0$. This means that u is ∞-superharmonic at x. Therefore (1) holds. We can similarly prove (2). □

Next proposition implies the Harnack inequality.

Proposition 3.2. *Let $x \in V$ and let u be a function on Nx. Let $c_x = \max_{y,z\in\partial x} r(x, y)/r(x, z)$.*

(1) *If u is ∞-superharmonic at x and $u \geq 0$ on Nx, then $u(y) \leq (1 + c_x)u(x)$ for $y \in \partial x$.*
(2) *If u is ∞-subharmonic at x and $u \leq 0$ on Nx, then $u(y) \geq (1 + c_x)u(x)$ for $y \in \partial x$.*

Proof. We shall prove (1) only. Theorem 3.1 shows that there is $z \in \partial x$ such that $\nabla u(x, z) = -M_u(x)$. Since $\nabla u(x, y) \leq M_u(x)$ for $y \in \partial x$ and $u(z) \geq 0$, it follows that

$$\frac{u(y) - u(x)}{r(x,y)} = \nabla u(x, y) \leq -\nabla u(x, z) = \frac{u(x) - u(z)}{r(x,z)} \leq \frac{u(x)}{r(x,z)}.$$

This implies that $u(y) \leq (1 + r(x, y)/r(x, z))u(x)$, and the assertion. □

Lemma 3.3. *Let $x \in V$. Let u and v be functions on ∂x with $u \leq v$. Then $\mathrm{H}^\infty_x u \leq \mathrm{H}^\infty_x v$.*

Proof. On the contrary we assume that $\mathrm{H}^\infty_x u > \mathrm{H}^\infty_x v$. Then $u(y) - \mathrm{H}^\infty_x u < v(y) - \mathrm{H}^\infty_x v$ for each $y \in \partial x$. Let $\tilde{v}$ be the function with $\tilde{v}(x) = \mathrm{H}^\infty_x v$ and $\tilde{v} = v$ on ∂x. Let $\tilde{u}$ be the function with $\tilde{u}(x) = \mathrm{H}^\infty_x u$ and $\tilde{u} = u$ on ∂x. Since $\tilde{u}$ is ∞-harmonic at x, Theorem 3.1 implies that there is $y_1 \in \partial x$ such that

$$\begin{aligned} M_{\tilde{u}}(x) = \nabla \tilde{u}(x, y_1) = \frac{\tilde{u}(y_1) - \tilde{u}(x)}{r(x,y_1)} &= \frac{u(y_1) - \mathrm{H}^\infty_x u}{r(x,y_1)} < \frac{v(y_1) - \mathrm{H}^\infty_x v}{r(x,y_1)} \\ &= \frac{\tilde{v}(y_1) - \tilde{v}(x)}{r(x,y_1)} = \nabla \tilde{v}(x, y_1) \leq M_{\tilde{v}}(x). \end{aligned}$$

Similarly, using $y_2 \in \partial x$ with $\nabla\tilde{v}(x, y_2) = -M_{\tilde{v}}(x)$, we obtain $M_{\tilde{v}}(x) < M_{\tilde{u}}(x)$, and a contradiction. □

Lemma 3.4. *Let $D \subset V$. Let u be a function in $\overline{D}$ and $x \in D$. Let*

$$\tilde{u}(y) = \begin{cases} \mathrm{H}^\infty_x u & \text{if } y = x; \\ u(y) & \text{if } y \neq x. \end{cases}$$

(1) *If u is an ∞-superharmonic function in D, then $\tilde{u}$ is an ∞-superharmonic function in D such that $\tilde{u}$ is ∞-harmonic at x and that $\tilde{u} \le u$.*
(2) *If u is an ∞-subharmonic function in D, then $\tilde{u}$ is an ∞-subharmonic function in D such that $\tilde{u}$ is ∞-harmonic at x and that $\tilde{u} \ge u$.*

Proof. We shall prove (1) only. It is obvious that $\tilde{u}$ is ∞-harmonic at x. Since $\tilde{u}(x) = \mathrm{H}^\infty_x u \le u(x)$ and $\tilde{u}(z) = u(z)$ for $z \neq x$, it follows that $\tilde{u} \le u$. Lemma 3.3 shows that $\tilde{u}(z) = u(z) \ge \mathrm{H}^\infty_z u \ge \mathrm{H}^\infty_z \tilde{u}$ for $z \neq x$. This means that $\tilde{u}$ is ∞-superharmonic at z. □

Lemma 3.5. *Let $D \subset V$ and $\{u_\lambda\}_{\lambda\in\Lambda}$ a family of functions in $\overline{D}$.*

(1) *Suppose that u_λ is ∞-superharmonic in D for each $\lambda \in \Lambda$ and that $u := \inf_{\lambda\in\Lambda} u_\lambda$ is finite for each vertex in $\overline{D}$. Then u is ∞-superharmonic in D.*
(2) *Suppose that u_λ is ∞-subharmonic in D for each $\lambda \in \Lambda$ and that $u := \sup_{\lambda\in\Lambda} u_\lambda$ is finite for each vertex in $\overline{D}$. Then u is ∞-subharmonic in D.*

Proof. We shall prove (1) only. Let $x \in D$. Lemma 3.3 shows that $\mathrm{H}^\infty_x u \le \mathrm{H}^\infty_x u_\lambda \le u_\lambda(x)$ for $\lambda \in \Lambda$. Hence $\mathrm{H}^\infty_x u \le u(x)$. This means that u is ∞-superharmonic at x. □

Lemma 3.6. *Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be a family of functions in V.*

(1) *Suppose that $u := \inf_{\lambda\in\Lambda} u_\lambda$ is finite for each vertex in V. Then $D_\infty[u] \le \sup_{\lambda\in\Lambda} D_\infty[u_\lambda]$.*
(2) *Suppose that $u := \sup_{\lambda\in\Lambda} u_\lambda$ is finite for each vertex in V. Then $D_\infty[u] \le \sup_{\lambda\in\Lambda} D_\infty[u_\lambda]$.*

We remark that $\sup_{\lambda\in\Lambda} D_\infty[u_\lambda] \le \infty$.

Proof. We shall prove (1) only. Let $(x, y) \in E$. We may assume $u(y) \ge u(x)$. For $\varepsilon > 0$ there is $\lambda \in \Lambda$ such that $u_\lambda(x) \le u(x) + \varepsilon$. Since $u_\lambda(y) \ge u(y)$, it follows that

$$\begin{aligned} 0 \le \nabla u(x, y) &= \frac{u(y) - u(x)}{r(x, y)} \le \frac{u_\lambda(y) - u_\lambda(x) + \varepsilon}{r(x, y)} \\ &= \nabla u_\lambda(x, y) + \frac{\varepsilon}{r(x, y)} \le \sup_{\lambda\in\Lambda} D_\infty[u_\lambda] + \frac{\varepsilon}{r(x, y)}. \end{aligned}$$

Letting $\varepsilon \to 0$ we have that $|\nabla u(x, y)| \le \sup_{\lambda\in\Lambda} D_\infty[u_\lambda]$, so that $D_\infty[u] \le \sup_{\lambda\in\Lambda} D_\infty[u_\lambda]$. □

4. The ideal boundary of a network

For $x, y \in V$ let $R(x, y)$ be the *geodesic distance* between x and y, i.e.,

$$R(x,y) = \inf\{\sum_i r(z_{i-1}, z_i); \{z_i\}_i \text{ is a path from } x \text{ to } y\} \qquad \text{if } x \neq y,$$

$$R(x,x) = 0.$$

Then R is a metric in V. An *infinite path* is an infinite sequence $\{x_i\}_{i=0}^{\infty}$ of distinct vertices such that $(x_{i-1}, x_i) \in E$ for $i = 1, 2, \dots$. Let $\boldsymbol{P}$ be the set of all infinite paths and let

$$\boldsymbol{P}_0 = \{\{z_i\}_i \in \boldsymbol{P}; \sum_{i=1}^{\infty} r(z_{i-1}, z_i) < \infty\}.$$

For $x \in V$ and for two infinite paths $\boldsymbol{x} = \{x_i\}_i, \boldsymbol{y} = \{y_j\}_j \in \boldsymbol{P}_0$ we let

$$R(x, \boldsymbol{y}) = R(\boldsymbol{y}, x) = \lim_{n\to\infty} R(x, y_n), \qquad R(\boldsymbol{x}, \boldsymbol{y}) = \lim_{\substack{m\to\infty \\ n\to\infty}} R(x_m, y_n).$$

It is obvious that the right-hand side of each exists and that R satisfies the triangle inequality in $V \cup \boldsymbol{P}_0$. However it is not a metric in general; it happens that $R(\boldsymbol{x}, \boldsymbol{y}) = 0$ for distinct $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{P}_0$. We identify $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{P}_0$ whenever $R(\boldsymbol{x}, \boldsymbol{y}) = 0$. We let $[\boldsymbol{x}]$ be the equivalence class containing $\boldsymbol{x} \in \boldsymbol{P}_0$ and let Ξ be the set of equivalence classes:

$$[\boldsymbol{x}] = \{\boldsymbol{y} \in \boldsymbol{P}_0; R(\boldsymbol{x}, \boldsymbol{y}) = 0\}, \qquad \Xi = \{[\boldsymbol{x}]; \boldsymbol{x} \in \boldsymbol{P}_0\}.$$

For $x, y \in V$ and $\xi, \eta \in \Xi$ we let

$$\rho(x,y) = R(x,y), \qquad \rho(x, \eta) = \rho(\eta, x) = R(x, \boldsymbol{y}), \qquad \rho(\xi, \eta) = R(\boldsymbol{x}, \boldsymbol{y}),$$

where $\boldsymbol{x} \in \xi$ and $\boldsymbol{y} \in \eta$. It is easy to see that ρ is well-defined. Also we have that, for $\{x_m\}_m \in \xi$ and $\{y_n\}_n \in \eta$,

$$\rho(x, \eta) = \rho(\eta, x) = \lim_{n\to\infty} \rho(x, y_n), \qquad \rho(\xi, \eta) = \lim_{\substack{m\to\infty \\ n\to\infty}} \rho(x_m, y_n),$$

and that ρ is a metric in $V \cup \Xi$. We call Ξ the *ideal boundary* of the network (V, E, r).

Lemma 4.1. *Let $u \in \mathbf{D}^{(\infty)}$ and $\xi \in \Xi$. Then there exists a finite limit $\lim_{n\to\infty} u(x_n)$ for $\{x_j\}_j \in \xi$, which is independent of the choice of the representative.*

Proof. Let $\{x_j\}_j \in \xi$. It is easy to see that $|u(x_m) - u(x_n)| \le D_\infty[u]\rho(x_m, x_n)$, and that $\{u(x_n)\}_n$ is a Cauchy sequence. There is a finite limit $\lim_{n\to\infty} u(x_n)$.

Let $\boldsymbol{x}^{(i)} = \{x_n^{(i)}\}_n \in \xi$ for $i = 1, 2$. Then $|u(x_m^{(1)}) - u(x_n^{(2)})| \le D_\infty[u]\rho(x_m^{(1)}, x_n^{(2)})$, and the right-hand side tends to 0 as $m, n \to \infty$. Therefore $\lim_{m\to\infty} u(x_m^{(1)}) = \lim_{n\to\infty} u(x_n^{(2)})$. □

We simply write $u(\xi) = \lim_{n\to\infty} u(x_n)$ for $u \in \mathbf{D}^{(\infty)}$ and $\{x_j\}_j \in \xi \in \Xi$.

Proposition 4.2. *Let $u \in \mathbf{D}^{(\infty)}$. Then $|u(\xi) - u(\eta)| \le D_\infty[u]\rho(\xi, \eta)$ for $\xi, \eta \in \Xi$.*

Proof. Let $\{x_n\}_n \in \xi$ and $\{y_m\}_m \in \eta$. Then $|u(x_n) - u(y_m)| \le D_\infty[u]\rho(x_n, y_m)$. It follows that $|u(\xi) - u(\eta)| \le D_\infty[u]\rho(\xi, \eta)$. □

Next theorem implies the maximal principle.

Theorem 4.3. *Let u be a function in V and $x_0 \in V$ with $M_u(x_0) > 0$.*

(1) *If u is ∞-superharmonic, then there is an infinite path $\boldsymbol{x} = \{x_i\}_{i=0}^\infty \in \boldsymbol{P}$ such that*

$$u(x_n) \le u(x_0) - M_u(x_0) \sum_{i=1}^n r(x_{i-1}, x_i) \quad \text{for each } n.$$

Moreover, if u is bounded from below, then $\boldsymbol{x} \in \boldsymbol{P}_0$.

(2) *If u is ∞-subharmonic, then there is an infinite path $\boldsymbol{x} = \{x_i\}_{i=0}^\infty \in \boldsymbol{P}$ such that*

$$u(x_n) \ge u(x_0) + M_u(x_0) \sum_{i=1}^n r(x_{i-1}, x_i) \quad \text{for each } n.$$

Moreover, if u is bounded from above, then $\boldsymbol{x} \in \boldsymbol{P}_0$.

Proof. We shall prove (2) only. Theorem 3.1 shows that there is $x_1 \in \partial x_0$ such that $\nabla u(x_0, x_1) = M_u(x_0)$. Note that $u(x_1) = u(x_0) + \nabla u(x_0, x_1) r(x_0, x_1) > u(x_0)$. Again Theorem 3.1 shows that there is $x_2 \in \partial x_1$ such that

$$\nabla u(x_1, x_2) = M_u(x_1) \ge \nabla u(x_0, x_1) = M_u(x_0).$$

Note that $u(x_2) = u(x_1) + \nabla u(x_1, x_2) r(x_1, x_2) > u(x_1)$, and that $x_2 \ne x_0, x_1$. Repeating this argument we obtain an infinite path $\boldsymbol{x} = \{x_i\}_i$ such that $\nabla u(x_{i-1}, x_i) \ge M_u(x_0)$ and $u(x_i) > u(x_{i-1})$ for each i. Therefore

$$u(x_n) - u(x_0) = \sum_{i=1}^n \nabla u(x_{i-1}, x_i) r(x_{i-1}, x_i) \ge M_u(x_0) \sum_{i=1}^n r(x_{i-1}, x_i)$$

for each n.

If u is bounded from above, then $\sum_{i=1}^\infty r(x_{i-1}, x_i) < \infty$, so that $\boldsymbol{x} \in \boldsymbol{P}_0$. □

Lemma 4.4. *Let $u \in \mathbf{D}^{(\infty)}$ and $x_0 \in V$ with $M_u(x_0) > 0$.*

(1) *If u is bounded from below and ∞-superharmonic, then there is $\xi \in \Xi$ such that*

$$u(\xi) \le u(x_0) - M_u(x_0)\rho(x_0, \xi).$$

(2) *If u is bounded from above and ∞-subharmonic, then there is $\xi \in \Xi$ such that*

$$u(\xi) \ge u(x_0) + M_u(x_0)\rho(x_0, \xi).$$

Proof. We shall prove (2) only. Theorem 4.3 shows that there is $\{x_n\}_n \in \boldsymbol{P}_0$ such that

$$u(x_n) \ge u(x_0) + M_u(x_0) \sum_{i=1}^n r(x_{i-1}, x_i) \ge u(x_0) + M_u(x_0)\rho(x_0, x_n).$$

Letting $\xi = [\{x_n\}_n] \in \Xi$ and tending $n \to \infty$ we have the assertion. □

Corollary 4.5. *Let* $u \in \mathbf{D}^{(\infty)}$.

(1) *If* u *is bounded from below and* ∞*-superharmonic, then* $\inf_{V\cup\Xi} u = \inf_{\Xi} u$.

(2) *If* u *is bounded from above and* ∞*-subharmonic, then* $\sup_{V\cup\Xi} u = \sup_{\Xi} u$.

Proof. We shall prove (1) only. If u is constant, then the assertion trivially holds. We assume that u is not constant. It suffices to show that $\inf_V u \geq \inf_\Xi u$. Let $z_0 \in V$. We need to show that $u(z_0) \geq \inf_\Xi u$. Let $A = \{x \in V; u(x) = u(z_0)\}$. Since u is not constant, it follows that there is $x_0 \in A$ with $M_u(x_0) > 0$. Lemma 4.4 implies that there is $\xi \in \Xi$ such that $u(\xi) \leq u(x_0) - M_u(x_0)\rho(x_0, \xi)$. Therefore $\inf_\Xi u \leq u(\xi) \leq u(x_0) = u(z_0)$. □

Lemma 4.6. *Let* $\zeta \in \Xi$ *and let* $u(x) = \rho(\zeta, x)$. *Then* u *is* ∞*-superharmonic and* $M_u \equiv 1$ *in* V. *Especially* $D_\infty[u] = 1$.

Proof. Let $x \in V$ and $y \in \partial x$. Since $|u(x) - u(y)| = |\rho(\zeta, x) - \rho(\zeta, y)| \leq \rho(x, y) \leq r(x, y)$, it follows that $|\nabla u(x, y)| \leq 1$, and that $M_u(x) \leq 1$.

Let $\{z_n\}_n \in \zeta$. Take a path $\{x_i\}_{i=0}^l$ from x to z_n. Then

$$\begin{aligned}\sum_{i=1}^{l} r(x_{i-1}, x_i) &= r(x, x_1) + \sum_{i=2}^{l} r(x_{i-1}, x_i) \geq r(x, x_1) + \rho(x_1, z_n) \\ &\geq \min_{y\in\partial x}(r(x, y) + \rho(y, z_n)).\end{aligned}$$

It follows that $\rho(x, z_n) \geq \min_{y\in\partial x}(r(x, y) + \rho(y, z_n))$. Letting $n \to \infty$ we have $\rho(x, \zeta) \geq \min_{y\in\partial x}(r(x, y) + \rho(y, \zeta))$. This means that $u(x) \geq u(y) + r(x, y)$ for some $y \in \partial x$, or $\nabla u(x, y) \leq -1$. Therefore $\nabla u(x, y) = -1$, and $M_u(x) = 1$. Theorem 3.1 shows that u is ∞-superharmonic at x. □

5. The Dirichlet problem

A network is said to be ∞-*hyperbolic* if $\boldsymbol{P}_0 \neq \emptyset$; otherwise a network is said to be ∞-*parabolic*.

First we shall show a Liouville type theorem for an ∞-parabolic network, namely Theorem 5.2, which immediately follows from the next proposition.

Proposition 5.1. *Suppose that* (V, E, r) *is an* ∞*-parabolic network.*

(1) *Let* u *be an* ∞*-superharmonic function such that* $\liminf_{n\to\infty} u(x_n) > -\infty$ *for each* $\{x_n\}_n \in \boldsymbol{P}$. *Then* u *must be constant.*

(2) *Let* u *be an* ∞*-subharmonic function such that* $\limsup_{n\to\infty} u(x_n) < \infty$ *for each* $\{x_n\}_n \in \boldsymbol{P}$. *Then* u *must be constant.*

Proof. We shall prove (2) only. If u is not constant, then there is $x_0 \in V$ such that $M_u(x_0) > 0$. Theorem 4.3 implies that there is $\boldsymbol{x} = \{x_n\}_n \in \boldsymbol{P}$ such that

$$u(x_n) \geq u(x_0) + M_u(x_0) \sum_{i=1}^{n} r(x_{i-1}, x_i).$$

Since $\limsup_{n\to\infty} u(x_n) < \infty$, it follows that $\sum_{i=1}^{\infty} r(x_{i-1}, x_i) < \infty$, and therefore $\boldsymbol{x} \in \boldsymbol{P}_0$, which is impossible because $\boldsymbol{P}_0 = \emptyset$. □

Theorem 5.2. *Suppose that (V, E, r) is an ∞-parabolic network. Let u be an ∞-harmonic function which is either bounded from above or bounded from below. Then u must be constant.*

There may be an unbounded ∞-harmonic function on an ∞-parabolic network.

Example 5.3. Let $V = \{x_n\}_{n=-\infty}^{\infty}$, $E = \{(x_{n-1}, x_n)\}_{n=-\infty}^{\infty}$ and $r \equiv 1$. Then (V, E, r) is an ∞-parabolic network. Let $u(x_n) = n$. Then u is an ∞-harmonic function in V. □

From here to the end of this section we always assume that (V, E, r) is an ∞-hyperbolic network. We formulate the Dirichlet problem for ∞-harmonic functions as follows:

$$\text{For a bounded function } f \text{ on } \Xi,$$
$$\text{find a bounded } \infty\text{-harmonic function } h \in \mathbf{D}^{(\infty)} \text{ such that } h \equiv f \text{ on } \Xi.$$

We define the *upper class* $\mathcal{U}_f$, the *lower class* $\mathcal{L}_f$, the *upper solution* $\overline{\mathcal{H}}_f$ and the *lower solution* $\underline{\mathcal{H}}_f$ by

$$\mathcal{U}_f = \left\{ u \in \mathbf{D}^{(\infty)};\ \begin{array}{l} u \text{ is a bounded from below and } \infty\text{-superharmonic function} \\ \text{such that } u \geq f \text{ on } \Xi \end{array} \right\},$$
$$\mathcal{L}_f = \left\{ v \in \mathbf{D}^{(\infty)};\ \begin{array}{l} v \text{ is a bounded from above and } \infty\text{-subharmonic function} \\ \text{such that } v \leq f \text{ on } \Xi \end{array} \right\},$$
$$\overline{\mathcal{H}}_f(x) = \inf\{u(x); u \in \mathcal{U}_f\}, \qquad \underline{\mathcal{H}}_f(x) = \sup\{v(x); v \in \mathcal{L}_f\} \quad \text{for } x \in V.$$

If $\mathcal{U}_f = \emptyset$, then we let $\overline{\mathcal{H}}_f \equiv \infty$. If $\mathcal{L}_f = \emptyset$, then we let $\underline{\mathcal{H}}_f \equiv -\infty$.

Proposition 5.4. *Let f be a constant function on Ξ. Then a solution to the Dirichlet problem for f must be constant.*

Proof. Suppose that there is a nonconstant solution h to the Dirichlet problem. Let $x_0 \in V$ with $M_h(x_0) > 0$. Lemma 4.4 shows that there are $\xi, \eta \in \Xi$ such that

$$f(\xi) = h(\xi) \leq h(x_0) - M_h(x_0)\rho(x_0, \xi),$$
$$f(\eta) = h(\eta) \geq h(x_0) + M_h(x_0)\rho(x_0, \eta).$$

Then

$$0 = f(\eta) - f(\xi) \geq M_h(x_0)(\rho(x_0, \eta) + \rho(x_0, \xi)) > 0,$$

which is a contradiction. □

Proposition 4.2 shows that the boundary function must be a Lipschitz function on Ξ whenever a solution to the Dirichlet problem exists. By Proposition 5.4 we may assume that the boundary function is not constant. Therefore we restrict a boundary function to a nonconstant Lipschitz function on Ξ. This also means that Ξ contains at least two points.

For a nonconstant Lipschitz function f on Ξ we let L_f be the Lipschitz constant:

$$L_f = \sup_{\substack{\xi,\eta\in\Xi \\ \xi\neq\eta}} \frac{|f(\xi)-f(\eta)|}{\rho(\xi,\eta)}.$$

We define

$$L_{f,\xi} = \sup_{\eta\in\Xi\setminus\{\xi\}} \frac{|f(\xi)-f(\eta)|}{\rho(\xi,\eta)},$$
$$\varphi_{f,\xi}(x) = f(\xi) + L_{f,\xi}\rho(\xi,x), \qquad \psi_{f,\xi}(x) = f(\xi) - L_{f,\xi}\rho(\xi,x),$$
$$\varphi_f(x) = \inf_{\xi\in\Xi} \varphi_{f,\xi}(x), \qquad \psi_f(x) = \sup_{\xi\in\Xi} \psi_{f,\xi}(x)$$

for $\xi \in \Xi$ and $x \in V$.

Lemma 5.5. *Let f be a nonconstant bounded Lipschitz function on Ξ. Then $\varphi_{f,\xi}, \varphi_f \in \mathcal{U}_f$ and $\psi_{f,\xi}, \psi_f \in \mathcal{L}_f$ for $\xi \in \Xi$. Moreover*

$$M_{\varphi_{f,\xi}} \equiv M_{\psi_{f,\xi}} \equiv L_{f,\xi} \qquad \text{in } V,$$
$$D_\infty[\varphi_{f,\xi}] = D_\infty[\psi_{f,\xi}] = L_{f,\xi},$$
$$\varphi_{f,\xi}(\xi) = \psi_{f,\xi}(\xi) = f(\xi),$$
$$D_\infty[\varphi_f] = D_\infty[\psi_f] = L_f,$$
$$\varphi_f \equiv \psi_f \equiv f \qquad \text{on } \Xi.$$

Proof. Lemma 4.6 shows that $\varphi_{f,\xi}$ is ∞-superharmonic, that $M_{\varphi_{f,\xi}} \equiv L_{f,\xi}$, and that $D_\infty[\varphi_{f,\xi}] = L_{f,\xi}$. It is easy to see that $\varphi_{f,\xi}(\xi) = f(\xi)$. Clearly $\varphi_{f,\xi} \geq f(\xi)$ in V, which means that $\varphi_{f,\xi}$ is bounded from below. Let $\eta \in \Xi$. Then $\varphi_{f,\xi}(\eta) = f(\xi) + L_{f,\xi}\rho(\xi,\eta)$. Since $f(\eta) - f(\xi) \leq L_{f,\xi}\rho(\xi,\eta)$, it follows that $\varphi_{f,\xi}(\eta) \geq f(\eta)$. Therefore $\varphi_{f,\xi} \in \mathcal{U}_f$.

Since $\varphi_{f,\xi} \geq f(\xi) \geq \inf_\Xi f$ in V, it follows that $\varphi_f \geq \inf_\Xi f$ in V and that φ_f is finite at each vertex in V. Lemmas 3.5 and 3.6 show that φ_f is ∞-superharmonic and that $D_\infty[\varphi_f] \leq \sup_{\xi\in\Xi} D_\infty[\varphi_{f,\xi}] = \sup_{\xi\in\Xi} L_{f,\xi} = L_f$. For $\xi \in \Xi$ and $\{y_n\}_n \in \eta \in \Xi$

$$\begin{aligned}\varphi_{f,\xi}(y_n) &\geq \varphi_{f,\xi}(\eta) - D_\infty[\varphi_{f,\xi}]\rho(y_n,\eta) = \varphi_{f,\xi}(\eta) - L_{f,\xi}\rho(y_n,\eta) \\ &\geq f(\eta) - L_f\rho(y_n,\eta).\end{aligned}$$

Taking the infimum with respect to ξ and tending $n \to \infty$ we obtain $\varphi_f(\eta) \geq f(\eta)$. Since $\varphi_f(\eta) \leq \varphi_{f,\eta}(\eta) = f(\eta)$, it follows that $\varphi_f \equiv f$ on Ξ and that $\varphi_f \in \mathcal{U}_f$. The fact $|f(\xi) - f(\eta)| = |\varphi_f(\xi) - \varphi_f(\eta)| \leq D_\infty[\varphi_f]\rho(\xi,\eta)$ gives that $D_\infty[\varphi_f] \geq L_f$, and that $D_\infty[\varphi_f] = L_f$.

We can similarly prove the assertion for $\psi_{f,\xi}$ and ψ_f. □

Lemma 5.6. *Let f be a nonconstant bounded Lipschitz function on Ξ with Lipschitz constant L_f. Let*

$$\tilde{\mathcal{U}}_f = \{u \in \mathcal{U}_f; u \leq \varphi_f \text{ in } V\}, \qquad \tilde{\mathcal{L}}_f = \{v \in \mathcal{L}_f; v \geq \psi_f \text{ in } V\}.$$

Then $D_\infty[u] \leq L_f$ for $u \in \tilde{\mathcal{U}}_f \cup \tilde{\mathcal{L}}_f$.

Proof. Let $u \in \tilde{\mathcal{U}}_f$ and $x_0 \in V$. We shall show that $M_u(x_0) \le L_f$. We may assume $M_u(x_0) > 0$. Lemma 4.4 shows that there is $\xi \in \Xi$ such that $u(\xi) \le u(x_0) - M_u(x_0)\rho(x_0, \xi)$, or

$$M_u(x_0) \le \frac{u(x_0) - u(\xi)}{\rho(x_0, \xi)}.$$

Lemma 5.5 shows that $f(\xi) \le u(\xi) \le \varphi_f(\xi) = f(\xi)$, so that $u(\xi) = f(\xi)$. Also

$$u(x_0) \le \varphi_f(x_0) \le \varphi_{f,\xi}(x_0) = f(\xi) + L_{f,\xi}\rho(\xi, x_0) \le f(\xi) + L_f\rho(\xi, x_0).$$

Combining these we have $M_u(x_0) \le L_f$. This means $D_\infty[u] \le L_f$.

We can similarly prove $D_\infty[u] \le L_f$ for $u \in \tilde{\mathcal{L}}_f$. □

Theorem 5.7. *Let f be a nonconstant bounded Lipschitz function on Ξ with Lipschitz constant L_f. Then both $\overline{\mathcal{H}}_f$ and $\underline{\mathcal{H}}_f$ are bounded ∞-harmonic functions with*

$$D_\infty[\overline{\mathcal{H}}_f] \le L_f, \qquad D_\infty[\underline{\mathcal{H}}_f] \le L_f, \qquad \overline{\mathcal{H}}_f \equiv \underline{\mathcal{H}}_f \equiv f \quad \text{on } \Xi.$$

In particular, both $\overline{\mathcal{H}}_f$ and $\underline{\mathcal{H}}_f$ are solutions to the Dirichlet problem for f.

Proof. First we shall show that $\inf_\Xi f \le \overline{\mathcal{H}}_f \le \sup_\Xi f$ in V. Since the constant function $\sup_\Xi f$ is in $\mathcal{U}_f$, it follows that $\overline{\mathcal{H}}_f \le \sup_\Xi f$ in V. Let $u \in \mathcal{U}_f$. Corollary 4.5 shows that $\inf_{V\cup\Xi} u = \inf_\Xi u \ge \inf_\Xi f$, so that $u \ge \inf_\Xi f$ in V. Therefore $\overline{\mathcal{H}}_f \ge \inf_\Xi f$ in V.

Lemma 3.5 shows that $\overline{\mathcal{H}}_f$ is ∞-superharmonic in V. Let $x \in V$. Let $u(x) = \mathrm{H}_x^\infty \overline{\mathcal{H}}_f$ and $u = \overline{\mathcal{H}}_f$ in $V \setminus \{x\}$. Then Lemma 3.4 shows that $u \le \overline{\mathcal{H}}_f$, that $u \in \mathcal{U}_f$, and that u is ∞-harmonic at x. Therefore $\overline{\mathcal{H}}_f \equiv u$, and that $\overline{\mathcal{H}}_f$ is ∞-harmonic at x.

It is easy to see that $\overline{\mathcal{H}}_f(x) = \inf\{u(x); u \in \tilde{\mathcal{U}}_f\}$ for $x \in V$, where $\tilde{\mathcal{U}}_f$ is defined as in Lemma 5.6. Lemmas 3.6 and 5.6 show that $D_\infty[\overline{\mathcal{H}}_f] \le \sup\{D_\infty[u]; u \in \tilde{\mathcal{U}}_f\} \le L_f$.

Next we claim that $\overline{\mathcal{H}}_f(\xi) = f(\xi)$ for $\xi \in \Xi$. Lemma 5.5 shows that $\overline{\mathcal{H}}_f(\xi) \le \varphi_f(\xi) = f(\xi)$. For the converse, let $\{x_n\}_n \in \xi$. Then

$$\overline{\mathcal{H}}_f(\xi) \ge \overline{\mathcal{H}}_f(x_n) - D_\infty[\overline{\mathcal{H}}_f]\rho(x_n, \xi) \ge \overline{\mathcal{H}}_f(x_n) - L_f\rho(x_n, \xi).$$

For $n \in \mathbb{N}$ and $\varepsilon > 0$ there is $u \in \tilde{\mathcal{U}}_f$ such that $\overline{\mathcal{H}}_f(x_n) \ge u(x_n) - \varepsilon$. Lemma 5.6 implies that

$$u(x_n) \ge u(\xi) - D_\infty[u]\rho(x_n, \xi) \ge f(\xi) - L_f\rho(x_n, \xi).$$

Combining these we obtain $\overline{\mathcal{H}}_f(\xi) \ge f(\xi) - 2L_f\rho(x_n, \xi) - \varepsilon$. Tending $n \to \infty$ and $\varepsilon \to 0$ we have $\overline{\mathcal{H}}_f(\xi) \ge f(\xi)$. Therefore $\overline{\mathcal{H}}_f(\xi) = f(\xi)$.

Similarly we can prove the assertion for $\underline{\mathcal{H}}_f$. □

Proposition 5.8. *Let f be a nonconstant bounded Lipschitz function on Ξ. Let h be a solution to the Dirichlet problem for f. Then*

$$\psi_f \le \overline{\mathcal{H}}_f \le h \le \underline{\mathcal{H}}_f \le \varphi_f \qquad \text{in } V.$$

Proof. We note that the set of solutions to the Dirichlet problem coincides with $\mathcal{U}_f \cap \mathcal{L}_f$, especially $h \in \mathcal{U}_f \cap \mathcal{L}_f$. It follows that $\overline{\mathcal{H}}_f = \inf_{u \in \mathcal{U}_f} u \le \inf_{u \in \mathcal{U}_f \cap \mathcal{L}_f} u \le h$ in V.

We shall prove $\underline{\mathcal{H}}_f \le \varphi_{f,\xi}$ in V for $\xi \in \Xi$. On the contrary we assume that $A := \{y \in V; \underline{\mathcal{H}}_f(y) > \varphi_{f,\xi}(y)\} \neq \emptyset$ for a fixed $\xi \in \Xi$. Let $y_0 \in A$. Lemma 5.5 and Theorem 4.3 show that there is $\{y_n\}_n \in \eta \in \Xi$ such that

$$\varphi_{f,\xi}(y_n) \le \varphi_{f,\xi}(y_0) - L_{f,\xi} \sum_{i=1}^{n} r(y_{i-1}, y_i).$$

Also

$$\begin{aligned} \underline{\mathcal{H}}_f(y_0) &= \underline{\mathcal{H}}_f(y_n) + \sum_{i=1}^{n} \nabla \underline{\mathcal{H}}_f(y_i, y_{i-1}) r(y_{i-1}, y_i) \\ &\le \underline{\mathcal{H}}_f(y_n) + \sum_{i=1}^{n} M_{\underline{\mathcal{H}}_f}(y_{i-1}) r(y_{i-1}, y_i). \end{aligned}$$

Combining these and the fact that $y_0 \in A$ we have

$$\varphi_{f,\xi}(y_n) + L_{f,\xi} \sum_{i=1}^{n} r(y_{i-1}, y_i) < \underline{\mathcal{H}}_f(y_n) + \sum_{i=1}^{n} M_{\underline{\mathcal{H}}_f}(y_{i-1}) r(y_{i-1}, y_i). \tag{3}$$

Since $f(\eta) \le \varphi_{f,\xi}(\eta)$ and $\underline{\mathcal{H}}_f(\eta) = f(\eta)$, it follows that $L_{f,\xi} \sum_{i=1}^{\infty} r(y_{i-1}, y_i) \le \sum_{i=1}^{\infty} M_{\underline{\mathcal{H}}_f}(y_{i-1}) r(y_{i-1}, y_i)$. There is $n \ge 0$ with $L_{f,\xi} \le M_{\underline{\mathcal{H}}_f}(y_n)$. We take the smallest such n. If $n \ge 1$, then, since $L_{f,\xi} > M_{\underline{\mathcal{H}}_f}(y_{i-1})$ for $i = 1, 2, \ldots, n$, the inequality (3) implies that

$$\varphi_{f,\xi}(y_n) + L_{f,\xi} \sum_{i=1}^{n} r(y_{i-1}, y_i) < \underline{\mathcal{H}}_f(y_n) + \sum_{i=1}^{n} L_{f,\xi} r(y_{i-1}, y_i),$$

so that $y_n \in A$. This also holds if $n = 0$.

Let $z_0 = y_n$. Then $M_{\underline{\mathcal{H}}_f}(z_0) \ge L_{f,\xi}$ and $z_0 \in A$. Lemma 4.4 shows that there is $\zeta \in \Xi$ such that

$$f(\zeta) = \underline{\mathcal{H}}_f(\zeta) \ge \underline{\mathcal{H}}_f(z_0) + M_{\underline{\mathcal{H}}_f}(z_0) \rho(z_0, \zeta) \ge \underline{\mathcal{H}}_f(z_0) + L_{f,\xi} \rho(z_0, \zeta).$$

Lemma 5.5 shows that

$$f(\zeta) \le \varphi_{f,\xi}(\zeta) \le \varphi_{f,\xi}(z_0) + D_\infty[\varphi_{f,\xi}] \rho(z_0, \zeta) = \varphi_{f,\xi}(z_0) + L_{f,\xi} \rho(z_0, \zeta).$$

These imply that $\varphi_{f,\xi}(z_0) \ge \underline{\mathcal{H}}_f(z_0)$, which contradicts $z_0 \in A$. This means that $\underline{\mathcal{H}}_f \le \varphi_{f,\xi}$ in V for $\xi \in \Xi$, and that $\underline{\mathcal{H}}_f \le \varphi_f$ in V.

The other inequalities can be proved similarly. □

Theorem 5.9. *Let f be a nonconstant bounded Lipschitz function on Ξ with Lipschitz constant L_f. Then a solution to the Dirichlet problem for f is a solution to the variational problem:*

$$\textit{Minimize } D_\infty[u] \textit{ subject to } u \in \mathbf{D}^{(\infty)} \textit{ and } u \equiv f \textit{ on } \Xi.$$

More precisely, if $u \in \mathbf{D}^{(\infty)}$ *satisfies* $u \equiv f$ *on* Ξ*, then* $D_\infty[u] \geq L_f$*, and the equality holds when* u *is bounded and* ∞*-harmonic.*

We remark that the equality above can hold even when u is not ∞-harmonic. See Example 5.10.

Proof. Let $u \in \mathbf{D}^{(\infty)}$ with $u \equiv f$ on Ξ. Let $\xi, \eta \in \Xi$. Then $|f(\xi) - f(\eta)| = |u(\xi) - u(\eta)| \leq D_\infty[u]\rho(\xi, \eta)$. This means $D_\infty[u] \geq L_f$.

Let h be a solution to the Dirichlet problem for f. Proposition 5.8 and Lemma 5.6 show that $h \in \tilde{\mathcal{U}}_f$ and that $D_\infty[h] \leq L_f$. □

A solution to the variational problem in Theorem 5.9 is not necessarily unique.

Example 5.10. We note that Lemma 5.5 shows that φ_f is a solution to the variational problem in Theorem 5.9. Let

$$V = \{o\} \cup \{x_n, y_n, z_n\}_{n=1}^\infty,$$
$$E = \{(x_{n-1}, x_n), (y_{n-1}, y_n), (z_{n-1}, z_n)\}_{n=1}^\infty,$$
$$r(x_{n-1}, x_n) = r(y_{n-1}, y_n) = r(z_{n-1}, z_n) = 2^{-n},$$

where $x_0 = y_0 = z_0 = o$. Let $\xi = [\{x_n\}_n]$, $\eta = [\{y_n\}_n]$, $\zeta = [\{z_n\}_n]$. Let $f(\xi) = -1$, $f(\eta) = 1$, $f(\zeta) = 0$. Then $L_{f,\xi} = L_{f,\eta} = 1$ and $L_{f,\zeta} = 1/2$. We have

$$\overline{\mathcal{H}}_f(x) = \begin{cases} 0 & \text{if } x = o; \\ 2^{-n} - 1 & \text{if } x = x_n; \\ 1 - 2^{-n} & \text{if } x = y_n; \\ 0 & \text{if } x = z_n, \end{cases} \qquad \varphi_f(x) = \begin{cases} 0 & \text{if } x = o; \\ 2^{-n} - 1 & \text{if } x = x_n; \\ 1 - 2^{-n} & \text{if } x = y_n; \\ 2^{-n-1} & \text{if } x = z_n \end{cases}$$

for $n \geq 1$. Note that φ_f is not ∞-harmonic at z_1. □

Here we show a uniqueness result for a solution to the Dirichlet problem. Let

$$\delta(x) = \inf_{\eta \in \Xi} \rho(x, \eta),$$
$$\boldsymbol{Q} = \{\{x_n\}_n \in \boldsymbol{P}; \liminf_{n\to\infty} \delta(x_n) > 0 \text{ and } \limsup_{n\to\infty} \delta(x_n) < \infty\}.$$

Lemma 5.11. *Let* v_1 *and* v_2 *be bounded from above and* ∞*-subharmonic functions with* $v_1, v_2 \in \mathbf{D}^{(\infty)}$*. Suppose that* $\limsup_{n\to\infty}(v_1(x_n) + v_2(x_n)) \leq 0$ *for all* $\{x_n\}_n \in \boldsymbol{P}_0 \cup \boldsymbol{Q}$*. Then* $v_1 + v_2 \leq 0$ *in* V*.*

Proof. Let $u = v_1 + v_2$. It suffices to show that $A := \{x \in V; u(x) > \alpha\} = \emptyset$ for all $\alpha > 0$. On the contrary we assume $A \neq \emptyset$ for some $\alpha > 0$. Let $M_i(x) = M_{v_i}(x)$ for $i = 1, 2$ and $M(x) = \max\{M_1(x), M_2(x)\}$.

If $M(x) = 0$ for each $x \in A$, then u is constant on $\overline{A}$, so that $A = V$. For $\{x_n\}_n \in \boldsymbol{P}_0$, it follows that $\alpha < \limsup_{n\to\infty} u(x_n) \leq 0$, which contradicts $\alpha > 0$. We may assume that $M(x_0) > 0$ for some $x_0 \in A$.

We shall show that there exists $x_1 \in \partial x_0$ such that either

(1a) $u(x_1) > u(x_0)$ and $M(x_1) \geq M(x_0)$; or
(1b) $u(x_1) = u(x_0)$, $v_1(x_1) > v_1(x_0)$ and $M(x_1) \geq M(x_0)$.

First we consider the case $M_1(x_0) > M_2(x_0)$. By Theorem 3.1, there exists $x_1 \in \partial x_0$ such that

$$\nabla v_1(x_0, x_1) = M_1(x_0) > M_2(x_0) \geq -\nabla v_2(x_0, x_1),$$

and hence $\nabla u(x_0, x_1) > 0$, i.e., $u(x_1) > u(x_0)$. Since $M(x_1) \geq M_1(x_1) \geq \nabla v_1(x_0, x_1) = M_1(x_0) = M(x_0)$, it follows that x_1 satisfies (1a). In case $M_1(x_0) < M_2(x_0)$, similarly there is $x_1 \in \partial x_0$ with $\nabla v_2(x_0, x_1) = M_2(x_0)$, which satisfies (1a). Next we consider the case $M_1(x_0) = M_2(x_0)$. By Theorem 3.1, there exists $x_1 \in \partial x_0$ such that

$$\nabla v_1(x_0, x_1) = M_1(x_0) = M_2(x_0) \geq -\nabla v_2(x_0, x_1),$$

and hence $\nabla u(x_0, x_1) \geq 0$, i.e., $u(x_1) \geq u(x_0)$. If $u(x_1) > u(x_0)$, then an argument similar to the first case shows that x_1 satisfies (1a). If $u(x_1) = u(x_0)$, then, since $\nabla v_1(x_0, x_1) = M(x_0) > 0$, it follows that $v_1(x_1) > v_1(x_0)$. The fact $M(x_1) \geq \nabla v_1(x_0, x_1) = M(x_0)$ implies that x_1 satisfies (1b).

Since $x_1 \in A$ and $M(x_1) > 0$, there is $x_2 \in \partial x_1$ such that either

(2a) $u(x_2) > u(x_1)$ and $M(x_2) \geq M(x_1)$; or

(2b) $u(x_2) = u(x_1)$, $v_1(x_2) > v_1(x_1)$ and $M(x_2) \geq M(x_1)$.

Repeating this argument we obtain a sequence $\{x_n\}_n$ such that $x_n \in \partial x_{n-1}$ and that either

(na) $u(x_n) > u(x_{n-1})$ and $M(x_n) \geq M(x_{n-1})$; or

(nb) $u(x_n) = u(x_{n-1})$, $v_1(x_n) > v_1(x_{n-1})$ and $M(x_n) \geq M(x_{n-1})$.

Suppose that $x_k = x_l$ for some $k < l$. If (ia) holds for some i with $k < i \leq l$, then

$$u(x_l) \geq \cdots \geq u(x_i) > u(x_{i-1}) \geq \cdots \geq u(x_k) = u(x_l),$$

a contradiction; if (ib) holds for all i with $k < i \leq l$, then $v_1(x_l) > \cdots > v_1(x_k) = v_1(x_l)$, a contradiction. Therefore $\boldsymbol{x} := \{x_n\}_n$ is an infinite path.

If $M_1(x_n) \geq M_2(x_n)$, then, since $M_1(x_n) = M(x_n) \geq M(x_0) > 0$, Lemma 4.4 shows that there is $\eta_n \in \Xi$ such that $v_1(\eta_n) \geq v_1(x_n) + M_1(x_n)\rho(x_n, \eta_n) \geq v_1(x_n) + M(x_0)\rho(x_n, \eta_n)$. Let $s = \sup v_1 \vee \sup v_2$. Since $v_1(x_n) = u(x_n) - v_2(x_n) \geq u(x_0) - s$, it follows that

$$s \geq u(x_0) - s + M(x_0)\rho(x_n, \eta_n),$$

and that

$$\delta(x_n) \leq \rho(x_n, \eta_n) \leq \frac{2s - u(x_0)}{M(x_0)}.$$

The same inequality also holds when $M_1(x_n) < M_2(x_n)$. Therefore

$$\limsup_{n\to\infty} \delta(x_n) \leq \frac{2s - u(x_0)}{M(x_0)} < \infty.$$

Let $\zeta \in \Xi$. Then $u(x_n) - u(\zeta) \leq D_\infty[u]\rho(x_n, \zeta)$. Since $u(\zeta) \leq 0$ and $u(x_n) > \alpha$, it follows that $D_\infty[u]\rho(x_n, \zeta) \geq \alpha$, so that

$$\liminf_{n\to\infty} \delta(x_n) \geq \frac{\alpha}{D_\infty[u]} > 0.$$

Therefore $\boldsymbol{x} \in \boldsymbol{Q}$. The assumption shows that

$$\alpha < \limsup_{n\to\infty} u(x_n) \le 0,$$

which contradicts $\alpha > 0$. □

Theorem 5.12. *Suppose that $\boldsymbol{Q} = \emptyset$. Let f be a nonconstant bounded Lipschitz function on Ξ. Then there exists a unique solution to the Dirichlet problem for f.*

Proof. Let $u \in \mathcal{U}_f$ and $v \in \mathcal{L}_f$. We apply Lemma 5.11 to v and $-u$ and obtain $v - u \le 0$ in V. This implies $\overline{\mathcal{H}}_f \ge \underline{\mathcal{H}}_f$. Proposition 5.8 shows that a solution h to the Dirichlet problem satisfies $h \equiv \overline{\mathcal{H}}_f \equiv \underline{\mathcal{H}}_f$. □

Now we address the question:

Can we replace $\boldsymbol{P}_0 \cup \boldsymbol{Q}$ in the condition of Lemma 5.11 by $\boldsymbol{P}_0$?

Let w be a nonnegative function on E with $w(x,y) = w(y,x)$ and $\boldsymbol{R} \subset \boldsymbol{P}$. Let

$$t[w,\boldsymbol{R}] = \inf\{\sum_j r(x_{j-1},x_j) w(x_{j-1},x_j); \{x_j\}_j \in \boldsymbol{R}\},$$
$$\mathcal{M}_\infty(\boldsymbol{R}) = \inf\{\sup_E w; t[w,\boldsymbol{R}] \ge 1\}.$$

We see that $\mathcal{M}_\infty$ is an outer measure on $\boldsymbol{P}$ and we call it the ∞-*modulus*. It is easy to see that $\mathcal{M}_\infty(\boldsymbol{P} \setminus \boldsymbol{P}_0) = 0$ and $\boldsymbol{Q} \subset \boldsymbol{P} \setminus \boldsymbol{P}_0$, so the above question seems to be affirmative. However the author has no idea to answer the question.

6. An ∞-harmonic functions as a limit of p-harmonic functions

Let $1 < p < \infty$ and let

$$\varphi_p(t) = |t|^{p-1} \operatorname{sgn} t = \begin{cases} t^{p-1} & \text{if } t > 0; \\ 0 & \text{if } t = 0; \\ -(-t)^{p-1} & \text{if } t < 0. \end{cases}$$

For $x \in V$ and $u \in L(\partial x)$ we define

$$\nu^p_{x,u}(t) = \sum_{y \in \partial x} \varphi_p\Big(\frac{u(y) - t}{r(x,y)}\Big).$$

Since $\nu^p_{x,u}$ is strictly decreasing and $\lim_{t\to\pm\infty} \nu^p_{x,u}(t) = \mp\infty$, there is a unique value $\mathrm{H}^p_x u$ such that $\nu^p_{x,u}(\mathrm{H}^p_x u) = 0$. Let $D \subset V$ and $u \in L(\overline{D})$. If u satisfies $u(x) \le \mathrm{H}^p_x u$ ($u(x) \ge \mathrm{H}^p_x u$, $u(x) = \mathrm{H}^p_x u$, resp.) for each $x \in D$, then u is said to be p-*subharmonic* (p-*superharmonic*, p-*harmonic*, resp.) in D.

Now we shall show that a limit of p-harmonic functions as $p \to \infty$ is an ∞-harmonic function.

Lemma 6.1. *Let $x \in V$ and $u \in L(\partial x)$. Then*

$$\lim_{p\to\infty} \mathrm{H}^p_x u = \mathrm{H}^\infty_x u.$$

Proof. Let $t_p = \mathrm{H}_x^p u$ and $t_\infty = \mathrm{H}_x^\infty u$. If u is constant on ∂x, then $t_p = t_\infty = u$ and the assertion holds. We assume that u is not constant on ∂x. Let $\tilde{u}$ be the function such that $\tilde{u}(x) = t_\infty$ and $\tilde{u} = u$ on ∂x. Then $M_{\tilde{u}}(x) > 0$. Let

$$J_+ = \{y \in \partial x; u(y) > t_\infty\}, \qquad J_- = \{y \in \partial x; u(y) < t_\infty\}.$$

Theorem 3.1 shows that there is $y_1 \in \partial x$ such that

$$\frac{u(y_1) - t_\infty}{r(x, y_1)} = \frac{\tilde{u}(y_1) - \tilde{u}(x)}{r(x, y_1)} = \nabla \tilde{u}(x, y_1) = M_{\tilde{u}}(x) > 0.$$

This means $y_1 \in J_+$, especially $J_+ \neq \emptyset$. Since

$$\frac{u(y) - t_\infty}{r(x, y)} = \frac{\tilde{u}(y) - \tilde{u}(x)}{r(x, y)} = \nabla \tilde{u}(x, y) \leq M_{\tilde{u}}(x)$$

for $y \in \partial x$, it follows that

$$\max_{y \in J_+} \frac{u(y) - t_\infty}{r(x, y)} = M_{\tilde{u}}(x) = \mu^\infty_{x,\tilde{u}}(\tilde{u}(x)) = \mu^\infty_{x,u}(t_\infty).$$

Similarly $J_- \neq \emptyset$ and

$$\max_{y \in J_-} \frac{t_\infty - u(y)}{r(x, y)} = \mu^\infty_{x,u}(t_\infty).$$

Let $\varepsilon > 0$ with $\varepsilon < |u(y) - t_\infty|$ for every $y \in J_+ \cup J_-$. Let $J_0 = \{y \in \partial x; u(y) = t_\infty\}$, which may be an empty set. We consider

$$\nu^p_{x,u}(t_\infty + \varepsilon) = \sum_{y \in J_+} \Big(\frac{u(y) - t_\infty - \varepsilon}{r(x, y)}\Big)^{p-1} - \sum_{y \in J_- \cup J_0} \Big(\frac{t_\infty + \varepsilon - u(y)}{r(x, y)}\Big)^{p-1}.$$

Let

$$\alpha_p = \sum_{y \in J_+} \Big(\frac{u(y) - t_\infty - \varepsilon}{r(x, y)}\Big)^{p-1}, \qquad \beta_p = \sum_{y \in J_- \cup J_0} \Big(\frac{t_\infty + \varepsilon - u(y)}{r(x, y)}\Big)^{p-1}.$$

Let q be a number with $(p-1)(q-1) = 1$. Then

$$\lim_{p \to \infty} \alpha_p^{q-1} = \max_{y \in J_+} \frac{u(y) - t_\infty - \varepsilon}{r(x, y)} < \mu^\infty_{x,u}(t_\infty),$$

$$\lim_{p \to \infty} \beta_p^{q-1} = \max_{y \in J_- \cup J_0} \frac{t_\infty + \varepsilon - u(y)}{r(x, y)} > \mu^\infty_{x,u}(t_\infty).$$

Therefore $\nu^p_{x,u}(t_\infty + \varepsilon) < 0$ for sufficiently large p. Similarly $\nu^p_{x,u}(t_\infty - \varepsilon) > 0$. Since $\nu^p_{x,u}$ is strictly decreasing, it follows that $t_\infty - \varepsilon < t_p < t_\infty + \varepsilon$, and hence $t_p \to t_\infty$. □

Theorem 6.2. *Let $D \subset V$. Let $\{p_n\}_n$ be a sequence such that $1 < p_n < \infty$ and $\lim_{n\to\infty} p_n = \infty$. Let $\{u_n\}_n$ be a sequence of functions in $\overline{D}$ such that u_n is p_n-harmonic in D and converges pointwise to a function u in $\overline{D}$. Then u is ∞-harmonic in D.*

Proof. Let $x \in D$ and $\varepsilon > 0$. By Lemma 6.1 there is n such that $|\mathrm{H}_x^{p_n} u - \mathrm{H}_x^{\infty} u| < \varepsilon$. We may assume that $|u(y) - u_n(y)| < \varepsilon$ for all $y \in Nx$. Then $|\mathrm{H}_x^{p} u - \mathrm{H}_x^{p} u_n| < \varepsilon$. Since $\mathrm{H}_x^{p_n} u_n = u_n(x)$, it follows that

$$\begin{aligned} |u(x) - \mathrm{H}_x^{\infty} u| &\le |u(x) - u_n(x)| + |\mathrm{H}_x^{p_n} u_n - \mathrm{H}_x^{p_n} u| + |\mathrm{H}_x^{p_n} u - \mathrm{H}_x^{\infty} u| \\ &\le \varepsilon + \varepsilon + \varepsilon, \end{aligned}$$

which means that u is ∞-harmonic at x. □

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